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MATHEMATICS
FOR
TECHNICAL STUDENTS

BY

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BOOK I

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J. S. BOWMAN
B.Sc., A.R.C.S.

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PREFACE

The aim of this book is to provide a complete course of study in mathematics for the first year of the National Certificate in Mechanical, Structural, and Electrical Engineering. Revision problems are introduced at all stages of the course so that tedious revision towards the end of the session is avoided. Every set of examples, except the first in each Part, contains some problems dealing with previous work which may or may not have a bearing on the subject just completed, but in every case a marginal reference is given so that appropriate bookwork may be found immediately. These chapter references will be of assistance also to lecturers whose syllabuses exclude some chapters of the book, or who wish to depart from the sequence given.

A chapter contains enough material for a single lecture, and a set of problems estimated to be sufficient for a first understanding of the facts together with revision of selected points.

Many problems are technical, but the analytical type of problem favoured by public examining bodies, and essential for the development of mathematical skill, is stressed.

Part 1 contains arithmetic, algebra and the use of logarithms, while Part 2 deals with geometry, mensuration and graphical work. *It is intended that Parts* 1 *and* 2 *should be worked concurrently, for at certain stages corresponding chapters in both Parts deal with the same topic from different points of view. In fact this sequence is assumed in some text references and in some of the proofs.*

I have tried to adjust problems to have equal value, lack of difficulty being balanced by increase in content. By this means the student should soon become accustomed to the type of question he is likely to meet in his examinations. The arrangement, I hope, will prove beneficial to students and lecturers.

My sincere thanks are due to the Syndics of the Cambridge University Press for suggestions and helpful criticism, and I am greatly indebted to N. Harwood, Esq., B.Sc., A.M.I.M.E.,

who retired from the Principalship of Chesterfield Technical College in April 1950, for his continual help and advice in all stages of the work.

My thanks are also due to Messrs Macmillan and Company for permission to reproduce copyright matter from *Logarithmic and Other Tables* by Frank Castle.

J.D.N.G.

CHESTERFIELD 1950

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PART 1

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CHAPTER 1

FRACTIONS

Four rules

A quantity which consists of one integer (i.e. whole number) divided by another integer is called a vulgar fraction, the top integer is called the *numerator* of the fraction, and the bottom integer is called the *denominator* of the fraction. Thus $\frac{3}{7}$, $\frac{1}{2}$, $\frac{27}{8}$, $\frac{101}{19}$ are vulgar fractions; the first two are referred to as proper fractions because the numerator is less than the denominator, while the third and fourth are called improper fractions.

Addition. In order to add fractions it is necessary to express them with the same denominator. Consider the addition of $\frac{2}{5}$ and $\frac{1}{4}$. Fig. 1 shows two equal areas, one divided into fifths and the other into quarters. We notice that both fractions may be expressed conveniently with the denominator 20, viz. $\frac{8}{20}$ and $\frac{5}{20}$. Hence the sum of $\frac{2}{5}$ and $\frac{1}{4}$ is $\frac{8}{20}+\frac{5}{20}=\frac{13}{20}$.

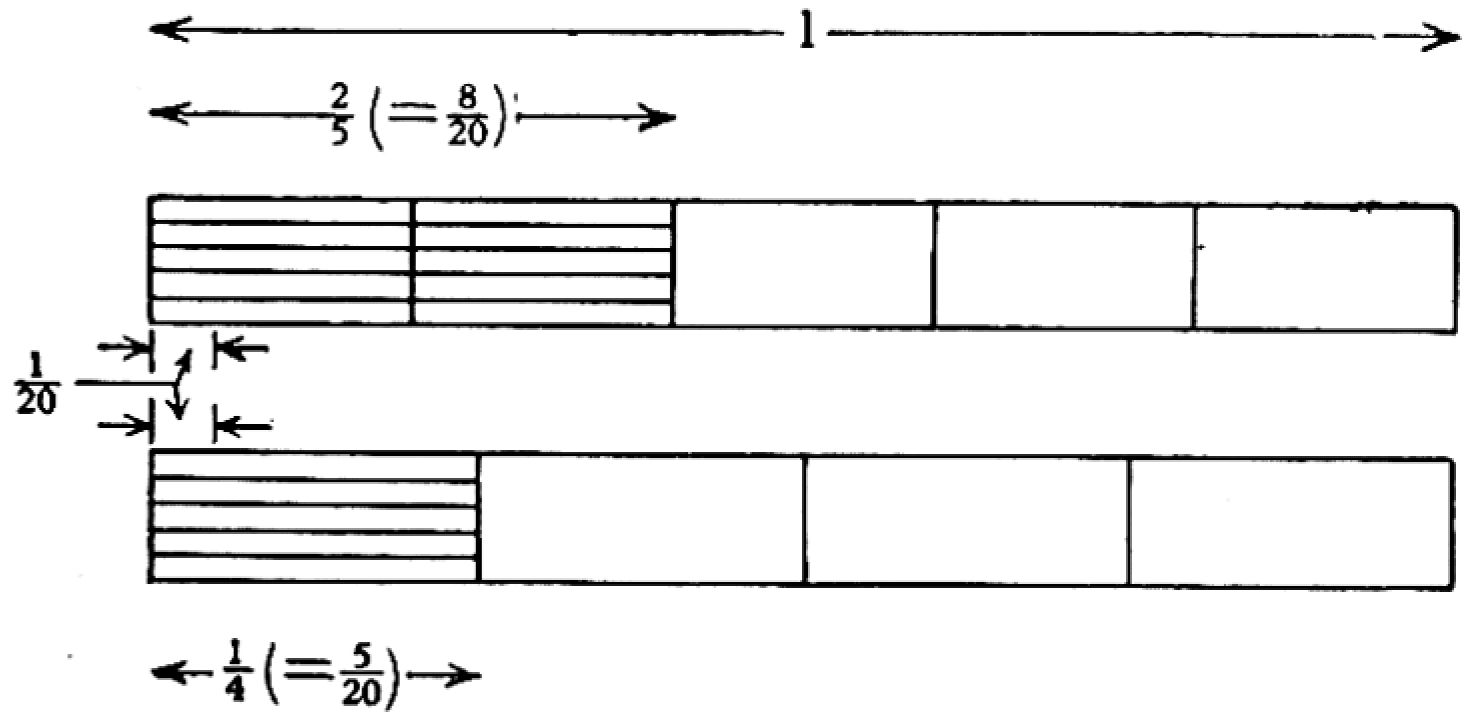

1
$\frac{2}{5}\left(=\frac{8}{20}\right)$
$\frac{1}{20}$
$\frac{1}{4}\left(=\frac{5}{20}\right)$

Fig. 1

This is the principle involved in every case of addition. Find the least number into which all the denominators will divide exactly (i.e. the Least Common Multiple, or L.C.M., of the denominators) and express all fractions to be added in terms of this denominator. The process will involve the multiplication of the numerator by the same quantity as that by which its denominator must be multiplied, but that does not alter the value of the fraction.

Example. $\frac{5}{12}+\frac{1}{3}+\frac{4}{9}$.

$$\frac{5}{12}+\frac{1}{3}+\frac{4}{9}=\frac{15+12+16}{36}=\frac{43}{36}=1\frac{7}{36}.$$

When the quantities to be added consist of mixed numbers, i.e. whole numbers with fractions added, the whole numbers may be dealt with separately.

Example. $5\frac{2}{7}+1\frac{1}{4}+\frac{4}{9}$.

$$5\frac{2}{7}+1\frac{1}{4}+\frac{4}{9}=6\frac{72+63+112}{252}=6\frac{247}{252}.$$

Subtraction. To subtract one fraction from another it is again necessary to express the fractions in terms of the same denominator, and, as in addition, whole numbers may be dealt with separately.

Example. $7\frac{3}{5}-3\frac{1}{7}$.

$$7\frac{3}{5}-3\frac{1}{7}=4\frac{21-5}{35}=4\frac{16}{35}.$$

Example. $5\frac{2}{9}-3\frac{7}{8}$.

$$5\frac{2}{9}-3\frac{7}{8}=5\frac{16}{72}-3\frac{63}{72}.$$

It is obvious that $\frac{63}{72}$ cannot be subtracted arithmetically from $\frac{16}{72}$, so it is necessary to write $5\frac{16}{72}$ as $4\frac{72+16}{72}$, i.e. $4\frac{88}{72}$, then

$$4\frac{88}{72}-3\frac{63}{72}=1\frac{25}{72}.$$

Multiplication. The meaning of the symbols '$\frac{2}{7}$' is '$\frac{2}{7}$ of 1', and the process is understood to mean that we take some quantity, divide it into seven equal parts, and then take any two of these parts. It may be, however, that the quantity which has to be subdivided into seven equal parts is not unity. We may wish to find $\frac{3}{4}$ of $\frac{2}{3}$ of 1, that is, $\frac{3}{4}$ of $\frac{2}{3}$. In this case, some whole quantity is thought of as divided into three equal parts, and two of them taken. Then these two parts ($\frac{2}{3}$) are taken as a new whole, divided into four equal parts, and any three of them taken for the answer.

This is multiplication, for '$\frac{3}{4}$ of $\frac{2}{3}$' and '$\frac{3}{4}\times\frac{2}{3}$' have the same meaning.

Consider fig. 2. The area $ABCD$ represents the whole quantity, and thus $AEFD$ represents $\frac{2}{3}$. Since $AGHD$ represents $\frac{3}{4}$ of $AEFD$, therefore the answer to the problem is the figure $AGHD$ which represents $\frac{3}{4}$ of $\frac{2}{3}$. But $AGHD$ is one-half of the original figure, so $\frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$.

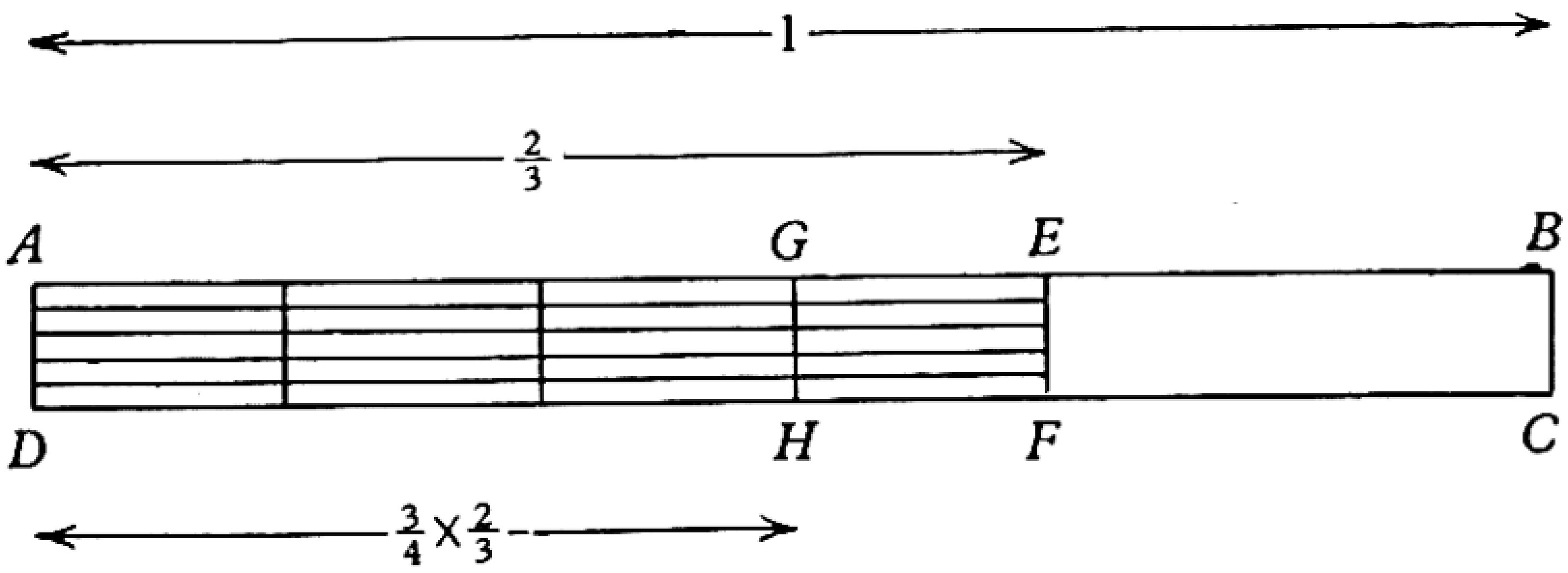


Fig. 2

The same result is obtained by multiplication of numerator by numerator, and denominator by denominator,

$\frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{1}{2}$ (Dividing numerator and denominator by 6).

This process of dividing numerator and denominator by the same quantity is called *cancelling*, a process which leaves the fraction unaltered in value. Students should note particularly that this can be done only when the whole numerator and the whole denominator are so divided.

The following cancelling is correct:

$$\frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{7}}} \times \frac{\overset{1}{\cancel{8}}}{\underset{5}{\cancel{15}}} \times \frac{\overset{3}{\cancel{21}}}{\underset{4}{\cancel{32}}} = \frac{3}{20}.$$

It would not be correct to attempt to cancel the corresponding figures in the following: $\frac{3}{7} + \frac{8}{15} + \frac{21}{32}$,

because each division of top and bottom would not be a division of the whole numerator or denominator.

When mixed numbers occur in multiplication problems it is advisable to convert them into improper fractions.

Example. $3\frac{2}{3} \times 2\frac{1}{5} \times \frac{1}{3}$.

$$3\tfrac{2}{3} \times 2\tfrac{1}{5} \times \tfrac{1}{3} = \tfrac{11}{3} \times \tfrac{11}{5} \times \tfrac{1}{3} = \tfrac{121}{45} = 2\tfrac{31}{45}.$$

Division. Fig. 3 shows the division of 1 by $\frac{1}{8}$. It is obvious that $\frac{1}{8}$ will divide into 1 exactly eight times.

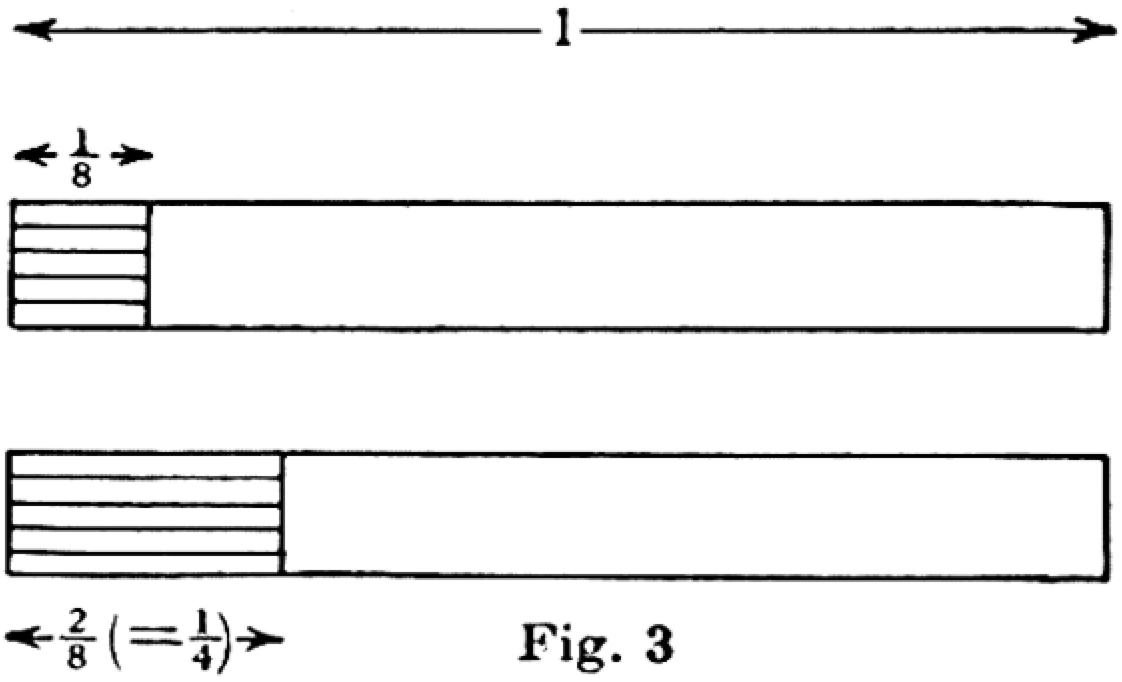


Fig. 3

Hence $\frac{1}{(\frac{1}{8})} = 8$. Also if we divide 1 by $\frac{1}{4}$ $(= \frac{2}{8})$ it is obvious that the result is half as great as before, i.e.

$$\frac{1}{(\frac{2}{8})} = 4.$$

Consequently $\quad \frac{1}{(\frac{1}{8})} = 1 \times \frac{8}{1} \quad$ and $\quad \frac{1}{(\frac{2}{8})} = 1 \times \frac{8}{2}.$

The result is true for all fractions. To divide by a fraction, *invert* the *divisor* and *multiply*.

Example. $\frac{3}{5} \div \frac{2}{3} = \frac{3}{5} \times \frac{3}{2} = \frac{9}{10}.$

Example. $5\frac{1}{3} \times 4\frac{1}{2} \div \frac{8}{27} = 5\frac{1}{3} \times 4\frac{1}{2} \times \frac{27}{8}$

$$= \frac{\overset{\overset{1}{2}}{\cancel{16}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{3}{\cancel{9}}}{\underset{1}{\cancel{2}}} \times \frac{27}{\underset{1}{\cancel{8}}} = 81.$$

Order of operations

When the problem contains no brackets, multiplication and division should be performed before addition and subtraction. (Note that the word 'of' signifies multiplication.) For example, '$\frac{7}{16} - \frac{1}{2}$ of $\frac{3}{8}$' means that $\frac{1}{2}$ of $\frac{3}{8}$ is to be subtracted from $\frac{7}{16}$; the multiplication must be done first.

Example. $1\frac{1}{3} \times \frac{9}{11} - \frac{7}{16} \div \frac{3}{4}.$

$$1\frac{1}{3} \times \frac{9}{11} - \frac{7}{16} \div \frac{3}{4} = \frac{4}{3} \times \frac{9}{11} - \frac{7}{16} \times \frac{4}{3}$$

$$= \frac{12}{11} - \frac{7}{12} \quad \text{(multiplication and division first)}$$

$$= \frac{144 - 77}{132} = \frac{67}{132}.$$

Brackets

Any numbers or fractions enclosed in a bracket must be considered as a single entity, and any number or fraction *immediately* preceding or following that bracket is to operate on the result of the bracket when evaluated. Thus $\frac{3}{4}(3\frac{1}{3}-\frac{2}{3}+\frac{1}{4})$ means that the value of the bracket is to be multiplied by $\frac{3}{4}$. In cases like this, when there is no sign between the quantity outside the bracket and the bracket, the operation is understood to be multiplication. Similarly, $(3\frac{1}{3}-\frac{2}{3}+\frac{1}{4})\div\frac{2}{3}$ means that the value of the bracket is to be divided by $\frac{2}{3}$, i.e. multiplied by $\frac{3}{2}$.

When brackets occur within brackets, distinguishing types of brackets are used as shown below.

Example. $\frac{2}{3}[5\frac{1}{4}-\frac{1}{4}(\frac{3}{8}-\frac{1}{6})+2\frac{1}{2}\{\frac{3}{8}+\frac{1}{4}(\frac{1}{2}+\frac{1}{3})\}+3]$.

$$\frac{2}{3}[5\tfrac{1}{4}-\tfrac{1}{4}(\tfrac{3}{8}-\tfrac{1}{6})+2\tfrac{1}{2}\{\tfrac{3}{8}+\tfrac{1}{4}(\tfrac{1}{2}+\tfrac{1}{3})\}+3]$$

$$=\frac{2}{3}[5\tfrac{1}{4}-\tfrac{5}{96}+2\tfrac{1}{2}\{\tfrac{3}{8}+\tfrac{5}{24}\}+3]$$

(Evaluating innermost brackets first)

$$=\frac{2}{3}[5\tfrac{1}{4}-\tfrac{5}{96}+2\tfrac{1}{2}\times\tfrac{14}{24}+3] \quad \text{(Evaluating innermost bracket)}$$

$$=\frac{2}{3}\left[5\frac{1}{4}-\frac{5}{96}+\frac{5}{\underset{1}{\cancel{2}}}\times\frac{\overset{7}{\cancel{14}}}{24}+3\right]$$

$$=\frac{2}{3}[5\tfrac{1}{4}-\tfrac{5}{96}+\tfrac{35}{24}+3] \quad \text{(Performing multiplication first)}$$

$$=\frac{2}{3}[9+\tfrac{1}{4}-\tfrac{5}{96}+\tfrac{11}{24}] \quad \text{(Adding whole numbers)}$$

$$=\frac{2}{3}\left[9\frac{24-5+44}{96}\right]=\frac{2}{3}\times 9\frac{63}{96}=\frac{2}{3}\times 9\frac{21}{32}$$

$$=\frac{2}{3}\times\frac{309}{32}=\frac{103}{16}=6\tfrac{7}{16}.$$

Example. In a works, $\frac{1}{3}$ of the total employees are engaged in the fitting shop, $\frac{1}{5}$ in the erecting shop, and $\frac{2}{9}$ in the foundry. The fraction employed in the drawing office comprises $\frac{3}{17}$ of this total. Find (*a*) the fraction of the total employees engaged in the drawing office, (*b*) the fraction of the total employees engaged in other departments:

$$\text{Fraction of employees in drawing office}=\frac{3}{17}(\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{2}{9})$$

$$=\frac{3}{17}\times\frac{15+9+10}{45}$$

$$=\frac{3}{17}\times\frac{34}{45}$$

$$=\frac{2}{15}.$$

Fraction of total employees engaged in departments mentioned

$$=\frac{2}{15}+\frac{1}{3}+\frac{1}{5}+\frac{2}{9}=\frac{6+15+9+10}{45}=\frac{40}{45}=\frac{8}{9}.$$

Therefore, fraction employed in other departments

$$=1-\tfrac{8}{9}=\tfrac{1}{9}.$$

Reciprocals

The reciprocal of a number is obtained by dividing the number into 1. Thus, the reciprocal of 2 is $\frac{1}{2}$, the reciprocal of $\frac{1}{2}$ is $1\div\frac{1}{2}=2$, and, similarly, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

Examples for Chapter 1

1. Evaluate:

(*a*) $5\frac{1}{4}-\frac{1}{3}\times 1\frac{1}{5}+3$. (*b*) $1\frac{1}{5}(5\frac{1}{4}-\frac{1}{3})+3$. (*c*) $5\frac{1}{4}-\frac{1}{3}(1\frac{1}{5}+3)$.
(*Chap.* 1)

2. (*a*) Evaluate $\frac{1}{2}\{3\frac{3}{8}+\frac{1}{4}(\frac{2}{3}+1\frac{1}{2})\}$.

(*b*) The articles produced by a certain firm are subjected to two examinations. At the first examination of one batch, $\frac{1}{8}$ were rejected, and ultimately $\frac{5}{6}$ of that batch were placed on the market. Find what fraction of the total product was rejected at the second examination. (*Chap.* 1)

3. (*a*) Evaluate $\frac{2}{3}(\frac{5}{6}$ of $3-\frac{1}{4}$ of $\frac{1}{7})\div\frac{2}{3}$.

(*b*) Four-fifths of the articles produced by a certain firm are for export. Of the rest, $\frac{2}{3}$ are guaranteed to one merchant, and $\frac{1}{9}$ similarly to another merchant. What fraction of the total articles produced goes on the open market? (*Chap.* 1)

4. An investor's capital is appreciating such that at the end of each year its value is $\frac{10}{9}$ of its value at the beginning of that year. What fraction of his original capital will he be able to withdraw at the end of the third year, so that only his original capital remains invested? (*Chap.* 1)

5. (*a*) $(2\frac{1}{4}-1\frac{1}{3})\div(\frac{2}{5}$ of $5\frac{1}{2})$.

(*b*) A train makes three stops before completing its journey. At the first stop it has covered $\frac{2}{9}$ of its journey; at the second stop it has covered a further $\frac{1}{11}$ of the journey, while the fraction of the total journey between second and third stops is $\frac{1}{3}$. Find what fraction of the journey remains to be covered.

If the total journey is 198 miles, how far is it from the third stop to the terminus? (*Chap.* 1)

6. A locomotive's water tank is full at the beginning of a journey, containing 1500 gallons. After covering $\frac{3}{4}$ of the journey, the tank contains 240 gallons. The locomotive then picks up 500 gallons and proceeds. Find what fraction of the tank is full of water at the end of the journey, assuming that the rate of water consumption is constant. (*Chap.* 1)

7. The value of the plant of a firm depreciates so that at the end of each year the value is $\frac{4}{5}$ of the value at the beginning of that year. The firm starts with plant of value £20,000; depreciation continues for three years, when £5000 worth of plant is added. Express the total value of the plant at this time as a fraction of the original value. (*Chap.* 1)

8. A speculator makes the following purchases: (*a*) 50 acres of land, (*b*) $\frac{1}{5}$ of a plot of land of $39\frac{1}{2}$ acres, (*c*) 4 plots of land each $\frac{3}{5}$ acre. He then sells $\frac{2}{3}$ of the land purchased. Find the acreage he has left. (*Chap.* 1)

9. One firm produces tractors, motor-cars, motor-cycles and cycles. The output of tractors is $\frac{8}{9}$ that of cars, the output of cars is $\frac{4}{5}$ that of motor-cycles, while the output of motor-cycles is $\frac{3}{7}$ that of cycles. Find what fraction the production of tractors is of that of cycles, and find what fraction the output of cycles is of the total production. (*Chap.* 1)

10. (*a*) $3\frac{1}{5}(\frac{1}{4} \text{ of } 5\frac{1}{8} - \frac{2}{3}) + \frac{2}{3} \div \frac{2}{5}$.

(*b*) Two-thirds of the employees of a firm are classed as skilled workers, the rest are unskilled. If $\frac{5}{9}$ of the former class have absences for illness during a certain year, and $\frac{7}{8}$ of the latter also have absences, what fraction of the total employees have no absences? (*Chap.* 1)

Answers to Examples for Chapter 1

1. (*a*) $7\frac{17}{20}$. (*b*) $8\frac{9}{10}$. (*c*) $3\frac{17}{20}$.
2. (*a*) $1\frac{23}{24}$. (*b*) $\frac{1}{24}$.
3. (*a*) $2\frac{13}{28}$. (*b*) $\frac{2}{45}$.
4. $\frac{271}{729}$.
5. (*a*) $\frac{5}{12}$. (*b*) $\frac{35}{99}$; 70 miles.
6. $\frac{16}{75}$.
7. $\frac{381}{500}$.
8. $20\frac{1}{10}$ acres.
9. $\frac{32}{105}$; $\frac{105}{218}$.
10. (*a*) $3\frac{19}{30}$. (*b*) $\frac{73}{216}$.

CHAPTER 2

DECIMALS

If we could restrict the denominators of all fractions to be multiples of 10, we should thereby simplify calculations. This is the basis of the decimal system. The denominator, however, is not written, but understood by the position of a dot, called a decimal point.

Our system of whole numbers uses a column for units, a column for tens, a column for hundreds, and so on. The decimal system is merely a logical extension of this system. Whole numbers are separated from fractions by a decimal point; the first figure following the point represents tenths, the next hundredths, the next thousandths, etc.

Consider 472·398. Starting from the left, there are 4 hundreds, 7 tens, 2 units, 3 tenths, 9 hundredths, and 8 thousandths.

In the number 5900, there are 5 thousands, 9 hundreds, no tens and no units. The two noughts must be written, otherwise the number becomes 59, one-hundredth of its actual value.

Similarly, 0·0059 contains 5 thousandths, and 9 ten-thousandths, but the two noughts after the point must not be omitted or the value of the number is multiplied by 100. Notice that the nought preceding the point can be omitted without altering the value of the number, but it is customary to write a nought when no whole numbers are present. The decimal point is made more obvious thereby, and errors due to misreading are avoided.

Further, consider 12·7 and 12·700. These numbers are both numerically equal, but it is not always correct to write 12·7 for either of them, because the use of the decimal system necessarily involves approximation. Now 12·7 means that no information is given regarding decimal fractions following the 7, but 12·700 means that the two figures following the 7 are known to be noughts, but no information is given about figures following the second nought. A result which is actually 12·712 may correctly be written as approximately 12·7, but not as 12·700.

Conversion of decimal fractions to vulgar fractions

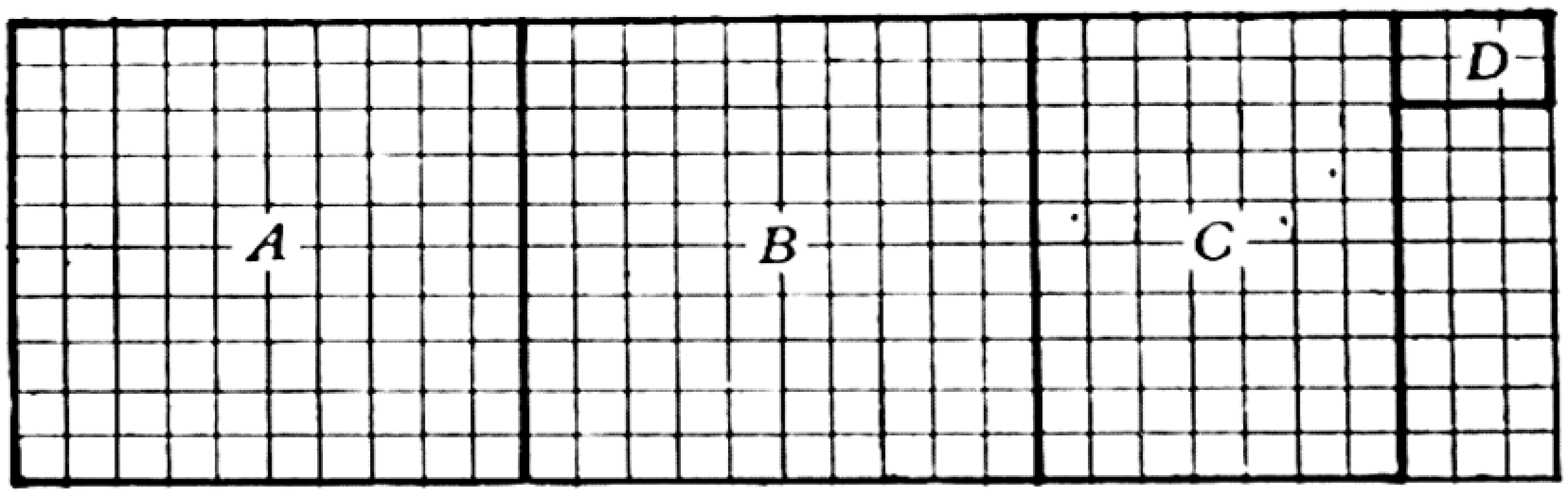
Fig. 4 illustrates the number 2·76; the areas A and B are each 1 unit, the area C is $\frac{7}{10}$ of a unit, and the area D is $\frac{6}{100}$ of a unit.

Thus 2·76 means $2 + \frac{7}{10} + \frac{6}{100}$, i.e.

$$2\frac{70+6}{100} \quad \text{or} \quad 2\frac{76}{100}.$$

Similarly, 14·327 means

$$14 + \frac{3}{10} + \frac{2}{100} + \frac{7}{1000} \quad \text{or} \quad 14\frac{300+20+7}{1000} \quad \text{or} \quad 14\frac{327}{1000}.$$



1 square inch = 1 unit

10 small squares = 0·1 unit $\left(=\frac{1}{10}\text{ unit}\right)$

1 small square = 0·01 unit $\left(=\frac{1}{100}\text{ unit}\right)$

Fig. 4

A general rule may be seen from these examples: To convert a decimal fraction into a vulgar fraction, write the actual figures which follow the decimal point in the numerator, and for the denominator write a figure 1 in place of the decimal point, and one nought for each figure in the numerator.

Thus $0{\cdot}3296 = \frac{3296}{10000}$ and $0{\cdot}0035 = \frac{35}{10000}$.

Accuracy of reading

When reading measurements of length (which includes readings of temperatures and spring balances) it is usual to read to one more place of decimals than that marked on the scale. Fig. 5 represents an object, whose length is required, placed on the scale of a ruler with markings of $\frac{1}{10}$ of an inch. We read off first 3 in. and $\frac{6}{10}$ of an inch. We then imagine the next tenth of an inch to be divided into 10 equal parts (which will be hundredths of an inch) and estimate how many of these hundredths will lie between the mark $3\frac{6}{10}$ and the end of the object. Estimating 5 from the diagram, the length of the object is $3+\frac{6}{10}+\frac{5}{100}$ in., or 3·65 in.

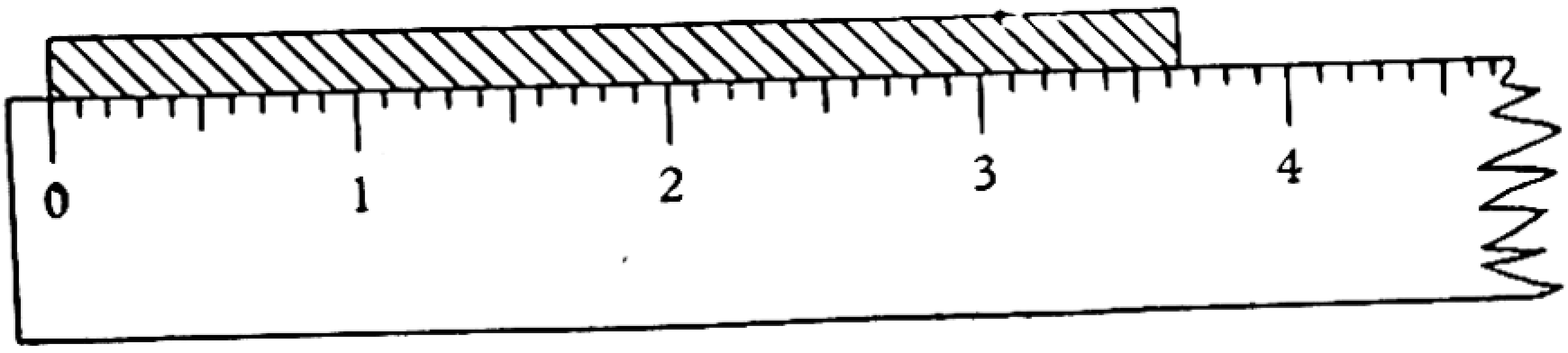


Fig. 5

Addition. Just as we add units to units, tens to tens, in the process of addition of whole numbers, so we add tenths to tenths, hundredths to hundredths in the process of addition of decimals. Hence the obvious method of setting down addition problems is to ensure that the decimal points in the several numbers are vertically underneath one another.

Subtraction. As above, the decimal points must be written beneath one another.

Example. Add 30·41 to 50·003:

$$\begin{array}{r} 30{\cdot}41 \\ 50{\cdot}003 \\ \hline 80{\cdot}413 \\ \hline\hline \end{array}$$

Example. Subtract 52·0063 from 109·651:

$$\begin{array}{r} 109{\cdot}651 \\ 52{\cdot}0063 \\ \hline 57{\cdot}6447 \\ \hline\hline \end{array}$$

Multiplication. Take first the case of multiplication by 10 or a multiple of 10. Hundredths multiplied by 10 become tenths, and tenths multiplied by 10 become units; thus, multiplication by 10 moves all figures one place to the left, or, what is the same thing, moves the decimal point one place to the right. Similarly, multiplication by 100 moves the decimal point two places to the right. Thus

$$1000\times 3{\cdot}1256 = 3125{\cdot}6 \quad\text{and}\quad 10{,}000\times 0{\cdot}000014 = 0{\cdot}14.$$

It will be obvious that division by 10 or multiples of 10 has the reverse effect. Thus

$$3{\cdot}1256 \div 1000 = 0{\cdot}0031256$$

(decimal point 3 places to left)

and

$$0{\cdot}000014 \div 10{,}000 = 0{\cdot}0000000014$$

(decimal point 4 places to left).

Numbers, such as the one above, which contain several noughts immediately after the decimal point, or numbers which are very large and contain several noughts preceding and including the units column, are frequently expressed as a number multiplied by a 'power' of 10. Since this notation is simple to understand, it will be convenient to use it here before we study the rules of 'indices' or 'powers'.

100 may be written as 10^2, which means 10×10, and is read as 'ten squared' or 'ten to the power two'. Also 1000 may be written as 10^3, which means $10 \times 10 \times 10$, and is read as 'ten cubed' or 'ten to the power three'. Similarly, 1,000,000 (one million) may be written as 10^6, so a number such as 53,600,000 may be written as $5{\cdot}36 \times 10^7$ to avoid too many noughts.

The same convention may be applied to numbers like 0·0000000014. Starting with 1·4 we have to move the decimal point 9 places to the left to produce 0·0000000014. This means that 0·0000000014 is the same as 1·4 divided by 10 *nine* times, i.e. 0·0000000014 is equal to 1·4 *divided by* 10^9. This is written as $1{\cdot}4 \times 10^{-9}$, the minus sign indicating that 1·4 is divided by 10^9 or multiplied by $\frac{1}{10^9}$.

At present, if the student will read statements like '$2{\cdot}1 \times 10^{-3}$' as '2·1 divided by 1000' or '2·1 multiplied by $\frac{1}{1000}$', he will find little difficulty in the notation.

Example.

$$0{\cdot}0000563 = 5{\cdot}63 \times 10^{-5}, \quad 0{\cdot}008421 = 8{\cdot}421 \times 10^{-3}.$$

The process of multiplying two numbers together when one or both of them contains a decimal point is the same as multiplication of whole numbers except for the location of the decimal point in the answer. The following examples will explain a simple rule which is used for this purpose.

Example. 5·823 × 0·0003.

Consider the last figure in each number, viz. 3 in both cases. The 3 in the first number represents $\frac{3}{1000}$, and the 3 in the second number represents $\frac{3}{10000}$. The product of these two numbers is $\frac{9}{10000000}$ or 0·0000009. Having found the position of the 9, we know that when 5·823 is multiplied by 0·0003, the last figure, 9, must occur in the *seventh* decimal place.

Thus $\qquad \underline{5{\cdot}823 \times 0{\cdot}0003 = 0{\cdot}0017469.}$

Otherwise,

$$5{\cdot}823 \times 0{\cdot}0003 = \frac{5823}{1000} \times \frac{3}{10000} = \frac{17469}{10000000} = 0{\cdot}0017469.$$

Example. 2·38 × 5·0002.

$$2{\cdot}38 \times 5{\cdot}0002 = \frac{238}{100} \times \frac{50002}{10000} = \frac{11900476}{1000000} = 11{\cdot}900476.$$

Notice that in each result the answer contains as many figures after the decimal point as the total of the decimal places in the two numbers which are multiplied.

Having understood this rule, multiplication problems may be set down as in the method of multiplication of whole numbers, not necessarily with decimal points underneath one another. Then to determine the position of the decimal point in the answer, we count from the right-hand end of the answer as many places as the sum of the decimal places in the two original numbers.

Example. 24·96 × 0·052.

```
  24·96
  0·052
 ------
12480
  4992
 ------
1·29792   (Count 5 decimal places from the right)
 ======
```

Checking. Students should cultivate the habit of checking multiplications and divisions mentally. In the above case the rough check is to consider 24·96 as approximately 25, and to consider 0·052 as approximately $\frac{5}{100}$ or $\frac{1}{20}$. Now $\frac{1}{20}$ of 25 is approximately 1·2, so the answer we have obtained by long multiplication may be assumed to be correct.

Division. The process is the same as long division with whole numbers, except that the question of a remainder does not arise, since we may continue dividing to as many decimal places as we like, unless the answer is exact. The one difficulty is the location of the decimal point in the answer, or quotient. There are several methods of overcoming this difficulty; the one we shall use consists in making the divisor a whole number by multiplying it by an appropriate power of 10.

Example. $642{\cdot}3 \div 5{\cdot}81$.

Now $\dfrac{642{\cdot}3}{5{\cdot}81} = \dfrac{64{,}230}{581}$ (Multiplying divisor and dividend by 100 in order to convert divisor into a whole number)

```
        110·5507
581)64230·0000(
    581
     613
     581
      3200
      2905
       2950
       2905
         4500
         4067
          433
```

Answer: 110·5507 to the fourth decimal place.

Note the following points in the problem:

(*a*) It is obvious that 581 will divide into 642 once, so the first figure, 1, of the quotient is placed over the figure 2 in the dividend, the next over the 3, and the next over the first 0. By this means the position of the decimal point in the answer is fixed over the decimal point in the dividend.

(*b*) We have not reached the end of our division; in fact, we may continue dividing indefinitely without an exact answer. It is customary, therefore, to divide to a number of decimal places convenient for the nature of the problem, the number being decided by the degree of accuracy of the divisor and dividend. If these are the results of physical measurements, we know with what accuracy they have been read.

(c) Division is continued by adding noughts in the dividend.

[Check: 5·81 is approximately 6, and 642 divided by 6 will give 107 approximately.]

Suppose the answer to the division problem above is required correct to the fourth decimal place, then we must find the *fifth* decimal place. This may be done by inspection, i.e. by considering the division of 4330 by 581. The fifth decimal place is therefore 7, so the answer correct to four decimal places is 110·5508, because 110·55077 is nearer to 110·5508 than it is to 110·5507. If the fifth decimal place had been 1, 2, 3 or 4, then the required answer correct to four places would have been 110·5507.

To write an answer correct to a given decimal place, examine the decimal place beyond this. If the figure there is 1, 2, 3 or 4, then the figure which occurs in the required place is left unaltered, but if the following figure is 5, 6, 7, 8 or 9, then the present figure in the required place is increased by one.

Applying this principle to the problem, the answer may be written as

(i) 110·5508 correct to 4 places of decimals,
(ii) 110·551 correct to 3 places of decimals,
(iii) 110·55 correct to 2 places of decimals,
(iv) 110·6 correct to 1 place of decimals,
(v) 111 correct to the nearest integer.

Significant figures

Sometimes an answer is requested correct to a certain number of significant figures. This means that the required number of figures, corrected if necessary, must be given in their proper places with respect to the decimal point.

Consider 0·0062305. This may be written as

0·006231 correct to 4 significant figures,
or 0·00623 correct to 3 significant figures,
or 0·0062 correct to 2 significant figures,
or 0·006 correct to 1 significant figure.

It could not be written to any less number of significant figures, but could be expressed as 0·01 correct to two decimal places.

Consider 7531·7, which may be written as

7532 correct to 4 significant figures,
or 7530 correct to 3 significant figures,
or 7500 correct to 2 significant figures,
or 8000 correct to 1 significant figure.

Significant figures are figures which must appear in the result for any position of the decimal point.

Example. Divide 0·000036 by 0·057 and give the result correct to three significant figures:

$$\frac{0\cdot000036}{0\cdot057} = \frac{0\cdot036}{57}$$

$$\begin{array}{l}
\cdot0006315 \\
57)0\cdot0360000(\\
\underline{342} \\
180 \\
\underline{171} \\
90 \\
\underline{57} \\
330 \\
\underline{285}
\end{array}$$

Answer: 0·000632 correct to three significant figures.

Conversion of vulgar fractions into decimal fractions

The meaning of a vulgar fraction is that the denominator is to be divided into the numerator, so in order to convert to a decimal fraction, this division is performed:

$$\tfrac{1}{4} = 1 \div 4 = 0\cdot25 \text{ by division,}$$

$$\tfrac{1}{8} = 1 \div 8 = 0\cdot125.$$

Also $\frac{17}{21} = 0\cdot8095$, correct to four significant figures.

Note that the reciprocal of 4 is thus 0·25, and the reciprocal of 8 is 0·125.

Recurring decimals

The fraction $\frac{1}{3}$ becomes 0·3333... on division, the figure 3 recurring in every subsequent division. Also $\frac{1}{6}$ becomes 0·16666... with a recurring 6, while $\frac{2}{11}$ becomes 0·181818..., the figures 1 and 8 recurring. Such decimals are written frequently as $0{\cdot}\dot{3}$, $0{\cdot}1\dot{6}$ and $0{\cdot}\dot{1}\dot{8}$ respectively, with dots over the figures which recur. Notice that $0{\cdot}\dot{3}$ correct to four significant figures is 0·3333, but $0{\cdot}1\dot{6}$ correct to the same number of figures is 0·1667, while the fraction $\frac{2}{3}$ correct to four significant figures is 0·6667.

The metric system

Units of length and of mass used commonly on the Continent of Europe, and used in the physical sciences, are the metre (m.) and the gramme (g.) respectively. Multiples and submultiples are given below:

10 millimetres (mm.) = 1 centimetre (cm.),
10 centimetres = 1 decimetre (dm.),
10 decimetres = 1 metre (m.),
1000 metres = 1 kilometre (km.),
1000 grammes = 1 kilogramme (kg.).

For conversion from the metric system to the English system, and vice versa, the following relationships may be used:

39·37 in. ≑ 1 m.
or 2·54 cm. ≑ 1 in.
1 lb. ≑ 453·6 g.

(The sign ≑ means 'is approximately equal to'.)

Accuracy of results

The technical student will be concerned with calculation of results from readings taken experimentally, and must decide how many figures must be included in the final answer obtained from multiplications or divisions.

Consider two readings, 3·72 and 5·39, both correct to three significant figures, which have to be multiplied together.

Now 3·72 × 5·39 gives 20·0508. How many of these figures must be retained in the answer, bearing in mind that the readings are only accurate to three figures? Actually 3·72 could have been anything from 3·715 to 3·724, and 5·39 could have been anything from 5·385 to 5·394:

$$3{\cdot}715 \times 5{\cdot}385 = 20{\cdot}005275 \quad \text{and} \quad 3{\cdot}724 \times 5{\cdot}394 = 20{\cdot}087256.$$

The three results, 20·0508, 20·005275 and 20·087256, expressed correct to three significant figures are 20·1, 20·0 and 20·1 respectively. The answer can be guaranteed only to two significant figures, but the error in writing 20·1 would not be very great. It is a rule that the result of an operation involving decimals should be expressed to as many significant figures as is the least accurate of the original numbers in the problem. Hence, if two numbers, one of which is correct to three significant figures, and the other correct to five significant figures, are multiplied, the result may only be given correct to three (or less) significant figures.

Examples for Chapter 2

1. (*a*) In order to find the average pressure in the cylinder of an engine, the following figures must be added and the total then divided by 8:

2·51, 2·36, 1·89, 1·56, 1·43, 1·39, 0·96, 0·54.

Find this result.

(*b*) To convert this result into the correct units, it is necessary to multiply the previous answer by 60. Give the answer correct to (i) three significant figures, (ii) two significant figures. (*Chap.* 2)

2. The length of a brass rod when heated to a temperature of 50° C. is given by the following expression:

$$\text{Length} = 100(1 + 0{\cdot}0000189 \times 50).$$

Find its length.

[*Note*. First multiply 0·0000189 by 50, then add 1, and then multiply the result by 100.] (*Chap.* 2)

3. (*a*) $3{\cdot}1 \times 10^4$ may be written as 31,000, and $3{\cdot}1 \times 10^{-4}$ may be written as 0·00031. Express in similar form $4{\cdot}2 \times 10^7$ and $16{\cdot}7 \times 10^{-6}$.

(*b*) Express 5,310,000 as 5·31 multiplied by a power of 10, and write similarly 0·00531. (*Chap.* 2)

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4. If mean value $= \dfrac{\text{area}}{\text{length}}$, find the mean value when area $= 1{\cdot}414$ and length $= 0{\cdot}251$. If the values are considered to be correct to the number of figures given, write the answer as accurately as can be expected. (*Chap.* 2)

5. The 'efficiencies' of seven machines are given by the following figures:

62·5, 81·4, 53·7, 29·5, 80·6, 32·3, 20·4.

The 'average' is obtained by adding these figures and dividing the total by the number of machines. Find the average efficiency correct to one decimal place, and find by how much the least efficient and the most efficient machines differ from that average. (*Chap.* 2)

6. (*a*) Calculate the reciprocals of the following, giving answers correct to three decimal places:

5, 0·51, 2·31.

(*b*) Add together 1·24 km., 20·3 m. and 67 cm., expressing the result in metres. (*Chap.* 2)

7. The work done by a machine may be measured as a product of feet and pounds weight, the answer then being expressed in foot-pounds. A certain machine does work equal to the product of 1200 lb. × 3·21 ft. in 1 sec. Express this work in foot-pounds per second. To find horse-power, divide work per second by 550. Give the horse-power of the machine correct to two decimal places. (*Chap.* 2)

8. Evaluate $3{\cdot}1 \times 10^{-4} + 2{\cdot}3 \times 10^{-3} + 7{\cdot}4 \times 10^{-4} + 0{\cdot}65 \times 10^{-3}$.
Express the answer in terms of 10^{-3}.
Divide the answer into 60. (*Chap.* 2)

9. (*a*) To one-quarter of two-thirds add one-ninth of one and a quarter, then divide this result into four and one-eighth. (*Chap.* 1)

(*b*) Evaluate the reciprocal of the answer to (*a*), correct to three significant figures. (*Chap.* 2)

10. (*a*) Evaluate $\{\frac{4}{9} \text{ of } 1\frac{1}{5} - \frac{1}{8}(\frac{1}{2} + \frac{3}{4})\} \div (1\frac{1}{2} - \frac{2}{3})$. (*Chap.* 1)

(*b*) Express the answer to (*a*) in decimal notation exactly. (*Chap.* 2)

Answers to Examples for Chapter 2

1. (*a*) 1·58. (*b*) (i) 94·8; (ii) 95.

2. 100·0945.

3. (*a*) 42,000,000; 0·0000167. (*b*) $5{\cdot}31 \times 10^6$; $5{\cdot}31 \times 10^{-3}$.

4. 5·63.

5. 51·5; 31·1; 29·9.

6. (*a*) 0·200; 1·961; 0·433. (*b*) 1260·97 m.

7. 3852 ft.lb., 7·00 horse-power.

8. 4×10^{-3}; 15,000.

9. (*a*) $13\frac{1}{2}$. (*b*) 0·0741.

10. (*a*) $\frac{181}{400}$. (*b*) 0·4525.

CHAPTER 3

RATIO AND PROPORTION. PERCENTAGE

Ratio

We may compare two quantities by expressing one of them as a fraction of the other. For example, $\frac{1}{3}$ stands for the relationship between 5 and 15, 17 and 51, in fact, for any two numbers the second of which is three times as great as the first. Stated differently, it may be said that the ratio of 17 to 51 is 1 to 3 (written as $\frac{1}{3}$ or 1:3). Scale-drawing makes use of ratios, for if a drawing is made to a scale of 1 in. to 1 ft., the ratio between the length of a line on the drawing and the actual length of the line it represents is 1:12 or $\frac{1}{12}$. Similarly, the ratio of £15,000 to £10,000 to £5000 is 3:2:1. (The fractional notation is not possible in this case.) A ratio should always be expressed in the smallest possible integers; thus 2:6 should be written 1:3. When, however, these are unwieldy it is convenient to divide the smallest into the others, so that the smallest shall be represented by 1, and the others by whole numbers and decimals. Thus 15:32:48 may equally well be written 1:2·13:3·2.

Example. Three partners have contributed £50,000, £10,000 and £4000 respectively to a business, and decide to share all profits in the ratio of capital contributed. How will they divide a profit of £2880?

The profit is divided in the ratio 50:10:4, i.e. 25:5:2. It must, therefore, be divided into 25 + 5 + 2 (32) equal parts, of which the first receives 25 parts, the second 5 parts, and the third 2 parts.

Now $\frac{1}{32}$ of £2880 = £90.

Hence the first receives $25 \times £90 = £2250$,
the second receives $5 \times £90 = £450$,
the third receives $2 \times £90 = £180$.

Example. Two firms *A* and *B* both start the year with capital valued at £300,000. During the year firm *A* adds another £85,000 capital, and the total depreciation is reckoned at £94,000 over the whole year. Firm *B* adds no new capital but depreciates by £80,000 over the year. Find the ratio of the capital values of the two firms at the end of the year:

Value of capital of A after 1 year

$$= £300{,}000 + £85{,}000 - £94{,}000 = £291{,}000,$$

Value of capital of B after 1 year

$$= £300{,}000 - £80{,}000 \qquad = £220{,}000.$$

Ratio required is $\dfrac{A}{B} = \dfrac{291}{220}$.

Now $\frac{291}{220} = 1{\cdot}323$ correct to four significant figures.

Thus, ratio of value of A to value of B is $1{\cdot}323 : 1$.

Proportion

The electrical resistance (measured in ohms) for different lengths of a sample of wire was measured, with the following results:

Length of wire in ft.	2	6	11	25
Resistance of wire in ohms	$\frac{1}{4}$	$\frac{3}{4}$	$1\frac{3}{8}$	$3\frac{1}{8}$

When the resistance is divided by the corresponding length, the same result, viz. $\frac{1}{8}$, is obtained in every case. The resistance is said to be proportional to the length of the wire (or the length is proportional to the resistance). If another sample of a different type of wire were taken, it would still be found that the resistance and length were proportional to each other (over a certain range), although the ratio of resistance to length would not be $\frac{1}{8}$.

Again, when a spring is loaded with different weights, and the corresponding extensions are measured, it is found that the extension is doubled when the load is doubled, and trebled when the load is trebled. The extension is then said to be proportional to the load.

These examples will show that two quantities are said to be proportional to each other if the ratios between corresponding values of the two quantities are the same.

Many examples of proportionality are to be found in everyday experience. For instance, the price of a quantity of a commodity is proportional to the quantity bought.

We shall choose a method of solving problems on proportion which will help in the study of variation later.

Example. A cargo boat covers 3200 miles in 12 days. Find (*a*) how many miles the boat would be expected to cover in 17 days, (*b*) the time the boat would take for a journey of 2200 miles.

(*a*)

Miles covered	3200	M
Time taken	12	17

(where M represents the number of miles required for the answer).

We take it that distance is proportional to time, so the figures and letter written above represent a part of a table similar to the table given in the electrical example.

Hence, to obtain miles from days we must multiply by the fraction $\frac{3200}{12}$ or $\frac{800}{3}$.

To determine the distance covered in 17 days, multiply 17 by $\frac{800}{3}$.

Therefore, distance covered in 17 days

$$= 17 \times \tfrac{800}{3} \text{ miles}$$
$$= 4533\tfrac{1}{3} \text{ miles}$$
$$= \underline{4533 \text{ miles approximately.}}$$

(*b*)

Miles covered	3200	2200
Time taken	12	D

(where D represents the number of days for a journey of 2200 miles).

To obtain days from miles we must multiply by $\frac{12}{3200}$, i.e. $\frac{3}{800}$.

Therefore, number of days $= 2200 \times \frac{3}{800} = \frac{66}{8} = \underline{8\tfrac{1}{4} \text{ days.}}$

The cases dealt with are all *direct* proportionalities, that is, as one quantity increases, the other increases in a definite ratio. Sometimes an increase of one quantity corresponds to a decrease in another. When this occurs, the quantities are said to be inversely proportional.

Consider the case of a firm of contractors wishing to complete an excavation in 12 days, knowing that with the present staff of labourers, the task will occupy 24 days. Other things being equal, they expect, therefore, that if the staff of labourers is doubled, the excavation will be completed in the required time. Quantity of labour and time taken are inversely proportional to each other.

The following table is constructed on the case just quoted:

Number of workers	20	30	40	60
Time to complete task (days)	24	16	12	8

It is obvious that the ratio between number of workers and days is not the same for any two pairs of values, for as one increases the other decreases.

Now consider rewriting the table using the reciprocals of the top line:

Reciprocals of number of workers	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{40}$	$\frac{1}{60}$
Time to complete task (days)	24	16	12	8

Comparing corresponding values in the two lines, it is seen that:

$$\tfrac{1}{20} \div 24 = \tfrac{1}{30} \div 16 = \tfrac{1}{40} \div 12 = \tfrac{1}{60} \div 8 = \tfrac{1}{480}.$$

Otherwise, we could leave the line showing number of workers unaltered, and write reciprocals of the values of time taken to complete the task.

Number of workers	20	30	40	60
Reciprocal of times	$\frac{1}{24}$	$\frac{1}{16}$	$\frac{1}{12}$	$\frac{1}{8}$

Here again,

$$20 \div \tfrac{1}{24} = 30 \div \tfrac{1}{16} = 40 \div \tfrac{1}{12} = 60 \div \tfrac{1}{8} = 480.$$

We conclude, therefore, that when two quantities are inversely proportional to each other, the reciprocal of one of them is directly proportional to the other quantity.

Example. A small building can be completed by 5 men in 8 days. Find (*a*) how long 8 men would take to complete it, (*b*) how many men would be required to complete it in 20 days.

(*a*) This is obviously an inverse relationship, so it will be best to write reciprocals of the values of the quantity for which two figures are given:

$\frac{1}{\text{Men}}$	$\frac{1}{5}$	$\frac{1}{8}$
Days	8	D

(where D represents the number of days for 8 men to complete the task).

Now the ratio of days to $\frac{1}{\text{Men}}$ is $8 \div \frac{1}{5}$ or 40.

Therefore

D = number of days required for 8 men = $40 \times \frac{1}{8} = \underline{5 \text{ days.}}$

(*b*)

$$\begin{array}{ccc} \text{Men} & 5 & M \\ \frac{1}{\text{Days}} & \frac{1}{8} & \frac{1}{20} \end{array}$$

(where M = number of men to complete task in 20 days).

Now the ratio of men to $\frac{1}{\text{Days}}$ is $5 \div \frac{1}{8} = 40$.

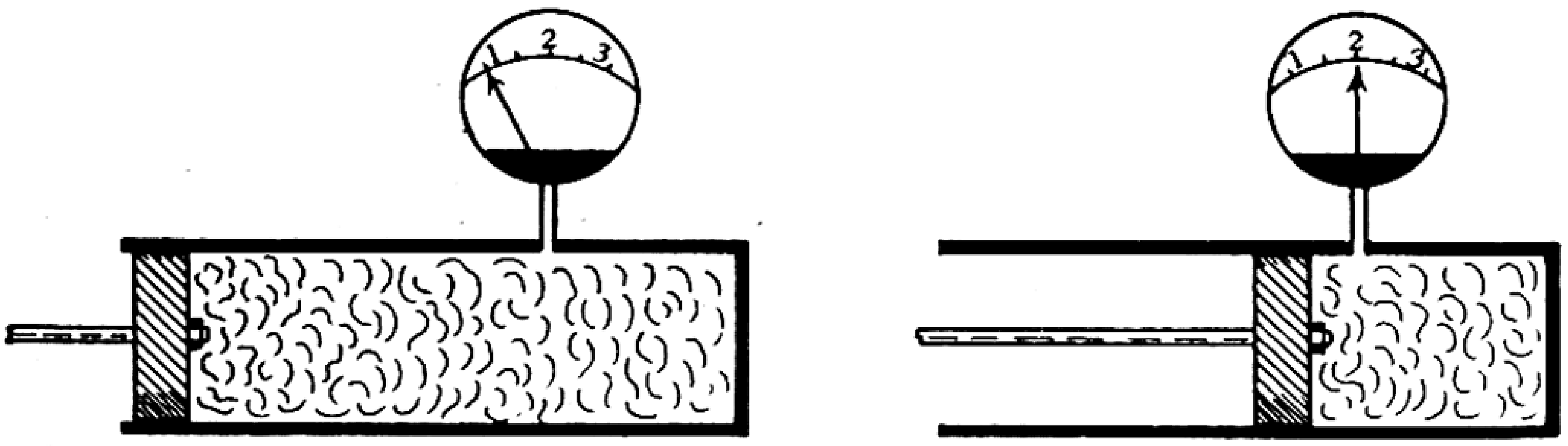
Therefore

M = number of men to complete task in 20 days = $40 \times \frac{1}{20}$

$= \underline{2 \text{ men.}}$

Boyle's Law

An important example of inverse proportion is the relationship between the volume of a gas and the pressure. Fig. 6 illustrates that doubling the pressure of a gas halves the volume. In the experiment a number of readings of pressures and corresponding volumes are taken, and the conclusion derived from the figures is that the volume of a given mass of gas is inversely proportional to the pressure (provided the temperature remains constant), or that the pressure is inversely proportional to the volume. This is Boyle's Law, which the student will use in Mechanics.



Original volume and pressure **Volume halved, pressure doubled**

Fig. 6

Example. The volume of gas in a cylinder is 1·6 cu.ft. when the pressure is 15 lb. per sq.in. Find (*a*) the volume when the pressure is 50 lb. per sq.in., (*b*) the pressure when the piston is pulled outwards to increase the volume to 2·2 cu.ft.

Pressure and volume are inversely proportional, and since we require volume for the answer to (a), write reciprocals of pressure:

$$\begin{array}{lcc} \dfrac{1}{\text{Pressure}} & \dfrac{1}{15} & \dfrac{1}{50} \\[2ex] \text{Volume} & 1{\cdot}6 & V \end{array}$$

(where V = volume when pressure is 50 lb. per sq.in.).

Ratio of volume to $\dfrac{1}{\text{Pressure}}$ is $1{\cdot}6 \div \frac{1}{15} = 24$.

Therefore

V = volume for pressure of 50 lb. per sq.in. $= 24 \times \frac{1}{50} = \underline{0{\cdot}48\,\text{cu.ft.}}$

For (b)

$$\begin{array}{lcc} \dfrac{1}{\text{Volume}} & \dfrac{1}{1{\cdot}6} & \dfrac{1}{2{\cdot}2} \\[2ex] \text{Pressure} & 15 & P \end{array}$$

(where P represents pressure for volume of 2·2 cu.ft.).

Ratio of pressure to $\dfrac{1}{\text{Volume}} = 15 \div \dfrac{1}{1{\cdot}6} = 24$.

Therefore

$$\begin{aligned} P &= \text{pressure required for volume of } 2{\cdot}2 \text{ cu.ft.} \\ &= 24 \times \frac{1}{2{\cdot}2} \doteqdot 10{\cdot}91 \text{ lb. per sq.in.} \end{aligned}$$

Percentage

To compare two fractions it is necessary to write them with the same denominator. Thus the difference between $\frac{7}{16}$ and $\frac{27}{64}$ is easily seen when they are written as $\frac{28}{64}$ and $\frac{27}{64}$.

Similarly, $\frac{17}{20}$ and $\frac{4}{5}$ are better compared when written as $\frac{85}{100}$ and $\frac{80}{100}$. A percentage is a fraction with a denominator 100, although, as we shall see, this denominator is not written. To express one quantity as a percentage of another write one quantity over the other and convert the fraction obtained to a fraction with 100 as denominator.

Thus $\quad \dfrac{1}{4} = \dfrac{25}{100} = 25\,\%$ (twenty-five per cent),

$$\frac{1}{3} = \frac{33\frac{1}{3}}{100} = 33\tfrac{1}{3}\,\%.$$

Now 25 is a quarter of 100, and $33\frac{1}{3}$ is a third of 100, so the process of expressing a fraction as a percentage consists in writing that fraction of 100.

To express $\frac{4}{9}$ as a percentage:

$\frac{4}{9} \times 100 = \frac{400}{9} = 44\frac{4}{9}\,\%$ (This might be written $44{\cdot}\dot{4}\,\%$).

To express $\frac{17}{21}$ as a percentage:

$$\tfrac{17}{21} \times 100 = \tfrac{1700}{21} \doteqdot 80{\cdot}95\,\%.$$

To express 86 % as a fraction:

$$86\,\% = \tfrac{86}{100} = \tfrac{43}{50}.$$

To express 32·5 % as a fraction:

$$32{\cdot}5\,\% = 32{\cdot}5 \div 100 = \tfrac{325}{10} \times \tfrac{1}{100} = \tfrac{13}{40}.$$

Cases in which the percentage is less than unity, for example, $\frac{1}{10000} = 0{\cdot}01\,\%$, are written in words as a fraction of one per cent, as this would be written '0·01 of one per cent', to avoid confusion.

Example. A business of capital value £8000 makes profits of £750 and £810 respectively in two successive years. Express the second year's profit as a percentage of the capital value, and find the percentage increase in profit in the second year.

For the first part of the question, £810 is to be expressed as a percentage of £8000, i.e.

$$\tfrac{810}{8000} = \tfrac{81}{800}.$$

$$\text{Percentage required} = \tfrac{81}{800} \times 100 = \tfrac{81}{8} = \underline{10{\cdot}125\,\%}.$$

For the second part, the increase in profit is £60, and is to be expressed as a percentage of £750.

$$\text{Percentage required} = \tfrac{60}{750} \times 100 = \tfrac{6}{75} \times 100 = \underline{8\,\%}.$$

Example. The manufacturer of a certain article makes a profit of $12\frac{1}{2}\,\%$ when selling to the wholesaler, while the wholesaler makes a profit of 20 % in selling to the retailer. The retailer buys for £86. 8*s*. Find the cost of manufacture and the price the wholesaler has to pay.

The wholesaler has made a profit of 20 % on what he paid by selling for £$86\frac{2}{5}$. Thus £$86\frac{2}{5}$ represents the price the wholesaler paid to the manufacturer plus 20 % of that price, i.e. £$86\frac{2}{5} = \frac{6}{5}$ of price the wholesaler paid. Hence this price is

$$\tfrac{5}{6} \text{ of } £86\tfrac{2}{5} = \tfrac{5}{6} \times \tfrac{432}{5} = \underline{£72}.$$

Similarly, £72 represents the cost of manufacture plus $12\frac{1}{2}$% (or $\frac{1}{8}$) of that cost, i.e. £72 = $\frac{9}{8}$ of cost of manufacture.

Hence, cost of manufacture = £72 × $\frac{8}{9}$ = £64.

Examples for Chapter 3

1. Three towns, *A*, *B* and *C*, are to have 1850 publicly owned houses built. It is decided to allot these between the towns in the ratio of their populations. If their populations are respectively 65,000, 40,000 and 80,000, find how many houses each town obtains. (*Chap.* 3)

2. Two rectangular plots of ground have dimensions 52 ft. by 18 ft., and 42 ft. by 26 ft. Express the ratio of their areas (*a*) in whole numbers, (*b*) when the smaller area is denoted by unity. (*Chap.* 3)

3. The extension of a spring is directly proportional to the load. For a load of $4\frac{1}{2}$ lb. the spring extends $5\frac{1}{4}$ in. Find what load must be added to make the total extension 6 in. Find also the extension produced by a total load of 5 lb. (*Chap.* 3)

4. The electrical resistance of a wire is directly proportional to its length. A wire 4 m. long has a resistance of $2\frac{1}{2}$ ohms. It is required to wind a coil with this wire, to have a resistance of $8\frac{1}{2}$ ohms. Find the length of wire required. What would be the resistance of a coil of this wire if it contained 30 turns and each turn were 10 cm.? (*Chap.* 3)

5. Air is compressed in a cylinder to a pressure of 25 lb. per sq.in., and the volume is then $4\frac{1}{2}$ cu.ft. The pressure is then increased to 60 lb. per sq.in. Find the volume if the compression has taken place according to Boyle's Law.

The air is now allowed to push out the piston until its pressure is the same as that of the atmosphere (15 lb. per sq.in.). Find the volume which the air now occupies. (*Chap.* 3)

6. A plant turns out 200 tons of castings in 5 days. It is decided to increase the equipment and labour to $3\frac{1}{2}$ times their previous amount. What weight of castings will they now expect to complete per month?

How long would an order which took 28 days to complete on the old system require on the new system? (Assume in both cases that the plant is in operation 7 days per week.) (*Chap.* 3)

7. A firm produces articles in lots of 25 tons. It is usual to reject $2\frac{1}{2}\%$ due to flaws, the rest going on to the market. If 85 % of the marketable stock is sold overseas, find, correct to the first place of decimals, the number of tons from each lot to be exported. (*Chap.* 3)

8. A dealer can make $11\frac{1}{9}\%$ profit by selling a car for £640. What would be his profit if he sold for £648? (*Chap.* 3)

9. Four political parties, A, B, C and D, were represented at a meeting, $\frac{1}{3}$ of those present belonging to A, $\frac{1}{4}$ to B, and $\frac{3}{10}$ to C. If $\frac{1}{2}$ of the members of A group, $\frac{1}{3}$ of B group, $\frac{1}{3}$ of C group, and all of D group voted in favour of the motion, express those in favour as a fraction of the total present at the meeting. (*Chap.* 1)

10. The speed with which light waves travel through space is 3×10^{10} cm. per sec. Calculate the speed in miles per second correct to three significant figures, taking 1 in. equal to 2·54 cm. (*Chap.* 2)

Answers to Examples for Chapter 3

1. 650; 400; 800.
2. $\frac{6}{7}$ or $1:1{\cdot}17$ (to 3 sig. fig.).
3. $\frac{9}{14}$ lb.; $5\frac{5}{6}$ in.
4. $13\frac{3}{5}$ m.; $1\frac{7}{8}$ ohms.
5. $1\frac{7}{8}$ cu.ft.; $7\frac{1}{2}$ cu.ft.
6. 3920 tons; 8 days.
7. 20·7 tons.
8. £72 or $12\frac{1}{2}\%$.
9. $\frac{7}{15}$.
10. 186,000 miles per sec.

CHAPTER 4

AVERAGES. SQUARE ROOTS

Averages

The average, or mean value, of several quantities, is obtained by adding together these quantities and dividing the total by the number of quantities considered. For instance, the average mark obtained by a class for an examination is the total number of marks obtained by all candidates, divided by the number of candidates. It happens frequently that not one of the quantities used coincides with the average value, but this is of no consequence since the result is used only to give a representative value. This type of average is not the only type in use; the science of statistics deals with three types of average. From the point of view of mathematics, however, an average is a mean value as defined above.

Example. For a 6-day working week, the number of employees attending a factory each day was as follows: 1250, 1320, 1320, 1290, 1252, 1189. Find the average attendance over the week.

Total attendances:

$$1250 + 1320 + 1320 + 1290 + 1252 + 1189 = 7621.$$

Average = total attendances ÷ 6.

Hence, average = 1270·1 or 1270 to the nearest integer.

Example. The average tonnage carried by a contractor over a period of 6 weeks is 162·3 tons. If he wishes his average over 7 weeks to be 165 tons, what must he attempt to carry in the seventh week? Actually he carries 170 tons in the seventh week. Find his average.

Total tonnage in 6 weeks = $162{\cdot}3 \times 6$ tons = 973·8 tons.

Total tonnage in 7 weeks to give average of 165 tons

$$= 7 \times 165 \text{ tons} = 1155 \text{ tons}.$$

Thus, tonnage to be carried in seventh week to reach average of 165 tons is $1155 - 973{\cdot}8$ tons, i.e. $\underline{181{\cdot}2 \text{ tons}}$.

$$\text{Actual average attained} = \frac{973{\cdot}8 + 170}{7}\text{ tons}$$

$$= \frac{1143{\cdot}8}{7}\text{ tons}$$

$$= \underline{163{\cdot}4\text{ tons.}}$$

Averages applied to areas. Fig. 7 represents a plot of ground bounded on one side by a straight road, and on the other side by an irregular hedge. A surveyor makes 'offsets' perpendicular to the road at the centre points of equal distances of 10 yd. along the road. The lengths of these offsets are given on the figure. The average distance from the road to the hedge is the average length of these lines.

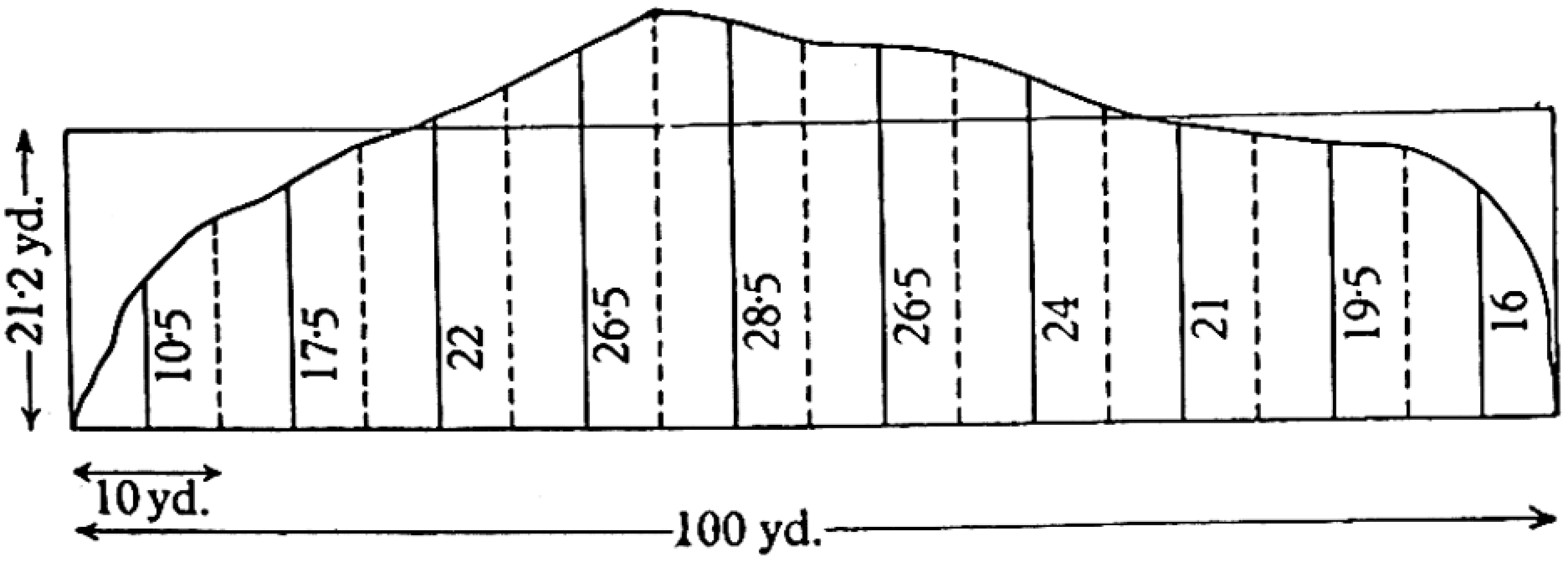


Fig. 7

Total lengths of lines

= 10·5 + 17·5 + 22 + 26·5 + 28·5 + 26·5 + 24 + 21 + 19·5 + 16 (yd.)

= 212·0 yd.

$$\text{Average distance of hedge from road} = \frac{212{\cdot}0}{10} = 21{\cdot}2\text{ yd.}$$

The result is only approximate because the boundary is not composed of straight lines. More equidistant offsets would give greater accuracy. As the average distance of the hedge from the road is 21·2 yd. we can replace the irregular hedge by a straight one distant 21·2 yd. from the road. The area of the plot is then approximately equal to the area of a rectangle 100 yd. by 21·2 yd.

Approximate area of plot = 100 × 21·2 sq.yd. = 2120 sq.yd.

Square root

A square of area 25 sq.in. has a side of length 5 in., and a square of area 81 sq.in. has a side of length 9 in. The number 5 is said to be the 'square root' of 25 (written $\sqrt{(25)}$), while 9 is the square root of 81, i.e. $\sqrt{(81)} = 9$. A square root does not necessarily refer to the side of a square; many cases occur in scientific work without reference to area.

A quantity whose square root can be obtained exactly is called a perfect square; thus 36, 64, 121, 625 are perfect squares, so are $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, the square roots of the last three being respectively $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$.

Numbers such as 2, 3, 5, 10 are not perfect squares because their square roots cannot be determined exactly.

The statement $5 \times 5 = 25$ may be written $5^2 = 25$ (five squared equals twenty-five), where the figure 2 is a power or index and means that 5 is multiplied by itself. (It does not mean that 5 is doubled.) Powers of 10 have been written in similar fashion in Chapter 2.

Square roots by factors. To find the square root of 11,025.

Resolve 11,025 into its prime factors:

$$\begin{aligned} 11{,}025 &= 3 \times 3 \times 5 \times 5 \times 7 \times 7 \\ &= 3^2 \times 5^2 \times 7^2. \end{aligned}$$

$$\begin{aligned} \text{Therefore } \sqrt{(11{,}025)} &= \sqrt{(3^2 \times 5^2 \times 7^2)} \\ &= 3 \times 5 \times 7 \\ &= \underline{105}. \end{aligned}$$

To find the square root of $\frac{484}{1225}$.

Resolve both numerator and denominator into prime factors:

$$\frac{484}{1225} = \frac{2 \times 2 \times 11 \times 11}{5 \times 5 \times 7 \times 7}.$$

$$\text{Therefore} \quad \sqrt{\frac{484}{1225}} = \sqrt{\left(\frac{2 \times 2 \times 11 \times 11}{5 \times 5 \times 7 \times 7}\right)} = \frac{2 \times 11}{5 \times 7} = \underline{\frac{22}{35}}.$$

Two important points are shown from these problems:

(i) The square root of the product of several numbers is the product of their square roots, for $\sqrt{(3^2 \times 5^2 \times 7^2)}$ is the same as $\sqrt{3^2} \times \sqrt{5^2} \times \sqrt{7^2}$, both equal to $3 \times 5 \times 7$.

(ii) The square root of a fraction is the square root of the numerator divided by the square root of the denominator. This is obvious when we consider that squaring a fraction means multiplying the numerator by itself and multiplying the denominator by itself, e.g.:

$$\left(\tfrac{22}{35}\right)^2 = \tfrac{22}{35} \times \tfrac{22}{35} = \tfrac{484}{1225},$$

therefore $$\sqrt{\tfrac{484}{1225}} = \tfrac{22}{35}.$$

The factor method of extracting square roots is only applicable when the prime factors occur in pairs. When this is not the case, the division method will have to be used, unless logarithms (Chapter 9) are employed.

Square root by division. Consider first the following facts:

(1) If a number consisting of one digit be squared, the square contains one or two digits. Hence the square root of a number consisting of one or two digits contains units or units and fractions only, e.g.

$$2^2 = 4, \quad 9^2 = 81, \qquad \text{i.e.} \quad \sqrt{4} = 2, \quad \sqrt{81} = 9.$$

(2) If a number containing two digits be squared, the square contains three or four digits, e.g.

$$13^2 = 169, \quad 86^2 = 7396.$$

(3) If a number consisting of three digits be squared, the square contains five or six digits, e.g.

$$121^2 = 14{,}641, \quad 998^2 = 996{,}004.$$

The sequence will now be obvious.

(4) When a quantity less than unity (numerically) is squared, the square is less than the square root. For example, the area of a square of side $\frac{1}{2}$ in., is $\frac{1}{4}$ sq.in.

(5) A number which may be expressed as one figure in the first decimal place gives two decimal places when squared, e.g.

$$0{\cdot}2^2 = 0{\cdot}04, \quad 0{\cdot}9^2 = 0{\cdot}81.$$

(6) A number which may be expressed as two figures in the first two decimal places gives four decimal places when squared, e.g.

$$0{\cdot}12^2 = 0{\cdot}0144, \quad 0{\cdot}65^2 = 0{\cdot}4225.$$

Rules for the division method are enumerated below, and will be referred to by number in the example which follows:

1. To determine the positions of the figures and of the decimal point in the answer mark off the digits in pairs starting from the decimal point. If the figures before the point are an odd number, then the first figure itself constitutes a 'pair'. If the number of figures following the decimal point is odd, then add a nought at the end to make the number even. Then each pair gives one figure in the answer.

2. Consider the first 'pair' in the dividend. Write over that pair the integer nearest below its square root, and write the same figure in the divisor.

3. Multiply answer by divisor (i.e. square the number written in 2). Subtract this from the first pair.

4. Double the answer and write as the new divisor, leaving room for an extra figure to be found by inspection.

5. Write the next pair beside the first remainder, and find by inspection a figure which, when written as the last figure of the divisor and used to multiply all the divisor, gives the highest number less than the remainder and pair.

Repeat rules 4 and 5 until the problem is completed.

Example. Find $\sqrt{(1369)}$.

	3 7 ·	
3	¦13¦69·	(Rule 1, mark off pairs. Rule 2 gives 3 in answer and 3 in divisor)
	9	(Rule 3 gives $3 \times 3 = 9$, and also $13 - 9 = 4$)
67	4 69	(Rule 4 gives 6 as divisor, with space left for next figure)
	4 69	(Rule 5 gives 69 as second pair beside figure 4, and also gives figure 7 which, multiplying 67, gives 469)
	· · ·	

Thus $\underline{\sqrt{(1369)} = 37}$.

Examine fig. 8, which helps to explain these processes.

ABCD is a square whose area is 1369 small squares. The square root of 1369 is the length of a side of this square (i.e. *AB*). The figure written in the answer from Rule 2 was the 3, but this represents 30 because it is over the pair of figures 13. Mark off $AE = 30$ units and complete the square *AEFG*. The rest of the answer to be found is the distance *EB*. To find *EB* remove the rectangle $GFDH$ and place it in the position $EBKJ$ shown dotted in the figure. Now the area of the rectangle $JKCH$, we know, is 1369 − 900 square units, i.e. 469 square units, and we know also that its length is more than 60 units (hence the multiplication of the divisor by 2 in Rule 4). It remains then to add to 60 a number which multiplies 60 plus itself in order to determine the length HC or EB. The figure required is 7, so we multiply 67 by 7 and obtain 469. The square root of 1369, i.e. the length AB, is 37 units.

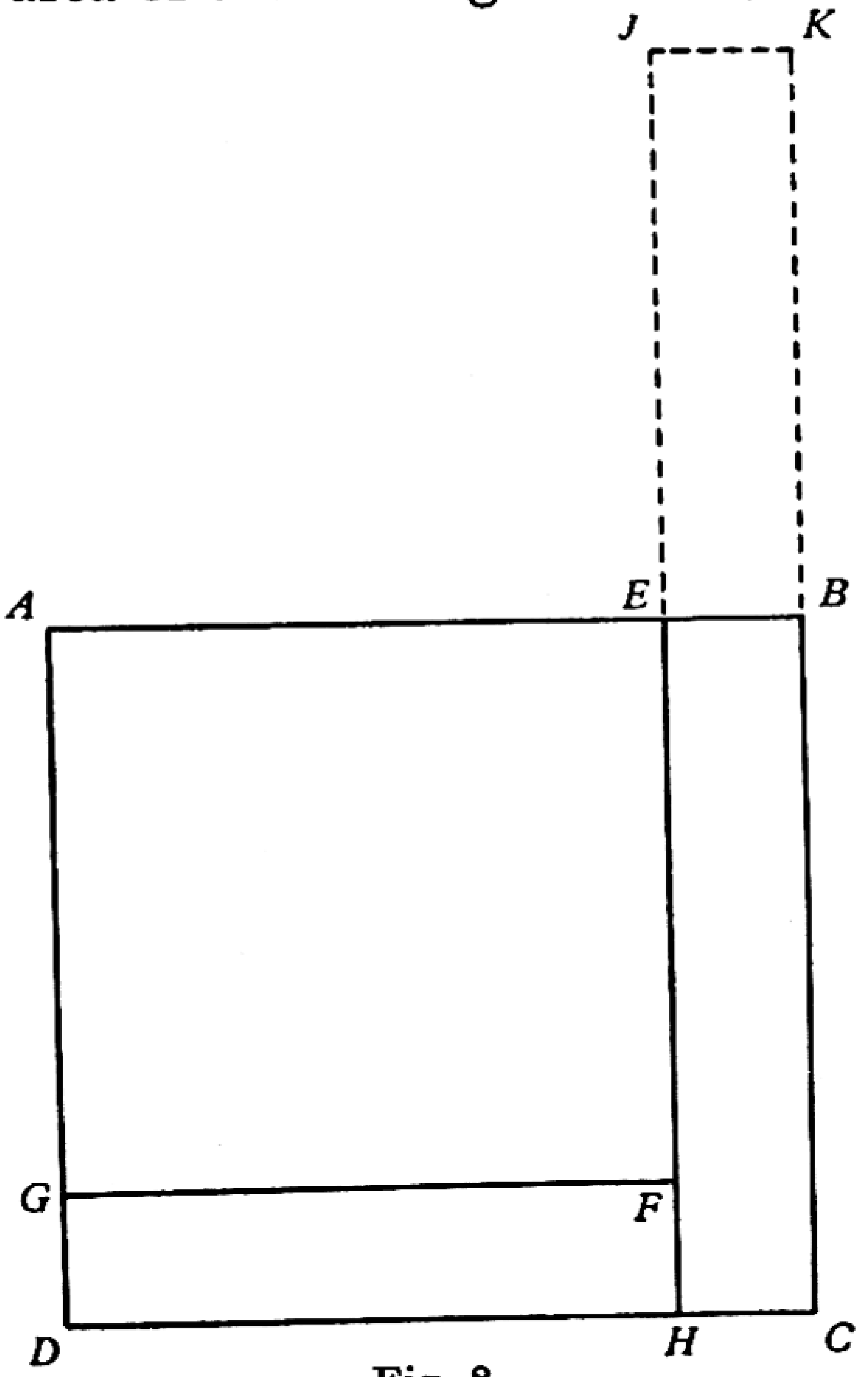


Fig. 8

In the following example the number of figures before the decimal point is odd, making the first 'pair' a single digit. The number of figures after the point is also odd, but an additional 0 makes them even.

Example. Find the square root of 11092·303.

```
          1 0  5 · 3  2
        ──────────────────
    1  | 1'10'92·30'30'
       | 1
       | ‾
  205  |   10 92
       |   10 25
       |   ─────
 2103  |      67 30
       |      63 09
       |      ─────
21062  |       4 21 30
       |       4 21 24
       |       ───────
                     6
```

(Note that 2 and another figure will not divide into 10, so we write 0 in the answer *and* in the divisor, bring down the next pair, and proceed)

Thus $\sqrt{(11092 \cdot 303)} = 105 \cdot 32$ correct to two decimal places.

Example. Determine $\sqrt{2}$ correct to four decimal places.

$$\begin{array}{r|l}
 & 1\cdot4\ \ 1\ \ 4\ \ 2\ \ 1 \\ \hline
1 & 2\cdot00\,00\,00\,00\,00 \\
 & 1 \\
24 & \overline{1}\ 00 \\
 & 96 \\
281 & 4\ 00 \\
 & 2\ 81 \\
2824 & 1\ 19\ 00 \\
 & 1\ 12\ 96 \\
28282 & 6\ 04\ 00 \\
 & 5\ 65\ 64 \\
282841 & 38\ 36\ 00
\end{array}$$

Thus $\sqrt{2} = 1\cdot4142$ correct to four decimal places.

For practical purposes $\sqrt{2}$ is taken as 1·414, $\sqrt{3}$ is taken as 1·732, and $\sqrt{5}$ is taken as 2·236. These roots, and many others, cannot be determined exactly, however many decimal places are taken. They belong to a class of numbers called surds or irrational quantities.

Some knowledge of approximate values of the square roots of 2, 3 and 5 will prove useful in evaluating square roots of larger numbers, as follows:

$$\begin{aligned}
\sqrt{8} &= \sqrt{(4\times2)} &&= \sqrt{4}\times\sqrt{2} &&= 2\times\sqrt{2} \doteq 2\times1\cdot414 = 2\cdot828,\\
\sqrt{12} &= \sqrt{(4\times3)} &&= \sqrt{4}\times\sqrt{3} &&= 2\times\sqrt{3} \doteq 2\times1\cdot732 = 3\cdot464,\\
\sqrt{45} &= \sqrt{(9\times5)} &&= \sqrt{9}\times\sqrt{5} &&= 3\times\sqrt{5} \doteq 3\times2\cdot236 = 6\cdot708,\\
\sqrt{300} &= \sqrt{(100\times3)} &&= \sqrt{100}\times\sqrt{3} &&= 10\times\sqrt{3} \doteq 10\times1\cdot732 = 17\cdot32.
\end{aligned}$$

Example. The speed of a body revolving about a vertical axle is given by the formula

$$\text{Velocity} = \sqrt{\left(\frac{0\cdot5\times4\times32}{6}\right)}\ \text{ft. per sec.}$$

Find the velocity correct to one decimal place.

$$\text{Velocity} = \sqrt{\left(\frac{0\cdot5\times4\times32}{6}\right)} = \sqrt{\left(\frac{1}{\underset{1}{2}}\times\frac{\overset{\overset{1}{2}}{4}}{\underset{3}{6}}\times32\right)} = \sqrt{\frac{32}{3}} = \sqrt{10\tfrac{2}{3}}.$$

To extract the square root of $10\frac{2}{3}$ to two decimal places, write 10·6667:

$$\begin{array}{r|r}
 & 3\cdot\ 2\ \ 6 \\ \hline
3 & 10\cdot66\,67 \\
 & 9 \\ \cline{2-2}
62 & 1\ 66 \\
 & 1\ 24 \\ \cline{2-2}
646 & 42\,67 \\
 & 38\,76 \\ \cline{2-2}
 & 3\,91
\end{array}$$

Velocity = 3·3 ft. per sec. correct to one decimal place.

Examples for Chapter 4

1. The pressure of steam in a cylinder, for eleven positions of the piston, was as follows:

100, 98, 90, 75, 60, 45, 36, 30, 28, 24, 20 lb. per sq.in.

Find the average pressure during the stroke of the piston, giving the answer to one decimal place. (*Chap.* 4)

2. (*a*) The areas of cross-section of a tapering pump-rod are measured at six equidistant points, as follows:

10·7, 7·5, 5·3, 3·7, 2·6, 1·8 sq.in.

Find the average cross-section correct to one decimal place.

(*b*) The radius of the average cross-section is given approximately by dividing the average cross-section by 3·14 and finding the square root of the answer. Find the average radius, in inches, correct to one decimal place. (*Chap.* 4)

3. (*a*) Find the square roots of the following numbers by the method of factors: (i) 196, (ii) 4356, (iii) 324.

(*b*) Evaluate the square roots of the following correct to four significant figures: (i) 9871·379, (ii) 0·0036051. (*Chap.* 4)

4. (*a*) Evaluate $\sqrt{(676)}$ by factors, and $\sqrt{7}$ by division, giving the latter correct to four significant figures.

(*b*) Hence, without further division, find $\sqrt{(700)}$ and $\sqrt{(112)}$. (*Chap.* 4)

5. (*a*) If $\sqrt{2} \doteqdot 1{\cdot}414$, and $\sqrt{3} \doteqdot 1{\cdot}732$, evaluate without division $\sqrt{(32)}$ and $\sqrt{(48)}$.

(*b*) The period of swing of a pendulum is given by

$$t = 6{\cdot}284\sqrt{\left(\tfrac{85}{981}\right)}\text{ sec.}$$

Find t to two decimal places. (*Chap.* **4**)

6. (*a*) A plot of ground is bordered on one side by a straight road, while the other borders are irregular. Ten equidistant 'offsets' are measured perpendicular to the road, in yards,

40, 42, 50, 53, 51, 49, 45, 44, 42, 41

Find the average width of the plot, and calculate its area approximately if the length is 120 yd.

(*b*) Find the side of a square which has the same area. (*Chap.* **4**)

7. The acceleration of a train is directly proportional to the effective force of the engine. When the force is 80 tons weight, the acceleration is 0·4 ft. per sec. per sec. Find the acceleration when the force is 25 tons weight, and find the force when the acceleration is 0·52 ft. per sec. per sec. (*Chap.* **3**)

8. (*a*) Evaluate $\sqrt{\{(1\tfrac{1}{2} \times 5 + \tfrac{5}{6} \times 4\tfrac{1}{3}) \div 1\tfrac{7}{9}\}}$.

(*b*) Express 0·0625 as a fraction and hence obtain its square root. (*Chaps.* **1** *and* **4**)

9. A sum of £150 was invested at such a rate of increase that at the end of any year its value was 104 % of its value at the beginning of that year. Find its value at the end of the third year. (*Chap.* **3**)

10. Write the next term of each of the following series:

(*a*) $\frac{1}{4} + \frac{7}{16} + \frac{5}{8} + \ldots.$ (*Chap.* **1**)

(*b*) $2{\cdot}3 + 0{\cdot}69 + 0{\cdot}207 + \ldots.$ (*Chap.* **2**)

(*c*) $7 + 7\sqrt{2} + 14 + \ldots.$ (*Chap.* **4**)

Answers to Examples for Chapter 4

1. 55·1 lb. per sq.in. 2. 5·3 sq.in.; 1·3 in.

3. (*a*) (i) 14; (ii) 66; (iii) 18. (*b*) (i) 99·35; (ii) 0·06004.

4. (*a*) 26; 2·646. (*b*) 26·46; 10·584.

5. (*a*) 5·656; 6·928. (*b*) 1·85 sec.

6. (*a*) 45·7 yd.; 5484 sq.yd. (*b*) 74 yd. to nearest integer.

7. 0·125 ft. per sec. per sec.; 104 tons weight.

8. (*a*) $\frac{5}{2}$. (*b*) $\frac{1}{4}$. 9. £168·7 to one decimal place.

10. (*a*) $\frac{13}{16}$. (*b*) 0·0621. (*c*) $14\sqrt{2}$.

CHAPTER 5

SYMBOLICAL REPRESENTATION

Use of symbols

One function of the branch of mathematics called algebra is to obtain a general solution to a number of problems of a similar type by means of a formula, and then to use this formula to solve any subsequent problem of that type. The area of a rectangle whose sides are 3 ft. and 2 ft., is 6 sq.ft., and the same process of multiplication must be performed whatever the lengths of those sides, so if we take the sides of a rectangle in general to be a and b, the area is the product of a and b, written ab. Multiplication of quantities denoted by letters is shown in algebra by writing these letters together. This rule is not true for numbers, for, as we know, 36 means 30 add 6, not 3×6. Again, in algebra, ab is the same as ba, for 3×6 is the same as 6×3, but the order of the figures 3 and 6 in the number 36 is important.

The same notation is used when a letter is multiplied by a number as when a letter multiplies a letter; thus $5a$ means $5 \times a$. (It is *not* written $a5$.)

When a letter is multiplied by itself the letter is not written twice, but a power or index number is used showing how many times the letter could be written down to give the correct value. Thus $a \times a$ is written a^2 (read 'a squared').

Similarly, $a \times a \times a$ is written a^3 (a cubed), and $a \times a \times a \times a$ is written a^4 (a to the power four).

Rules for symbolical representation

Addition. The plus sign (+) indicates addition. Thus $3a + 2b$ means that the quantity a is to be multiplied by 3, the quantity b multiplied by 2, and the results added. The multiplication is performed before the addition as in the rule for fractions in Chapter 1.

Subtraction. The minus sign (−) indicates subtraction.

Multiplication. Quantities to be multiplied together are written together except in the case of numbers multiplying numbers, when the multiplication sign (×) is used. The absence of a number in front of a letter or a group of letters implies that the letter (or group) is multiplied by 1, e.g. xy means $1 \times x \times y$. An index is used to imply that a number or letter is to be multiplied by itself a certain number of times.

Division. Division of the quantity a by the quantity b is written a/b. This is usually more convenient than $a \div b$, although the latter is not incorrect.

When brackets are used, the same rule applies as we adopted for the use of fractions in Chapter 1, namely, that any number or quantity immediately preceding the bracket multiplies the bracket as a whole. Similarly, the denominator of a fractional expression is taken as dividing the numerator as a whole. Hence in numerical evaluation of literal expressions (expressions containing letters) when numbers are given to replace the letters, the brackets are evaluated first.

Thus $x(a+b+c)$ implies that the value of the bracket as a whole is multiplied by x, and $\dfrac{a+b+c}{x}$ implies that x is to be divided into the sum of a, b and c.

If $x = 3$, $a = 2$, $b = 4$, $c = 6$, then the answer to the first expression is $3 \times 12 = 36$, and the answer to the second is $\frac{12}{3} = 4$.

Construction of formulae

Consider the quantity x tons. We wish to express this in (a) hundredweights, (b) pounds. Now to convert any number of tons into hundredweights multiplication by 20 is necessary, therefore x tons $= 20x$ cwt. Similarly, x tons $= 2240x$ lb.

To express a speed of v miles per hour in feet per second:

$$\begin{aligned} v \text{ m.p.h.} &= 5280v \text{ ft. per hr.} \\ &= \frac{5280v}{3600} \text{ ft. per sec.} \\ &= \frac{22v}{15} \text{ ft. per sec.} \end{aligned}$$

The statement is a formula to convert speeds expressed in miles per hour into speeds expressed in feet per second. To use this formula in order to express 60 m.p.h. in feet per second, write 60 for v.

Thus $$60\text{ m.p.h.} = \frac{22 \times 60}{15}\text{ ft. per sec.}$$
$$= 88\text{ ft. per sec.}$$

When a train is climbing an incline there are two main forces opposing its progress. First there is the resistance due to friction (rubbing of axles on bearings) which is present whenever the train is in motion. There is also the resistance due to the force of gravity which is greater the steeper the incline. The frictional force is usually expressed as a certain number of pounds weight per ton of train (because all forces may be expressed as weights). So if the train weighs W tons, and the frictional resistance is p lb. wt. per ton of train, then the total frictional resistance is Wp lb. wt. (This is not a case of multiplying tons by pounds as might be imagined at first sight, for the value W is merely a number. Let the resistance be 10 lb. wt. per ton of train, then if the train weighs 300 tons the total frictional resistance is 10 lb. wt. × 300, i.e. 3000 lb. wt.) The resistance due to gravity is the weight of the train multiplied by a quantity which is merely numerical, i.e. it has no dimensions. Thus if we multiply the weight of the train by a number, the answer is in tons weight. Let this number be k, then the resistance due to the force of gravity is Wk tons weight.

Total resistance to motion of the train
$$= Wp\text{ lb. wt.} + Wk\text{ tons wt.}$$
$$= (Wp + 2240\,Wk)\text{ lb. wt.}$$

Example. The total outlay for accessories for a car is as follows: x articles at £p each; y articles at s shillings each; z articles at half-a-crown each, and w articles at d pence each. Express the total cost in (i) pounds, (ii) shillings.

x articles at £p each $= £px$,

y articles at s shillings each $= ys$ shillings $= £\dfrac{ys}{20}$,

z articles at 2s. 6d. each $= 2\frac{1}{2} \times z$ shillings
$$= £\left(\frac{5z}{2} \times \frac{1}{20}\right) = £\frac{z}{8},$$

w articles at d pence each $= wd$ pence $= £\dfrac{wd}{240}$.

$$\text{Total cost in pounds} = \underline{£\left(px + \frac{ys}{20} + \frac{z}{8} + \frac{wd}{240}\right)}.$$

$$\text{Total cost in shillings} = \underline{\left(20px + ys + \frac{5z}{2} + \frac{wd}{12}\right) \text{ shillings.}}$$

Example. When an aircraft is landing the resistance per lb. wt. of the machine is the sum of two forces, (i) a constant force of a lb. wt., (ii) a force of $b \times$ (square of speed of aircraft in feet per second) lb. wt. Write an expression for the total resistance in lb. wt. on an aircraft weighing W tons coming in to land at V m.p.h.

Weight of aircraft $= W$ tons $= 2240W$ lb.

$$\text{Speed of aircraft} = V \text{ m.p.h.} = \frac{V \times 5280}{3600} \text{ ft. per sec.}$$

$$= \tfrac{22}{15}V \text{ ft. per sec.}$$

Hence, force of resistance per lb. wt. of machine (from original statement) $= a + b(\tfrac{22}{15}V)^2$ lb. wt.

Thus, total resistance $= 2240W\{a + b(\tfrac{22}{15}V)^2\}$ lb. wt.

$$= \underline{2240W\left\{a + \frac{484bV^2}{225}\right\} \text{ lb. wt.}}$$

Example. An express is travelling at a speed of u cm. per sec., and a slow train at v m.p.h. Find an expression for the difference in speed of the two trains in feet per second, given that 2·54 cm. = 1 in.

$$\text{Speed of express} = u \text{ cm. per sec.} = \frac{u}{2{\cdot}54} \text{ in. per sec.}$$

$$= \frac{u}{2{\cdot}54 \times 12} \text{ ft. per sec.}$$

Speed of slow train $= v$ m.p.h. $= v \times 5280$ ft. per hr.

$$= \frac{v \times 5280}{3600} \text{ ft. per sec.}$$

$$\text{Thus, difference in speeds} = \underline{\left(\frac{u}{30{\cdot}48} - \frac{22v}{15}\right) \text{ ft. per sec.}}$$

Evaluation of formulae

When a formula is given and numbers are to be inserted for the letters contained in it, the rules for addition, subtraction, multiplication, division and use of brackets are to be followed implicitly, but certain corrections may have to be made before evaluation. If, for instance, a symbol v in the formula means speed in miles per hour, then, if the figure given for speed is in ft. per sec., a conversion to the correct units must be made before evaluation.

Example. When a rod is swinging about an axle through one end, the time in seconds (t) of one oscillation is given by

$$t = 2\pi \sqrt{\frac{I}{Mgh}}.$$

Find the time of one oscillation when $\pi = \frac{22}{7}$, $I = 7\frac{1}{2}$, $M = 2\frac{1}{2}$, $g = 32$, $h = 3$.

By substitution, $t = 2 \times \frac{22}{7}\sqrt{\frac{7\frac{1}{2}}{2\frac{1}{2} \times 32 \times 3}} = \frac{44}{7}\sqrt{\frac{\frac{15}{2}}{\frac{5}{2} \times 32 \times 3}}$

$$= \tfrac{44}{7}\sqrt{(\tfrac{15}{2} \times \tfrac{2}{5} \times \tfrac{1}{32} \times \tfrac{1}{3})}$$

$$= \tfrac{44}{7}\sqrt{\tfrac{1}{32}}$$

$$= \frac{44}{7}\sqrt{\frac{1}{2 \times 16}}$$

$$= \tfrac{44}{7} \times \tfrac{1}{4} \times \sqrt{\tfrac{1}{2}} \quad (\text{since } \sqrt{16} = 4)$$

$$= \tfrac{11}{7}\sqrt{\tfrac{1}{2}}$$

$$= \frac{11}{7} \times \frac{1}{\sqrt{2}} \left(\text{since } \sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\right)$$

$$= \frac{11}{7 \times 1{\cdot}414} = \frac{11}{9{\cdot}898}$$

$$= \underline{1{\cdot}11 \text{ sec. to two decimal places.}}$$

Example. The following formulae are equations of motion for a body travelling with constant acceleration:

(1) $s = ut + \frac{1}{2}ft^2$,

(2) $v^2 = u^2 + 2fs$.

Find the value of s in (1) if $u = 30$, $t = 3$ and $f = 16$.
Find the value of v in (2) if $u = 14$, $f = 2$ and $s = 13$.

$$\begin{aligned} s &= ut + \tfrac{1}{2}ft^2 \\ &= 30 \times 3 + \tfrac{1}{2} \times 16 \times 3^2 \\ &= 90 + 8 \times 9 \\ &= 90 + 72 \\ &= \underline{162.} \end{aligned}$$

$$\begin{aligned} v^2 &= u^2 + 2fs, && \text{(A)} \\ &= 14^2 + 2 \times 2 \times 13, && \text{(B)} \\ &= 196 + 52, && \text{(C)} \\ &= 248, && \text{(D)} \end{aligned}$$

[This result is v^2, so, to obtain v, the square root of 248 must be calculated]

$$\begin{aligned} v &= \sqrt{(248)}, && \text{(E)} \\ &= \underline{15{\cdot}7 \text{ to one decimal place.}} \end{aligned}$$

Use of the sign of equality

The sign of equality (=) is perhaps the sign most frequently used in mathematics. It may be read as 'equals', or 'is equal to', and therefore represents a balance between two expressions. The sign means that the expression which precedes it *is equal to* the expression which follows it. Until its use becomes familiar it is well for a student to ask himself if he is showing equality of two expressions or statements every time he writes it. Only by this means will its misuse be prevented.

Referring to the problem just completed: Starting with line (A) we are stating that v^2 is equal to $u^2 + 2fs$, then the use of the sign in line (B) means that $u^2 + 2fs$ is equal to $14^2 + 2 \times 2 \times 13$, and so on, each successive line being equal to the one preceding it. After line (D) we have to extract the square root of 248. It would be a grave error to write line (E) as '$= \sqrt{(248)}$' immediately following line (D), because obviously 248 is not equal to $\sqrt{(248)}$. We therefore start with a new statement 'v' for line (E). If the successive signs of equality are kept under each other, the process of checking their validity will be easier.

Examples for Chapter 5

1. (*a*) Express a speed of x ft. per min. in miles per hour.

(*b*) Express a speed of y m. per sec. in miles per hour given 1 in. = 2·54 cm.

(*c*) Find an expression for the total cost in shillings of a articles at £1 each, b articles at 2*s*. 6*d*. each, and c articles at £p each. (*Chap.* 5)

2. The resistance to the motion of a train due to friction is x lb. wt. per ton of train. The resistance due to gravity is $1/y$ of the weight of the train. If the train weighs M tons, find an expression for the total resistance to motion, in lb. wt., and hence find the resistance when $M = 500$, $x = 10$, $y = 100$. (*Chap.* 5)

3. Two wires of electrical resistance R_1 and R_2 respectively, are connected in parallel. It is required to find the effective resistance (R) of the two together. The reciprocal of R is equal to the sum of the reciprocals of R_1 and R_2 when they are so connected. Express this as a formula and hence evaluate the reciprocal of R when $R_1 = 2\tfrac{1}{2}$, $R_2 = 1\tfrac{1}{5}$. From your answer find the value of R. (*Chap.* 5)

4. (*a*) Find the value of C from $C = M\left(gb - \dfrac{aV^2}{r}\right)$ given that $M = 250$, $g = 32$, $b = 2\tfrac{1}{2}$, $a = 1\tfrac{1}{2}$, $v = 22$ and $r = 50$.

(*b*) The moment of inertia of a plate is given by

$$I = \frac{bd^3}{12}.$$

Evaluate I when $b = 4\tfrac{1}{2}$, $d = 1\tfrac{1}{2}$. (*Chap.* 5)

5. The following formulae are used in problems on the expansions of rods and of gases respectively:

$$\text{(i)}\ L_t = L_0(1 + \alpha t); \quad \text{(ii)}\ V_1 = \left(\frac{t_1 + 273}{t_2 + 273}\right) V_2.$$

Find L_t when $L_0 = 120$, $t = 50$ and $\alpha = 2 \times 10^{-5}$.
Find V_1 when $V_2 = 50$, $t_1 = 0$ and $t_2 = 40$. (*Chap.* 5)

6. (*a*) Given that $t = 2\pi\sqrt{\left(\frac{k^2+h^2}{hg}\right)}$, find t when $\pi = \frac{22}{7}$, $k = 6$, $h = 8$, $g = 32$.

(*b*) If sand costs x shillings per cwt., cement y shillings per lb., and gravel z pence per cwt., find the total cost in £ of 1 ton of each commodity. (*Chap.* 5)

7. From the formula $v^2 = u^2 - 2gks$, show that $v = 0$ if $u = 14$, $g = 32$, $k = 0{\cdot}5$, $s = 6\frac{1}{8}$.

Find the value of v from the same formula if $u = 14$, $g = 32$, $k = 0{\cdot}9$, $s = 3$, and give the answer correct to three significant figures. (*Chaps.* 5 *and* 4)

8. The output from a quarry on each of the first five days of a week was 52·1, 63·2, 49·8, 58·4 and 65·3 tons. Find the average daily output. It is hoped to have an average of 60 tons per day over the 6 days. What would the output require to be on the sixth day to attain this average? How far does the average fall short of 60 tons if the output on the sixth day was 68·1 tons? (*Chap.* 4)

9. The pressure of an enclosed volume of gas is directly proportional to its temperature measured on the absolute scale. When the pressure is 40 lb. per sq.in., the absolute temperature is 280°. Find the pressure when the absolute temperature is 300°, and find what absolute temperature is necessary to raise the pressure to 50 lb. per sq.in. (*Chap.* 3)

10. Write the following expression in decimal form, and evaluate it:

$$\frac{5\frac{1}{2}(3\frac{1}{8}+2\frac{7}{20}-\frac{1}{2}\text{ of }1\frac{4}{9})}{\frac{1}{3}\times 10^{-2}}.$$

(*Chap.* 2)

Answers to Examples for Chapter 5

1. (*a*) $\frac{x}{88}$ m.p.h. (*b*) $2{\cdot}24y$ m.p.h. (*c*) $20a+\frac{5b}{2}+20pc$.

2. $R = Mx+\frac{2240M}{y}$ lb. wt.; $R = 16{,}200$ lb. wt.

3. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$; $\frac{1}{R} = 1\frac{7}{30}$; $R = \frac{30}{37}$.

4. (*a*) 16,370. (*b*) $1\frac{17}{64}$. 5. $L_1 = 120{\cdot}12$; $V_1 = 43{\cdot}61$.

6. (*a*) $3\frac{13}{14}$. (*b*) $x + 112y + \frac{z}{12}$.

7. 4·82. 8. 57·76 tons; 71·2 tons; 0·52 tons.

9. $42\frac{6}{7}$ lb. per sq.in.; 350°. 10. 7842·5.

CHAPTER 6

DIRECTED NUMBERS. ADDITION AND SUBTRACTION

The range of numbers in arithmetic starts at zero and increases without limit. In algebra this range is extended. Numbers from zero upwards are called positive, and numbers proceeding in the opposite direction from zero are called negative. Such quantities with a direction are called directed numbers, and when these are used it is useful to consider zero as lying in the middle of the range instead of at one end as previously. Directed numbers are not confined to algebra; several cases occur in ordinary usage. For instance, a temperature of minus 3 degrees implies a temperature of three degrees below zero, and a change in the value of stocks or shares of minus 3 means a fall of three points.

Consider fig. 9. Let distances above the level of the ground be called positive, then distances measured downwards into the ground, e.g. in a pit, will be called negative. This is merely an analogy to help to show that the methods to be used in dealing with directed numbers imply no actual new rules but merely an extension of the rules we have used already.

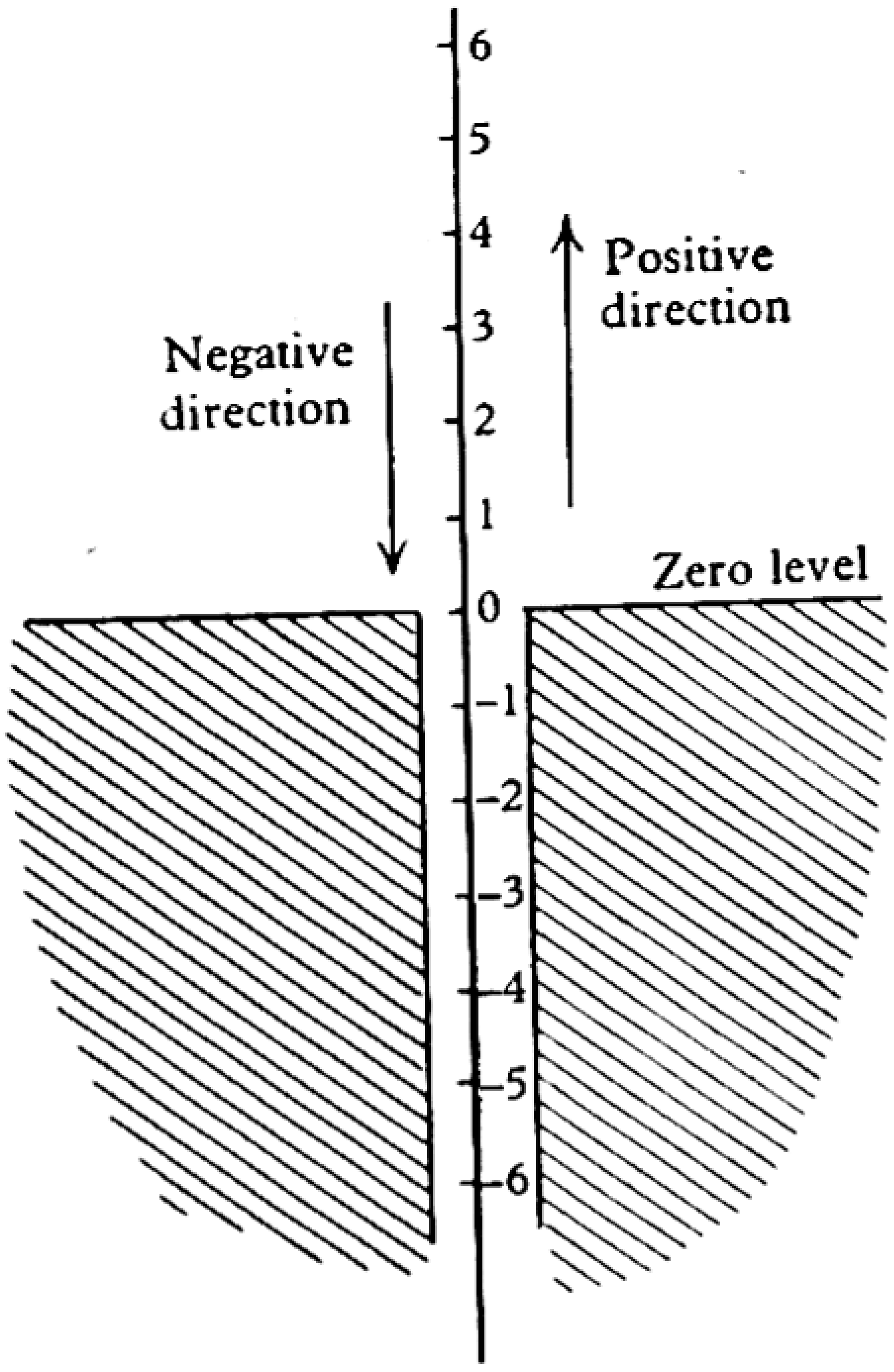


Fig. 9

Apply the statement $6+4=10$ to the diagram. It may be illustrated by considering that a person starts at the point $+6$ and takes 4 steps in the positive direction (i.e. upwards), finishing at $+10$, a distance of 10 units in the positive direction above zero (see fig. 10). The rule for addition is to start at the point of the scale represented by the first number, and then move the number of steps indicated by the second number in the direction shown by the *sign* of that number. If the sign of the second number is positive the steps are taken upwards, and if negative, downwards. To extend the system to the addition of several numbers, consider $-4+7-5$ (fig. 11).

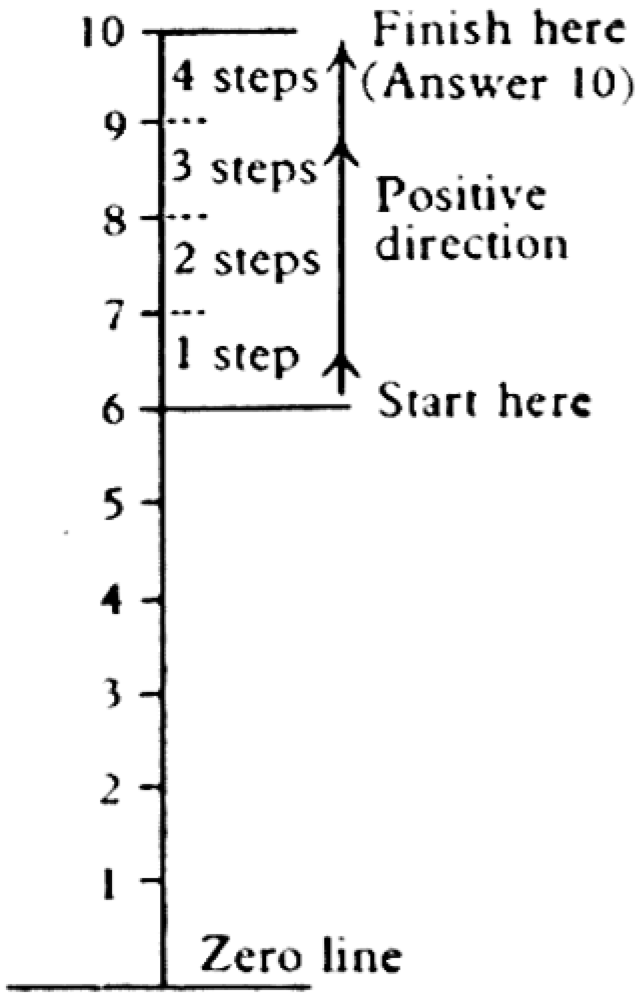


Fig. 10

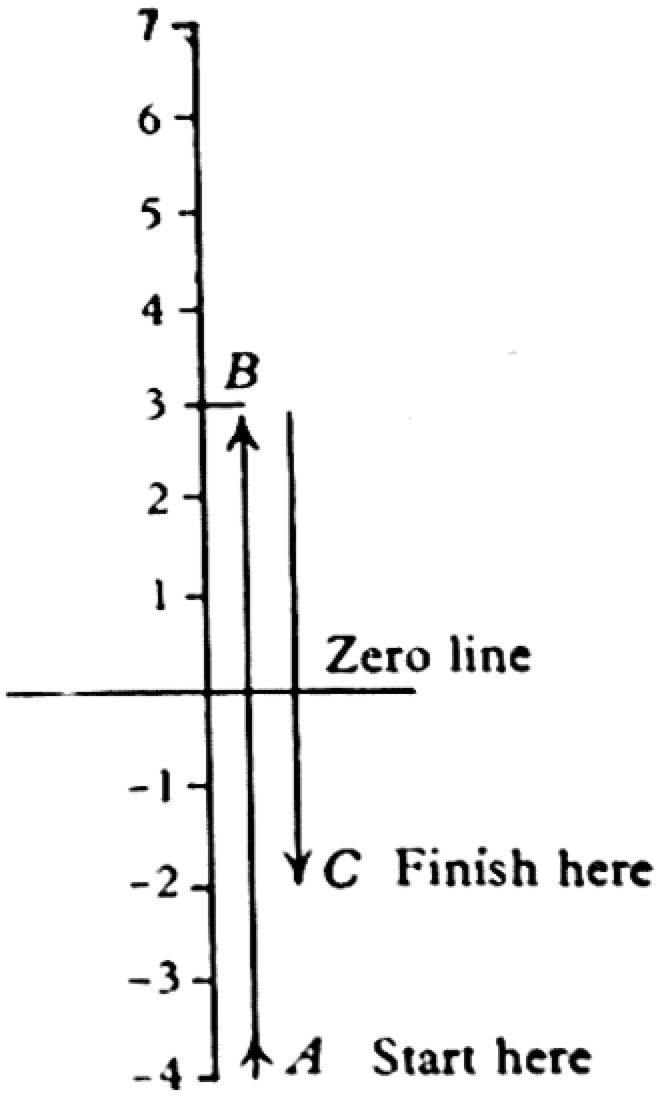


Fig. 11

Start at the point A (-4), take 7 steps in the positive direction to B, which is at the point 3, then take 5 steps in the negative direction to C, which is at the point -2. Hence, by addition,

$$-4+7-5=-2.$$

The rule for addition of directed numbers is therefore:

Add together the numbers which are preceded by a positive sign, then add together those preceded by a negative sign, take the smaller total from the greater, and the sign of the answer is the sign of the greater.

Example. $3-2+9-7+6-5$ (figure 3 is positive if no sign precedes it).

Total of positive numbers $= 18$
Total of negative numbers $= -14$
Answer $= 4$ (Subtracting 14 from 18)

Example. $-14-3+2\frac{1}{4}-6+19$.

Total of positive numbers $= 21\frac{1}{4}$
Total of negative numbers $= -23$
Answer $= -1\frac{3}{4}$ (Subtract $21\frac{1}{4}$ from 23)

The 'steps' used in the last examples have each been unity, but the process is the same when each step is x, or a, or any other letter.

For instance, $-14a-3a+2\frac{1}{4}a-6a+19a$ gives $-1\frac{3}{4}a$ by the same argument as that of the last problem.

We shall have to consider also cases where steps have different values in the same problem. Take $3x-4y$. This means that, starting at a point on the scale representing $3x$ units, $4y$ steps must be taken in the negative direction, but since the relation between the numerical values of x and y is not known, the result cannot be written any simpler than $3x-4y$.

Consider next

$$3x-2y+5x-3y+7x-y+x-8x+10y.$$

Deal first with all the movements expressed in terms of x, and then with those expressed in terms of y.

Total of terms in x:

$3x+5x+7x+x-8x = 8x$ (by addition of directed numbers).

Total of terms in y:

$$-2y-3y-y+10y = 4y.$$

Answer: $\underline{8x+4y.}$

A problem of this type is usually written in columns, one column for each different unit:

$$\begin{array}{r}
3x-2y\\
5x-3y\\
7x-y\\
x\\
-8x+10y\\
\hline
8x+4y\\
\hline\hline
\end{array}$$

Terms in a^2, or ab, or similar cases of terms of higher degree than the first, have been mentioned already. Now a^2 is not in the same units as a, for if a should be expressed in feet, then a^2 is in square feet, and as square feet and feet cannot be added together, we see that only terms containing exactly the same letter or letters shall appear in the same column in algebraic addition.

Example. Add together:

$$3a^2 - 4ab + b^2, \quad ab + 2a - b^2, \quad 3a^2 - 2b, \quad b^2 - 4ab + a^2:$$

$$\begin{array}{rrrrr}
3a^2 & -\,4ab & +\,b^2 & & \\
 & +\;\;ab & -\,b^2 & +\,2a & \\
3a^2 & & & & -\,2b \\
a^2 & -\,4ab & +\,b^2 & & \\
\hline
7a^2 & -\,7ab & +\,b^2 & +\,2a & -\,2b \\
\hline\hline
\end{array}$$

When an expression consists of terms using different powers of the same letter, it is usual to write the columns in 'descending powers' of that letter.

Example. Add together:

$$5x^4 - 3x^3 + 2x^2 - x + 3, \quad 3x^3 + 2x^2 - 5, \quad x^4 - 3x^2 + 6, \quad x^3 - 2x + 2:$$

$$\begin{array}{rrrrr}
5x^4 & -\,3x^3 & +\,2x^2 & -\;\;x & +\,3 \\
 & 3x^3 & +\,2x^2 & & -\,5 \\
x^4 & & -\,3x^2 & & +\,6 \\
 & x^3 & & -\,2x & +\,2 \\
\hline
6x^4 & +\;\;x^3 & +\;\;x^2 & -\,3x & +\,6 \\
\hline\hline
\end{array}$$

Rules for addition

(1) Write the terms in columns so that all terms in any one column contain exactly the same letter or letters.

(2) Starting at any column, add all the positive terms, add all the negative terms, subtract the numerically smaller total from the larger total, and give the result the sign of the larger.

Algebraic subtraction

Consider fig. 12. We wish to subtract 3 from 8. We therefore start from $+3$ and count the steps until we reach $+8$. The answer is 5, and as we are moving from 3 to 8 in the positive direction, the answer is $+5$.

Now consider fig. 13. We wish to subtract 8 from 3. Starting at $+8$ we move to $+3$ by taking 5 steps in the negative direction. The answer is thus -5.

Fig. 14 shows the subtraction of -2 from $+4$. The answer is seen to be 6 steps in the positive direction, i.e. $+6$.

Fig. 15 shows the subtraction of $+2$ from -5, and the answer is -7.

Let us set down these four subtractions as problems, with the number at which we started in the figure at the bottom:

$$\begin{array}{rrrrl} 8 & 3 & 4 & -5 & \text{(Upper)}\\ \underline{3} & \underline{8} & \underline{-2} & \underline{2} & \text{(Lower)}\\ \underline{\underline{5}} & \underline{\underline{-5}} & \underline{\underline{6}} & \underline{\underline{-7}} & \text{(Answer)}\end{array}$$

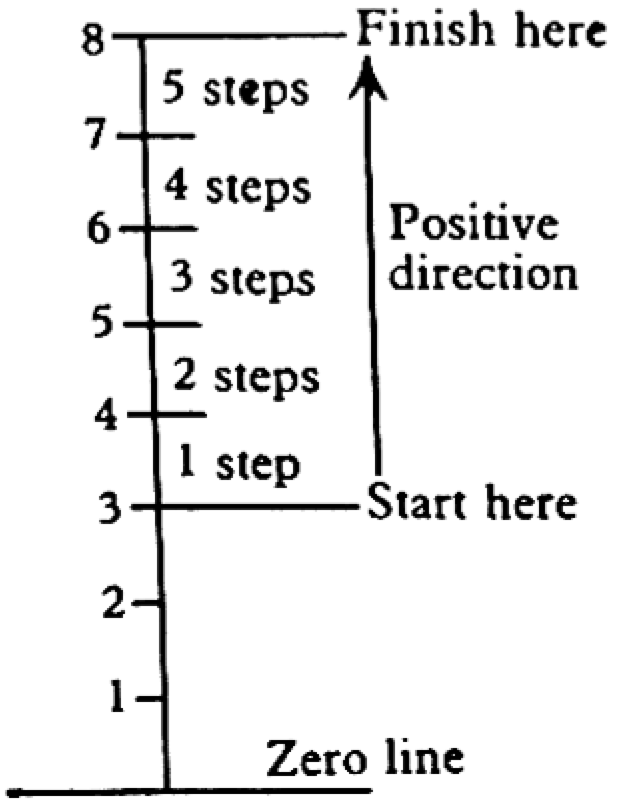


Fig. 12

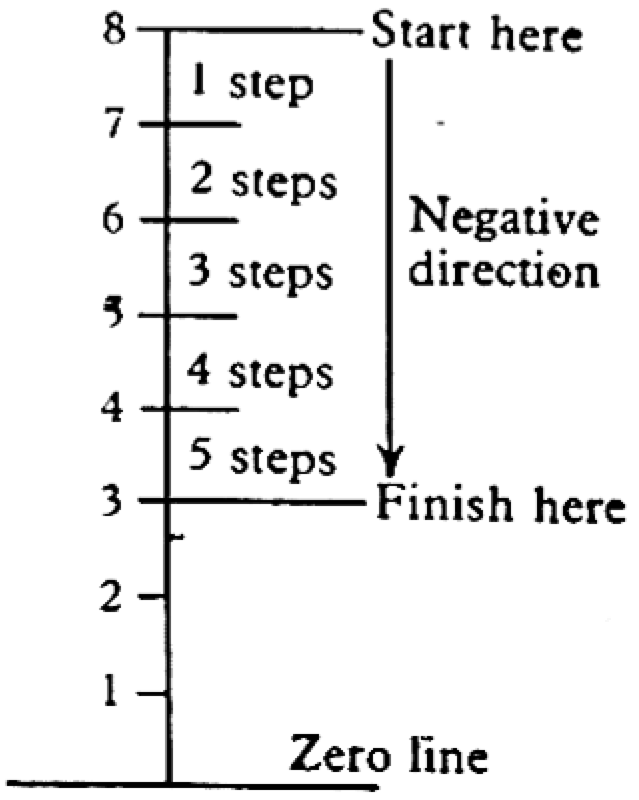


Fig. 13

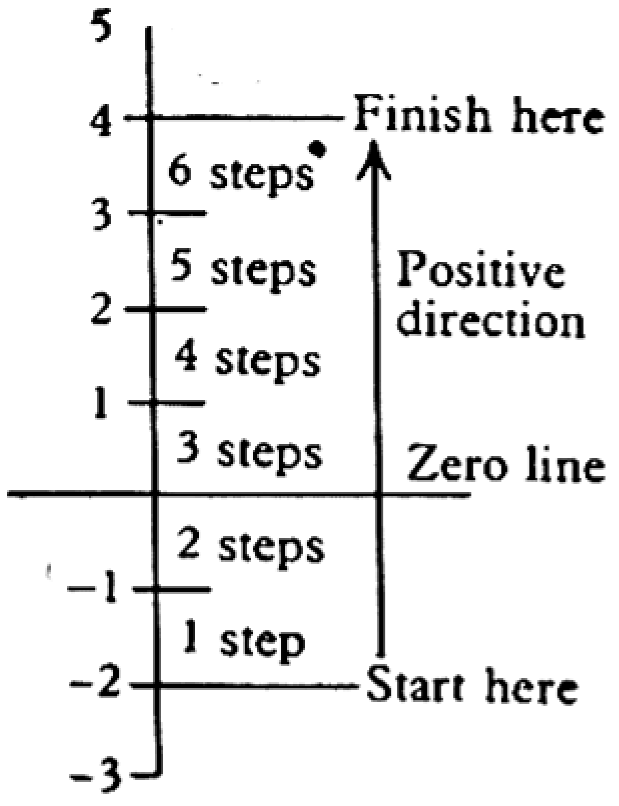


Fig. 14

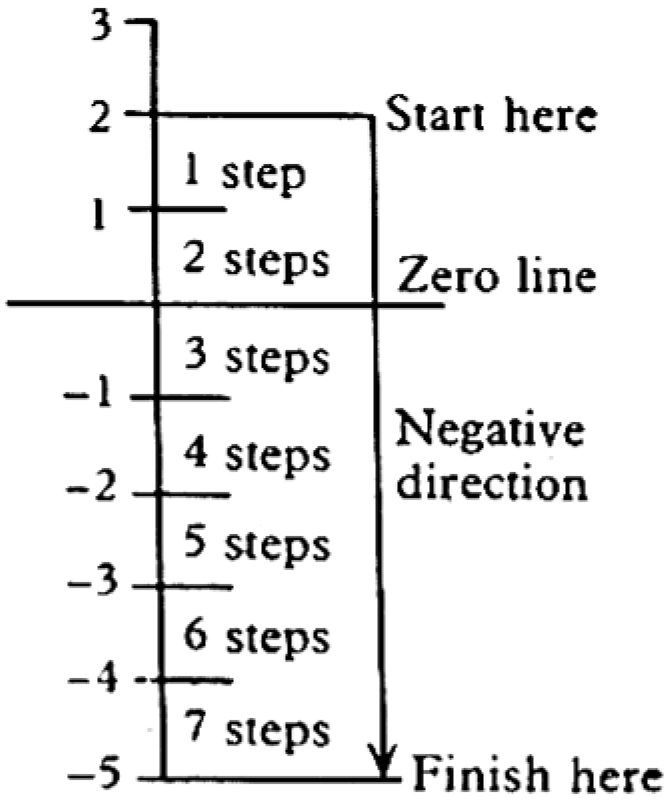


Fig. 15

Now rewrite the four problems with the sign of the lower number changed, and *add* algebraically:

$$\begin{array}{rrrrl} 8 & 3 & 4 & -5 & \\ \underline{-3} & \underline{-8} & \underline{2} & \underline{-2} & \text{(Signs changed)}\\ \underline{\underline{5}} & \underline{\underline{-5}} & \underline{\underline{6}} & \underline{\underline{-7}} & \end{array}$$

The answers produced by addition when the sign of the lower term is changed are the actual answers obtained by subtraction on the diagrams. We have, then, a rule for algebraic subtraction:

Imagine that the sign of the quantity to be subtracted is changed, then add.

Example. From 72 subtract 91:

$$\begin{array}{lr l} & 72 & \\ & 91 & \text{(Imagine this to be } -91\text{, and add)} \\ \cline{2-2} \text{Answer:} & \underline{\underline{-19}} & \end{array}$$

Example. Subtract $3x-2y+4z$ from $4y-2x+9z$:

$$\begin{array}{lr l} & 4y-2x+9z & \\ & -\;2y+3x+4z & \text{(Imagine this to be } 2y-3x-4z\text{, and add)} \\ \cline{2-2} \text{Answer:} & \underline{\underline{6y-5x+5z}} & \end{array}$$

Example. From $3x^3-2x^2+4x-1$ subtract $5x^4-3x^2-9$:

$$\begin{array}{lrrrrr} & & 3x^3 & -2x^2 & +4x & -1 \\ & 5x^4 & & -3x^2 & & -9 \\ \hline \text{Answer:} & -5x^4 & +3x^2 & +x^2 & +4x & +8 \\ \hline\hline \end{array}$$

Note that, when directed numbers are used, a larger quantity may be subtracted from a smaller. Also, a number may be subtracted from zero by writing it with its sign changed.

Examples for Chapter 6

1. (*a*) Add together: $3x-4y$, $6y-3x$, $y+2x$, $x-7y$.

(*b*) Add together: $3pq-2qr+pr$, $5pr+2pq$, $6qr-pr-pq$.

(*c*) Add together:

$$3x^3+2x^2-3x+1, \quad 4x^2-3, \quad x^4-3x^2+x-9.$$

(*d*) Add together:

$$xy-x^2+y^2, \quad 3y^2-2y, \quad x^2-3xy-2y, \quad y^2+xy.$$

(*Chap.* 6,

2. (*a*) Add together:

$$5m+4n, \quad 6l-5m-3n, \quad n-2m+l, \quad l+m+n.$$

(*b*) Add together:

$$12a^2-2ab, \quad b^2-4a^2, \quad 7ab-a^2-b^2, \quad 5b^2-5a^2.$$

(*c*) Add together:

$$z^3-5z^2+3z, \quad 6z^3-5, \quad z^2-3z+1, \quad 1-z^3.$$ (*Chap.* 6)

3. (*a*) From $5y-3x+6z$ subtract $x-y-z$.

(*b*) From $3a+2b-7c$ subtract $b-c+2a$.

(*c*) From $l-24m$ subtract $6l+9m$.

(*d*) From x^3-3x^2+2x-9 subtract $4x^3+x^2-5x+6$.

(*e*) From y^3-7y^2+6 subtract $3y^3+y^2-10y+8$.

(*f*) From $5-7k$ subtract $19-6k+2l$.

(*g*) From 51 subtract $8x-4y+9$.

(*h*) From $x+y+z$ subtract $-x-y-z$. (*Chap.* 6)

4. (*a*) By means of a diagram illustrate the processes of adding -5 to $+3$, of subtracting -5 from $+3$, and of subtracting $+3$ from -5.

(*b*) If a positive sign indicates distances to the north along a certain road, and a negative sign denotes distances in a southerly direction, find how far a man is from his starting point if he travels the following distances in miles: 8, -4, $+3$, -16, $+5$. (*Chap.* 6)

5. A man stands at the edge of a cliff 128 ft. high, and throws a stone vertically upwards. The distance above his level at any time is given by $s = ut-\frac{1}{2}gt^2$, where $u = 32$, $g = 32$, and t represents the time in seconds after he has thrown the stone. Show that when $t = 2$ the stone has fallen to his level, and that when $t = 4$ sec. the stone has reached the foot of the cliff. (*Chaps.* 6 *and* 5)

6. Add together $3a+2b-4c+d$ and $6a-7b+3c-3d$ and subtract $-10a-9b-c-2d$ from the answer.

Evaluate the answer when $a = 2$, and $b = 3$. (*Chaps.* 6 *and* 5)

7. The work done in foot-pounds in one stroke of an engine is given by $W = PLA$, where P = pressure in pounds per square inch, L = length of stroke in feet, and A = area of piston in square inches. Find the work done when the pressure is P tons per sq.in., length of stroke is L in., and area is $\frac{22}{28}d^2$ sq. in. (*Chap.* 5)

8. Given that $\sqrt{2} \doteqdot 1{\cdot}414$ and $\sqrt{3} \doteqdot 1{\cdot}732$, evaluate without division $\sqrt{(243)}$, $\sqrt{(48)}$, $\sqrt{(0{\cdot}18)}$, $\sqrt{(800)}$. (*Chap.* 4)

9. One ton of lead is cast and 3 % is lost in the process. It is then recast, due to errors, and again 3 % is lost in the process. Find the weight of the second casting in pounds correct to one decimal place. (*Chap.* 3)

10. Fifty labourers attempt to finish a contract in 6 days. By the end of the fifth day $\frac{4}{5}$ of the work has been completed. Assuming all labourers to be of equal value, how many extra hands must be employed on the sixth day to ensure completion of the task? (*Chap.* 3)

Answers to Examples for Chapter 6

1. (*a*) $3x - 4y$. (*b*) $4pq + 4qr + 5pr$.
(*c*) $x^4 + 3x^3 + 3x^2 - 2x - 11$. (*d*) $5y^2 - xy - 4y$.

2. (*a*) $8l - m + 3n$. (*b*) $2a^2 + 5ab + 5b^2$. (c) $6z^3 - 4z^2 - 3$.

3. (*a*) $6y - 4x + 7z$. (*b*) $a + b - 6c$. (*c*) $-5l - 33m$.
(*d*) $-3x^3 - 4x^2 + 7x - 15$. (*e*) $-2y^3 - 8y^2 + 10y - 2$.
(*f*) $-14 - k - 2l$. (*g*) $-8x + 4y + 42$. (*h*) $2x + 2y + 2z$.

4. (*a*) -2, 8, -8. (*b*) 4 miles to south.

6. $19a + 4b$; 50. 7. $\frac{440}{3}PLd^2$ ft. lb.

8. 15·588, 6·928, 0·4242, 28·28. 9. 2107·6 lb. 10. 13.

CHAPTER 7

DIRECTED NUMBERS. MULTIPLICATION AND DIVISION

Let us reconsider our diagrammatic representation of directed numbers and make one change. Let the positive numbers represent speeds in a direction from south to north, and let negative numbers represent speeds from north to south. Then the number 30 represents the speed of a train travelling northwards at 30 m.p.h. and -30 refers to a train travelling at the same speed in the opposite direction. Thus the minus sign reverses the direction of motion without altering the speed. But a minus sign has no meaning when standing alone, so we may write 30×-1 for -30, and $30 \times +1$ for 30. Multiplication by -1 represents *reversal* of direction. With this notation, 30×2 represents 60 m.p.h. northwards, while 30×-2, which is $30 \times 2 \times -1$, or -1×60 represents 60 m.p.h. southwards. Similarly, $-4 \times 28 = -112$.

Next, suppose we wish to multiply -6 by -3 (i.e. a speed of 6 m.p.h. southwards to be multiplied by 3 and by -1). Multiplying by 3 makes the speed 18 m.p.h. southwards, and multiplying 18 by -1 reverses the direction giving 18 m.p.h. northwards. Hence $-6 \times -3 = +18$, which illustrates a very important rule.

The rule of signs in multiplication

(1) A *positive* quantity multiplied by a *positive* quantity gives a *positive* result.

(2) A *negative* quantity multiplied by a *positive* quantity gives a *negative* result.

(3) A *positive* quantity multiplied by a *negative* quantity gives a *negative* result. [This is the same as (2), for 2×-3 is the same as -3×2.]

(4) A *negative* quantity multiplied by a *negative* quantity gives a *positive* result.

Examine the four statements, and notice that when quantities bearing similar signs are multiplied together the result is positive, while the result is negative when the signs of the quantities multiplied are not similar. Thus, in the case of two quantities:

Quantities bearing *like* signs, when multiplied, give a *positive* result.

Quantities bearing *unlike* signs, when multiplied, give a *negative* result.

For convenience, every algebraic term may be thought of as consisting of three parts, the *sign*, the *coefficient* (i.e. number) and the *letters* attached.

The sign of the answer has been explained above. Multiplication of the coefficients is ordinary arithmetical multiplication, but the letters must be multiplied too. We shall examine these now.

Instances will occur in mechanics where different units are multiplied together; for instance, feet may be multiplied by pounds to give foot-pounds. In the same manner, x may be multiplied by y to give xy, and a multiplied by b gives ab. To multiply letters we write them adjacent to each other.

Thus
$$3ab \times 5cd = 15abcd,$$
$$-3ac \times 5bd = -15abcd,$$
$$-3ad \times -5bc = 15abcd.$$

The rule of indices in multiplication

To multiply $3x^3$ by $4x^2$. Now $3x^3$ may be written $3xxx$, and $4x^2$ may be written $4xx$. Multiplying them we obtain $12xxxxx$, i.e. $12x^5$.

Similarly, $y^5 \times y^3 = yyyyy \cdot yyy = y^8$. A rule is necessary to avoid writing the letters separately in this way. Notice that in the case of $3x^3 \times 4x^2$, the power of x in the answer is the *sum* of the powers of x in the two terms. Also in the case of $y^5 \times y^3 = y^8$, the final power of y is $5+3=8$. The rule is:

When multiplying powers of the same quantity add indices.

Example. $-4x^3y^2z \times 7xy^4z^5 = -28x^4y^6z^6$.

(The power of x in the answer is $3+1 = 4$, the power of y is $2+4 = 6$, and the power of z is $1+5 = 6$.) Notice that if no index is written, as in x, then that quantity is raised to the power 1. Similarly, if no coefficient is written, the coefficient is 1.

Example. $-7p^3q^2rs \times a^2p^2qr^3 = -7a^2p^5q^3r^4s.$

Multiplication of expressions. The multiplication of an expression containing several terms by a single term is performed in the same way.

Example. Multiply $3a^2b-4abc$ by $5ab^2$.

$$\begin{array}{r} 3a^2b - 4abc \\ 5ab^2 \\ \hline 15a^2b^3 - 20a^2b^3c \\ \hline\hline \end{array}$$

[The term $5ab^2$ multiplies separately both the terms $3a^2b$ and $-4abc$.]

Example. Multiply $4xy-3xz+2yz$ by xyz.

$$\begin{array}{r} 4xy - 3xz + 2yz \\ xyz \\ \hline 4x^2y^2z - 3x^2yz^2 + 2xy^2z^2 \\ \hline\hline \end{array}$$

These two examples have been worked with the multiplier underneath the multiplicand to show a similarity with arithmetical multiplication. It is more convenient to write the multiplier outside a bracket which contains the multiplicand, as

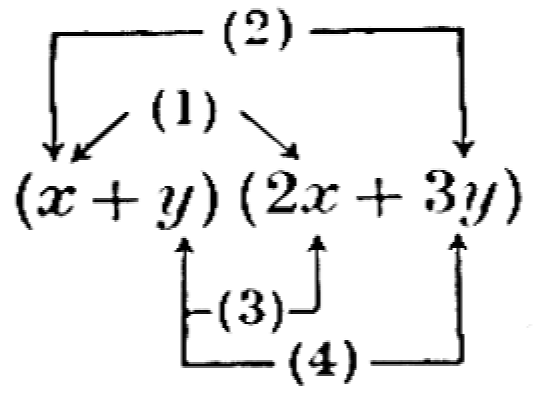
$$5ab^2(3a^2b-4abc) = 15a^3b^3-20a^2b^3c$$

and $$xyz(4xy-3xz+2yz) = 4x^2y^2z-3x^2yz^2+2xy^2z^2.$$

A term immediately preceding a bracket multiplies every term in the bracket.

Multiplication of two binomials (i.e. expressions containing two terms) is a very common process, and requires special attention.

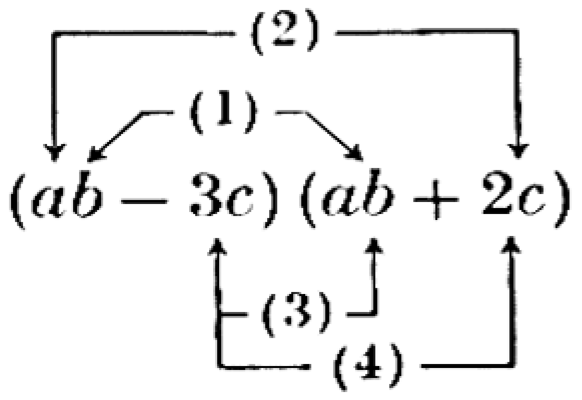
Consider $(x+y)(2x+3y)$. The presence of the brackets adjacent to each other signifies that all of the expression $x+y$ is to multiply all of the expression $2x+3y$. To do this, multiply both terms in the second bracket by the first term in the first bracket, and then multiply both terms in the second bracket by the second term in the first bracket:



$$(x+y)(2x+3y) = \underset{(1)}{2x^2} + \underset{(2)}{3xy} + \underset{(3)}{2xy} + \underset{(4)}{3y^2}$$

$$= \underline{2x^3 + 5xy + 3y^2} \quad \{\text{Adding terms (2) and (3)}\}.$$

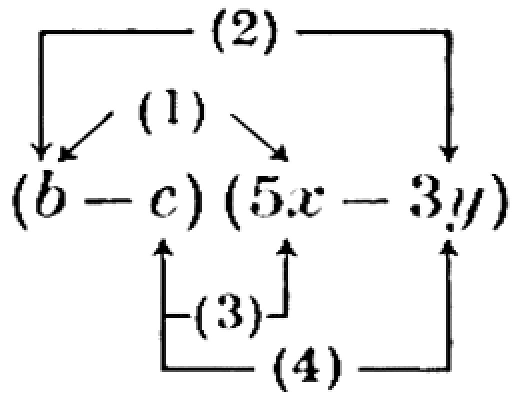
Again,



$$(ab-3c)(ab+2c) = \underset{(1)}{a^2b^2} + \underset{(2)}{2abc} - \underset{(3)}{3abc} - \underset{(4)}{6c^2}$$

$$= \underline{a^2b^2 - abc - 6c^2} \quad \{\text{Adding terms (2) and (3)}\}.$$

Also



$$(b-c)(5x-3y) = \underset{(1)}{5bx} - \underset{(2)}{3by} - \underset{(3)}{5cx} + \underset{(4)}{3cy} \quad \text{(No addition possible)}$$

and $(2x^3 - 5y^4)(y^3 - x^2) = 2x^3y^3 - 2x^5 - 5y^7 + 5x^2y^4$

(No addition possible).

Use of brackets

It frequently happens that brackets occur within other brackets. In these cases it is necessary that the innermost brackets shall be 'removed' first, that is, the multiplication which the presence of brackets implies is performed first on the innermost expressions. Different types of brackets must be used for distinguishing the closure of the various expressions, and it is customary to use the following types, and in that order: ( ), { }, [ ]. If four types are necessary, a line drawn over the expression may be used for the innermost bracket, viz. $\overline{x-2y}$.

An implication from the law of signs, which is worth noting, is that when a positive quantity stands immediately outside a bracket, the signs of all the terms which result from the removal of the bracket are *unchanged*, while a bracket preceded by a negative quantity results in a *change of sign* for all terms on removal of the bracket.

Example.

$$4x(2y+3x-5z) = 8xy+12x^2-20xz \quad \text{(Signs unchanged).}$$

Example.

$$-4x(2y+3x-5z) = -8xy-12x^2+20xz \quad \text{(All signs changed).}$$

Example. Simplify $3ab[2a\{4ac-3(a+2c)-b\}+5a]$.

$$3ab[2a\{4ac-3(a+2c)-b\}+5a]$$

$$=3ab[2a\{4ac-3a-6c-b\}+5a] \quad \text{(Removing inner brackets)}$$

$$=3ab[8a^2c-6a^2-12ac-2ab+5a]$$

(Removing inner brackets)

$$=\underline{24a^3bc-18a^3b-36a^2bc-6a^2b^2+15a^2b.}$$

Example. Simplify $4[5x-\{2y-3z(x-y)\}-5z]$.

$$4[5x-\{2y-3z(x-y)\}-5z]$$

$$=4[5x-\{2y-3xz+3yz\}-5z]$$

$$=4[5x-2y+3xz-3yz-5z] \quad \text{(Multiplier here was } -1\text{)}$$

$$=\underline{20x-8y+12xz-12yz-20z.}$$

Division

Rule of signs. The division of one quantity by another means the multiplication of the first by the reciprocal of the second. Thus $a \div b$ means $a \times \frac{1}{b}$, and $-a \div b$ means $-a \times \frac{1}{b}$; also $a \div -b$ means $a \times -\frac{1}{b}$, the answers being respectively, $\frac{a}{b}$, $-\frac{a}{b}$, and $-\frac{a}{b}$.

Hence the rule of signs for division is the same as the rule of signs in multiplication, e.g.

$$4x \div -5 = 4x \times -\tfrac{1}{5} = -\frac{4x}{5}$$

and
$$-6ab \div -2a = -6ab \times -\frac{1}{2a} = +\frac{6ab}{2a} = +3b.$$

Rule of indices. Take $x^5 \div x^2$, or $\dfrac{x^5}{x^2} = \dfrac{xxx\cancel{x}\cancel{x}}{\cancel{x}\cancel{x}} = x^3$.

Also $y^7 \div y^3$, or $\dfrac{y^7}{y^3} = \dfrac{yyyy\cancel{y}\cancel{y}\cancel{y}}{\cancel{y}\cancel{y}\cancel{y}} = y^4$.

We see from the examples that when a power of a number is divided by another power of the same number, the power of that number in the quotient is obtained by *subtracting* the power of the divisor from the power of the dividend.

Thus
$$-8a^5b^3c^2 \div 4a^3bc = -\frac{8a^5b^3c^2}{4a^3bc}$$
$$= -\frac{\overset{2}{\cancel{8}}aa\cancel{a}\cancel{a}\cancel{a}\cancel{b}bb\cancel{c}c}{\cancel{4}\cancel{a}\cancel{a}\cancel{a}\cancel{b}\cancel{c}} = -2a^2b^2c.$$

Now consider
$$\frac{x^5yz^3}{x^2y^2z^4} = \frac{xxx\cancel{x}\cancel{x}\cancel{y}\cancel{z}\cancel{z}\cancel{z}}{\cancel{x}\cancel{x}\cancel{y}y\cancel{z}\cancel{z}\cancel{z}z} = \frac{x^3}{yz}$$

by cancelling. Trying the same problem and applying the rule of subtraction of indices (bearing in mind that the subtraction is algebraic), we have
$$\frac{x^5yz^3}{x^2y^2z^4} = x^3y^{-1}z^{-1}$$
(since 2 subtracted from 1 gives -1, and 4 subtracted from 3 also gives -1). But both answers must be the same; therefore $x^3y^{-1}z^{-1}$ is the same as $\dfrac{x^3}{yz}$. We see that y^{-1} is another way of writing $\dfrac{1}{y}$ and z^{-1} is another way of writing $\dfrac{1}{z}$.

Similarly, x^{-2} means $\dfrac{1}{x^2}$ and y^{-4} means $\dfrac{1}{y^4}$.

In Chapter 2 we used the notation 5×10^6, which, we said, meant $5 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$, or 5 millions. We also used $1{\cdot}4 \times 10^{-9}$ for 0·0000000014. The result we have just found explains this notation, for $1{\cdot}4 \times 10^{-9}$ means $1{\cdot}4 \times \dfrac{1}{10^9}$ or
$$1{\cdot}4 \times \frac{1}{1{,}000{,}000{,}000}.$$

The use of indices will be extended in the next chapter, while the process of long division will be left until Chapter 11. A few worked examples will conclude this chapter.

Examples.

1. $(a^5b-c^3d^2)(ac-bd) = a^6bc-a^5b^2d-ac^4d^2+bc^3d^3.$

2. $$a^3b^2c\times -a^5b^7c^2\times -ab^6c^6 = +a^{3+5+1}b^{2+7+6}c^{1+2+6} = a^9b^{15}c^9.$$

3. $$-6x\times -2xy\times -x^2y^2\times -2y^2 = +24x^{1+1+2}y^{1+2+2} = 24x^4y^5.$$

4. $(x-y)(x+y) = x^2+xy-xy-y^2 = x^2-y^2.$

5. $$5p^4q^3r^2\div 15p^3qr^2 = \frac{5p^4q^3r^2}{15p^3qr^2} = \frac{pq^2}{3}.$$

6. $$16x^2yz^3\div 4xy^2z = \frac{16x^2yz^3}{4xy^2z} = 4xy^{-1}z^2 = \frac{4xz^2}{y}.$$

7. $$-a^2bcd^2\div -ab^2c^2d = +\frac{a^2bcd^2}{ab^2c^2d} = ab^{-1}c^{-1}d = \frac{ad}{bc}$$

8. $$3^{-2} = \frac{1}{3^2} = \frac{1}{9}.$$

9. $$2^{-4} = \frac{1}{2^4} = \frac{1}{16}.$$

10. $$\left(\frac{1}{4}\right)^{-2} = \frac{1}{(\frac{1}{4})^2} = \frac{1}{(\frac{1}{16})} = 16.$$

Examples for Chapter 7

1. (*a*) $5a^2bc\times 4ab^2c^2$. (*d*) $-\frac{1}{4}pq\times -\frac{2}{3}q^2r\times \frac{1}{3}pr^2$.
   (*b*) $-3x^2yz\times 5xyz^3$. (*e*) $-4a\times -2ab\times -3bc\times -ac$.
   (*c*) $lmn\times m^2n^2\times -3l^2mn$. (*f*) $x^2\times -xy\times xyz$. (*Chap.* 7)

2. (*a*) $(a-b)(a+b)$. (*f*) $(l+2m)(n-2p)$.
   (*b*) $(3a-4b)(2a+3b)$. (*g*) $(a-c)(b-d)$.
   (*c*) $(4x-y)(x-4y)$. (*h*) $(4-x)(y-3x)$.
   (*d*) $(2p-q)(q-3p)$. (*i*) $(h+k)(3k-h)$.
   (*e*) $(15-3x)(2-x)$. (*j*) $(11l+3m)(l-4m)$. (*Chap.* 7)

3. (*a*) $4a^4b^5c^2 \div 2a^2bc$.
(*b*) $-3l^4m \div 2lm$.
(*c*) $\frac{1}{5}abcd \div \frac{1}{4}bcde$.
(*d*) $-p^2q^2r^2 \div -2p^3q^3r$.
(*e*) $-7xy^4z \div x^2y^2z^2$.
(*f*) $3\frac{1}{2} \div -\frac{7}{9}$.
(*g*) $-5x^4 \div -3xy^4$.
(*h*) $-x^4y^4z^4w^4 \div -x^5y^3z^5w^3$.
(*i*) $l^4m^4 \div -klmn$.
(*j*) $h^{16}j^{12}k^{11} \div j^{14}k^{12}l^{10}$. (*Chap.* 7)

4. (*a*) Multiply $4x+2y+3z$ by $x+y$.
(*b*) $(a-2b)(a^2+ab+b^2)$.
(*c*) $4a\{x-3a(b+c)-y\}$.
(*d*) $2x\{4-3b+2c(c-b)\}$.
(*e*) $p[q+r\{s-p(r-p)\}-p]$.
(*f*) $(a+b)\{a-2b(a-c)\}$. (*Chap.* 7)

5. (*a*) Multiply $4a^{-2}b^2c^{-3}$ by $-2ab^{-3}c$ by adding the indices algebraically, and write the result with positive indices.

(*b*) Multiply similarly $-3x^{-1}yz$ by $-2x^2y^{-2}z^3$.

(*c*) Divide $a^3b^{-1}c^{-2}$ by $a^{-2}bc^2$ by algebraic subtraction of indices and write the answer with positive indices.

(*d*) Divide pqr by $p^{-3}q^{-2}r^{-1}$ as in part (*c*) above. (*Chap.* 7)

6. (*a*) Since $(x+y)(2x-y) = 2x^2+xy-y^2$, the binomial expressions $x+y$ and $2x-y$ are said to be the factors of

$$2x^2+xy-y^2.$$

Show by multiplication that the factors of x^2-y^2 are $x-y$ and $x+y$, the factors of a^3+b^3 are $a+b$ and a^2-ab+b^2, and that the factors of a^3-b^3 are $a-b$ and a^2+ab+b^2.

(*b*) Draw a square of side $a+b$ units, where $a = 3$ in., $b = 1$ in., and show that the square may be divided into a square of side a units, a square of side b units, and two rectangles of sides a and b. Hence prove that $(a+b)^2 = a^2+b^2+2ab$. (*Chap.* 7)

7. (*a*) Add together $4x-3y$, $7x+4z$, $z-3x$, $y-2z$.

(*b*) Subtract $5a^2-3ab+b^2$ from $2b^2-4ab+a^2-3a$.

(*c*) In the equation $s = at-bt^2$, s represents the distance above its starting point, after t sec., of a body projected vertically upwards. Find the distances of the body from its starting point after 2, 3, 4 sec., if $a = 50$, $b = 16$. (*Chap.* 6)

8. (*a*) Write as a single number $3{\cdot}125 \times 10^7$.

(*b*) The formula $n = \dfrac{1}{2l}\sqrt{\dfrac{T}{M}}$ is used in experiments on vibrating wires. Find n if $l = 50$, $T = 3{\cdot}125 \times 10^7$, and $M = 0{\cdot}05$. (*Chaps.* 2 *and* 5)

9. Three partners *A*, *B* and *C* operate a business. From all profits *A* first takes 10 % as payment for managerial duties. The rest is shared among them in the ratio 5:3:2 respectively. How much will each receive from a profit of £1200? (*Chap.* 3)

10. A transport company employs $\frac{3}{5}$ of its staff on actual transport, $\frac{1}{3}$ on maintenance, and the rest in the offices. Ten of the staff retire from the office and thus reduce the office staff to $\frac{5}{6}$ of its former strength. Find the total number of employees formerly. (*Chap.* 1)

Answers to Examples for Chapter 7

1. (*a*) $20a^3b^3c^3$. (*b*) $-15x^3y^2z^4$. (*c*) $-3l^3m^4n^4$.
(*d*) $\frac{1}{18}p^2q^3r^3$. (*e*) $24a^3b^2c^2$. (*f*) $-x^4y^2z$.

2. (*a*) a^2-b^2. (*b*) $6a^2+ab-12b^2$. (*c*) $4x^2-17xy+4y^2$.
(*d*) $5pq-q^2-6p^2$. (*e*) $30-21x+3x^2$.
(*f*) $ln-2lp+2mn-4mp$. (*g*) $ab-ad-cb+cd$.
(*h*) $4y-12x+3x^2-xy$. (*i*) $3k^2+2hk-h^2$.
(*j*) $11l^2-41lm-12m^2$.

3. (*a*) $2a^2b^4c$. (*b*) $-\dfrac{3l^3}{2}$. (*c*) $\dfrac{4a}{5e}$. (*d*) $\dfrac{r}{2pq}$. (*e*) $-\dfrac{7y^2}{xz}$.
(*f*) $-4\frac{1}{2}$. (*g*) $\dfrac{5x^3}{3y^4}$. (*h*) $\dfrac{yw}{xz}$. (*i*) $-\dfrac{l^3m^3}{kn}$. (*j*) $\dfrac{h^{16}}{j^2kl^{10}}$.

4. (*a*) $4x^2+6xy+3xz+2y^2+3yz$. (*b*) $a^3-a^2b-ab^2-2b^3$.
(*c*) $4ax-12a^2b-12a^2c-4ay$. (*d*) $8x-6bx+4c^2x-4bcx$.
(*e*) $pq+prs-p^2r^2+p^3r-p^2$.
(*f*) $a^2-2a^2b+2abc+ab-2ab^2+2b^2c$.

5. (*a*) $\dfrac{-8}{abc^2}$. (*b*) $\dfrac{6xz^4}{y}$. (*c*) $\dfrac{a^5}{b^2c^4}$. (*d*) $p^4q^3r^2$.

7. (*a*) $8x-2y+3z$. (*b*) $b^2-ab-4a^2-3a$. (*c*) 36 ft. above, 6 ft. above, 36 ft. below.

8. (*a*) 31,250,000. (*b*) $n = 250$.

9. *A* receives £660, *B* receives £324, *C* receives £216.

10. 900.

CHAPTER 8

INDICES. COMMON LOGARITHMS

Indices

Certain facts about indices have been covered in the previous chapter. These will be repeated and other facts added.

Rule 1. When powers of the same quantity are multiplied, the indices are added, e.g. $x^4 \times x^7 = x^{11}$.

Rule 2. When a power of one quantity is divided by a power of the same quantity the power of the divisor is subtracted from the power of the dividend, e.g. $x^7 \div x^4 = x^3$.

Rule 3. When a power of a quantity is itself to be raised to a power, the indices are multiplied.

Consider $(x^3)^2$. Now $x^3 = xxx$, and

$$(x^3)^2 = (xxx)(xxx) = xxxxxx = x^6 (3 \times 2 = 6).$$

Also $(y^8)^3 = y^8 y^8 y^8 = y^{24} (8 \times 3 = 24)$.

Rule 4. When a root is required of a quantity which is already raised to a power, the number expressing the root is divided into the original power. (The number expressing a square root is 2, although this is not written outside the radical, or root, sign, but cube, and other roots have a number outside the radical sign.)

E.g. $\sqrt{x^6} = x^{6 \div 2} = x^3$, since $\sqrt{xxxxxx} = xxx = x^3$.

E.g. $\sqrt[3]{y^{12}} = y^{12 \div 3} = y^4$.

Negative indices. From Rule 2 above, $x^4 \div x^6 = x^{-2}$, by subtraction of indices. But $\dfrac{x^4}{x^6} = \dfrac{1}{x^2}$ by cancelling, therefore x^{-2} means $\dfrac{1}{x^2}$. Similarly x^{-1} means $\dfrac{1}{x}$, 4^{-3} means $\dfrac{1}{4^3}$ or $\dfrac{1}{64}$, and 10^{-6} means $\dfrac{1}{10^6}$ or $\dfrac{1}{1{,}000{,}000}$ or 1 millionth.

Zero index. Consider $x^3 \div x^3$. By subtraction of indices the answer is x^0, but, by cancelling, the answer is 1. Therefore $x^0 = 1$. Similarly, $a^5 \div a^5 = a^0 = 1$.

It follows that any quantity to the power 0 is 1.

Fractional indices. From Rule 4 above, $\sqrt{x^3} = x^{3\div 2} = x^{\frac{3}{2}}$. Also $\sqrt[3]{x^5} = x^{5\div 3} = x^{\frac{5}{3}}$. We shall therefore assign this meaning to fractional indices:

When a quantity is written with a fractional index, the numerator of the fraction represents the power to which the quantity is to be raised, while the denominator indicates the root which must be extracted of that answer.

Using this notation, we may express simple roots without using the radical sign, e.g.

$$\sqrt{x} = \sqrt{x^1} = x^{\frac{1}{2}}, \quad \sqrt[4]{y} = \sqrt[4]{y^1} = y^{\frac{1}{4}}.$$

Negative fractional indices. We have seen that x^{-4} means $\frac{1}{x^4}$, therefore $x^{-\frac{3}{2}}$ means $\frac{1}{x^{\frac{3}{2}}}$ or $\frac{1}{\sqrt{x^3}}$.

Note. $4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = 8$, performed by raising 4 to the power 3 and extracting the square root of the answer. The same result is obtained if we extract the square root of 4 first, and then cube the answer. So $\sqrt{4^3}$ is the same as $(\sqrt{4})^3$. It makes no difference whether the power is taken first and the root second, or the root first and the power second. Since understanding the use of directed numbers, however, we realise that 2 is not the only number which may be squared to give 4. The square of -2 is also 4. Hence the square roots of 4 are $+2$ and -2. In fact, there are two square roots to any number, but it is not always necessary to consider the negative square root in a problem.

Hence $4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = \pm 8$ (Read 'plus or minus eight').

Again $81^{-\frac{1}{2}} = \frac{1}{81^{\frac{1}{2}}} = \frac{1}{\sqrt{81}} = \pm\frac{1}{9}$.

Also $\left(\frac{1}{4}\right)^{-\frac{7}{2}} = \frac{1}{(\frac{1}{4})^{\frac{7}{2}}} = \frac{1}{\sqrt{(\frac{1}{4})^7}} = \frac{1}{\sqrt{\frac{1}{16384}}} = \pm\frac{1}{(\frac{1}{128})} = \pm 128.$

A cube root, in the case of ordinary numbers, has only one value, for $2 \times 2 \times 2 = 8$, but $-2 \times -2 \times -2 = -8$, so $\sqrt[3]{8} = 2$, and $\sqrt[3]{-8} = -2$.

Worked examples:

1. $(x^{\frac{3}{2}})^4 = x^{4 \times \frac{3}{2}} = x^{\frac{12}{2}} = x^6$.

2. $x^{\frac{3}{4}} \div x^{\frac{1}{2}} = x^{\frac{3}{4}-\frac{1}{2}} = x^{\frac{1}{4}}$.

3. $x^{\frac{3}{4}} \times x^{\frac{1}{2}} = x^{\frac{3}{4}+\frac{1}{2}} = x^{\frac{5}{4}}$ (or $\sqrt[4]{x^5}$).

4. $x^{\frac{3}{2}} \div x^{\frac{9}{5}} = x^{\frac{3}{2}-\frac{9}{5}} = x^{-\frac{3}{10}} = \dfrac{1}{x^{\frac{3}{10}}}$.

5. $\sqrt[3]{x^{\frac{9}{5}}} = (x^{\frac{9}{5}})^{\frac{1}{3}} = x^{\frac{3}{5}}$.

Use of indices to simplify calculations. The table below gives a list of numbers expressed as powers of 2:

2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
2	4	8	16	32	64	128	256	512	1024	2048	4096

By means of this table and the laws of indices, certain calculations may be simplified.

Examples. $32 \times 128 = 2^5 \times 2^7 = 2^{12} = 4096$,

$16 \times 64 = 2^4 \times 2^6 = 2^{10} = 1024$,

$2048 \div 128 = 2^{11} \div 2^7 = 2^4 = 16$,

$\sqrt{1024} = \sqrt{2^{10}} = 2^{\frac{10}{2}} = 2^5 = 32$,

$\sqrt[3]{4096} = \sqrt[3]{2^{12}} = 2^{\frac{12}{3}} = 2^4 = 16$.

Obviously the range of use is limited, because the list of numbers is not complete. If every number were written as a power of 2, including fractional powers as well, the complete table would simplify all calculations. In that case the number 2 would be called the 'base' of the system. The number 10 has proved to be a much more convenient base, and tables which express numbers from 10 to 99 as powers of 10 are called tables of common logarithms.

Common logarithms

Common logarithms (usually called logarithms) are powers of 10. Thus, since $1000 = 10^3$, the logarithm of 1000 is 3 (written $\log 1000 = 3$) and $1{,}000{,}000 = 10^6$, so $\log 1{,}000{,}000 = 6$.

Similarly, $2 = 10^{0 \cdot 3010}$, so $\log 2 = 0 \cdot 3010$, and $3 = 10^{0 \cdot 4771}$, so $\log 3 = 0 \cdot 4771$.

The advantage of choosing 10 as the base is obvious when we try to write numbers outside the range of 10 to 99 as powers of 10 (i.e. to write the logs of numbers outside this range):

$$200 = 2 \times 100 = 2 \times 10^2 = 10^{0\cdot3010} \times 10^2 = 10^{2\cdot3010}.$$

Thus $\log 200 = 2\cdot3010$.

$$3000 = 3 \times 1000 = 3 \times 10^3 = 10^{0\cdot4771} \times 10^3 = 10^{3\cdot4771}.$$

Thus $\log 3000 = 3\cdot4771$.

Now $10^0 = 1$, as shown previously, so $\log 1 = 0$.

Characteristic and mantissa. It is convenient to distinguish between the part of a logarithm which precedes the decimal point, and the part which follows the point, by two different names.

Consider $\log 200 = 2\cdot3010$ and $\log 3000 = 3\cdot4771$. The integers 2 and 3 which precede the points in the two cases are called the characteristics of the logarithms, while the figures 3010 and 4771 are called the mantissae (singular, mantissa) of the logarithms. The characteristic represents the whole, or integral, power of 10, while the mantissa represents the fractional power of 10.

A logarithm is defined as follows:

The logarithm of a number N to a given base a is the power to which the base a must be raised to give the number N.

In our case a is 10.

To obtain the characteristic of the logarithm of a number. Write the number concerned as a power of 10 multiplied by a number between 1 and 10. Thus

$391 = 3\cdot91 \times 10^2$. The characteristic of the log of 391 is 2.

$41 = 4\cdot1 \times 10$. The characteristic of the log of 41 is 1.

$98760 = 9\cdot876 \times 10^4$. The characteristic of the log of 98760 is 4.

$3\cdot75 = 3\cdot75 \times 10^0$ (since $10^0 = 1$). The characteristic of the log of 3·75 is 0.

$76\cdot93 = 7\cdot693 \times 10^1$. The characteristic of the log of 76·93 is 1.

This is the logical approach, but an alternative argument may be used: 391 lies between 100 and 1000, i.e. between 10^2 and 10^3, so the logarithm lies between 2 and 3. Hence the characteristic is 2.

41 lies between 10 and 100, i.e. between 10^1 and 10^2, so the logarithm lies between 1 and 2. Hence the characteristic is 1.

98760 lies between 10,000 and 100,000, i.e. between 10^4 and 10^5, so the logarithm lies between 4 and 5. Hence the characteristic is 4.

A convenient rule, which should be checked from the above arguments, covers these processes:

To find the characteristic of the logarithm of a number greater than unity, count the number of figures before the decimal point. One less than this total is the required characteristic.

To determine the mantissa of the logarithm of a number from the table. Part of a table of logarithms is shown below:

												Mean differences								
		0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
	10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
							0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
	11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
							0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
→	12	0792	**0828**	0864	0899	0934						3	7	11	14	18	21	**25**	28	32
							0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
	13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
							1303	1235	1367	1399	1430	3	7	10	13	16	19	22	25	29
	14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
→							1614	1644	1673	1703	**1732**	3	6	9	12	14	**17**	20	23	26

These are four-figure tables, i.e. the logarithms are given correct to four significant figures:

To find the logarithm of 1·217.

The characteristic is 0, since $1{\cdot}217 = 1{\cdot}217 \times 10^0$.

Once the characteristic is established, the decimal point may be ignored, so we think merely of the figures 1217.

(1) Find the line corresponding to the first two figures, 12, in the left-hand column.

(2) Move along this line to the main column which contains the next figure, 1, as its head. This reads 0828.

(3) Look in the 'differences' columns opposite 0828 and read the number in the column headed by the fourth figure in the original number, 7. This reads 25.

(4) Add the figures in the difference column to the figures in the main column, i.e.

$$0828 + 25 = 0853.$$

The mantissa of log 1·217 is thus 0853.

Hence log 1·217 is 0·0853.

(This means that $1{\cdot}217 = 10^{0{\cdot}0853}$.)

To find log 12·17.

The characteristic is 1.

For the mantissa, look up the figures 1217 in the table. But this will give exactly the same mantissa as in the previous case.

Hence log 12·17 is 1·0853.

Similarly, log 121700 is 5·0853.

The student will now realise why the table need only give logarithms of numbers from 10 to 99. This is the advantage of using 10 as a base.

To find the logarithm of 149·6.

Characteristic = 2.

(1) The appropriate line in the table is that opposite 14 on the left.

(2) The column headed by the third figure, 9, gives 1732.

(3) The 'differences' column headed by 6 gives 17.

(4) The mantissa of the logarithm required is $1732 + 17 = 1749$

Thus, log 149·6 = 2·1749 (which means that $149{\cdot}6 = 10^{2{\cdot}1749}$).

To find the logarithms of 59·17, 1009 and 90·06.

The student should check these using the full table of logarithms:

log 59·17: Characteristic 1. Mantissa $7716 + 5 = 7721$.

Thus log 59·17 = 1·7721.

log 1009: Characteristic 3. Mantissa $0000 + 38 = 0038$.

Thus log 1009 = 3·0038.

log 90·06: Characteristic 1. Mantissa $9542 + 3 = 9545$.

Hence log 90·06 = 1·9545.

Reverse use of the table of logarithms. (The student should check each of the figures given in the full table of logarithms.)

To find the number which is equal to $10^{0{\cdot}7657}$, i.e. to find the number whose logarithm is 0·7657.

In the main columns the figures 7657 are found opposite 58 on the left in the column headed 3. Thus the number whose logarithm is 0·7657 contains the figures 5830. Since the characteristic was given as 0, the number required is 5·830.

To find the number whose logarithm is 2·7657. The characteristic 2 indicates that the required number lies between 10^2 (i.e. 100) and 10^3 (i.e. 1000), and therefore contains three figures before the decimal point. The figures 7657 are in the same place as we found in the previous example, meaning that the required number contains the figures 5830. As three of these must precede the decimal point, the required number is 583·0.

To find the number whose logarithm is 1·3825. The characteristic 1 shows that the number lies between 10^1 and 10^2, i.e. between 10 and 100, and thus contains two figures before the decimal point. We cannot find 3825 in the main columns. The nearest below 3825 is 3820 which occurs opposite 24 in the main column headed 1. The extra 5 required is in the mean difference column opposite 24, with the heading 3. Our number thus contains the digits 2413. With two figures before the decimal point, the number required is 24·13.

The chapter will conclude with a few worked examples based on the use of indices and logarithms.

Example. If $2 = 10^{0\cdot3010}$, $3 = 10^{0\cdot4771}$ and $5 = 10^{0\cdot6990}$, write 6, 15, 75 and 12 as powers of 10:

$$6 = 3 \times 2 = 10^{0\cdot4771} \times 10^{0\cdot3010} = 10^{0\cdot4771+0\cdot3010} = 10^{0\cdot7781}.$$

$$15 = 5 \times 3 = 10^{0\cdot6990} \times 10^{0\cdot4771} = 10^{0\cdot6990+0\cdot4771} = 10^{1\cdot1761}.$$

$$\begin{aligned} 75 = 5 \times 5 \times 3 &= 10^{0\cdot6990} \times 10^{0\cdot6990} \times 10^{0\cdot4771} \\ &= 10^{0\cdot6990+0\cdot6990+0\cdot4771} = 10^{1\cdot8751}. \end{aligned}$$

$$\begin{aligned} 12 = 2 \times 2 \times 3 &= 10^{0\cdot3010} \times 10^{0\cdot3010} \times 10^{0\cdot4771} \\ &= 10^{0\cdot3010+0\cdot3010+0\cdot4771} = 10^{1\cdot0791}. \end{aligned}$$

Example. Given $\log 2 = 0\cdot3010, \log 7 = 0\cdot8451, \log 3 = 0\cdot4771$, find, without the use of tables, log 8, log 3·5, log 1·5, log 600.

$\log 2 = 0\cdot3010$, i.e. $2 = 10^{0\cdot3010}$, also

$$8 = 2^3 = (10^{0\cdot3010})^3 = 10^{3\times0\cdot3010} = 10^{0\cdot9030}.$$

$$\text{Therefore } \underline{\log 8 = 0\cdot9030.}$$

To find log 3·5:

$$3{\cdot}5 = \frac{7}{2} = \frac{10^{0{\cdot}8451}}{10^{0{\cdot}3010}} = 10^{0{\cdot}8451-0{\cdot}3010} = 10^{0{\cdot}5441}.$$

Therefore $\underline{\log 3{\cdot}5 = 0{\cdot}5441.}$

To find log 1·5:

$$1{\cdot}5 = \frac{3}{2} = \frac{10^{0{\cdot}4771}}{10^{0{\cdot}3010}} = 10^{0{\cdot}4771-0{\cdot}3010} = 10^{0{\cdot}1761}.$$

Therefore $\underline{\log 1{\cdot}5 = 0{\cdot}1761.}$

To find log 600. Now

$$600 = 3 \times 2 \times 100 = 10^{0{\cdot}4771} \times 10^{0{\cdot}3010} \times 10^{2{\cdot}0000} = 10^{2{\cdot}7781}.$$

Therefore $\underline{\log 600 = 2{\cdot}7781.}$

Examples to Chapter 8

1. (*a*) Express with positive indices, x^{-5}, x^{-2}, $\frac{1}{x^{-3}}$, $x^{-\frac{1}{2}}$.

(*b*) Express with fractional indices, $\sqrt{x}$, $\frac{1}{\sqrt[3]{x}}$, $\sqrt{x^3}$, $\frac{1}{\sqrt[5]{x^4}}$.

(*c*) Express using radical signs, $x^{\frac{2}{3}}$, $x^{-\frac{1}{2}}$, $x^{-\frac{5}{3}}$, $x^{\frac{3}{4}}$.

(*d*) Evaluate $4^{\frac{1}{2}}$, $9^{-\frac{1}{2}}$, $(\frac{1}{81})^{-\frac{1}{4}}$, $4^{\frac{3}{2}}$. (*Chap.* 8)

2. (*a*) Write as powers of x, $(x^{\frac{2}{3}})^{\frac{1}{2}}$, $(x^{\frac{1}{3}})^{3}$, $\sqrt{x^{\frac{3}{2}}}$, $\sqrt[3]{\left(\frac{1}{x}\right)^{2}}$.

(*b*) Simplify: $a^{\frac{1}{2}} \times a^{\frac{3}{2}}$, $\sqrt{b} \times b^{-1}$, $c^{\frac{5}{4}} \div c^{\frac{1}{4}}$, $d^{-\frac{1}{2}} \div d^{\frac{1}{2}}$.

(*c*) Evaluate: $100^{\frac{3}{2}} \times 100^{\frac{1}{2}}$, $2^3 \div 2^5$, $9^{\frac{3}{2}} \div 9$, $27^{\frac{1}{3}} \times 27^{-\frac{1}{3}}$.

(*Chap.* 8)

3. (*a*) If $5 = 10^{0{\cdot}6990}$ and $3 = 10^{0{\cdot}4771}$, write 15, $1\frac{2}{3}$ and 450 as powers of 10.

(*b*) Using the figures in (*a*) write as powers of 10 the answers to $50{,}000 \div 300$, $1{\cdot}5 \times 10^4 \div 3 \times 10^2$.

(*c*) Evaluate without tables, using the figures in (*a*): $10^{3{\cdot}6990}$, $10^{0{\cdot}9542}$, $10^{1{\cdot}3980}$. (*Chap.* 8)

4. (*a*) Using the table, find the logarithms of

3·142, 314·2, 789·3, 7·861, 10·03, 1,157,000.

(*b*) Using the table of logarithms, find the numbers which have the following logarithms:

0·5621, 1·5621, 2·7714, 1·9909, 5·0170.

(*c*) Write the logarithms of the following:

3×10^{10}, $4{\cdot}2 \times 10^7$, $7{\cdot}1 \times 10^8$. (*Chap.* 8)

5. (*a*) Use the table of logarithms to write 6 and 7 as powers of 10. Hence write 42, 420, 4·2 as powers of 10.

(*b*) Write 7·921 and 8·563 as powers of 10 using the table. Hence write 7·921 × 8·563 as a power of 10. From the same table write the answer to 7·921 × 8·563 to four significant figures. (*Chap.* 8)

6. (*a*) If $\log 2 = 0{\cdot}3010$, $\log 3 = 0{\cdot}4771$, $\log 5 = 0{\cdot}6990$, find without tables, $\log 2{\cdot}5$, $\log 3\frac{1}{3}$, $\log 4{\cdot}5$, $\log 81$.

(*b*) Using the tables, write 4·9 as a power of 10. Hence write $(4{\cdot}9)^3$ as a power of 10.

(*c*) Using the log of 4·9 from above, write log 49,000, and hence find the square root of 49,000 to four significant figures. (*Chap.* 8)

7. (*a*) $(4x-3y)(7x+2y)$, $(a-3b)(a^2-3ab+b^2)$,

$-4x(x^2+3x-1)$.

(*b*) $\frac{1}{4}a^2b^3c \div -3ab^4c^5$, $\sqrt{x^9} \times x^{\frac{3}{2}}$.

(*c*) Write an expression for the area of a rectangular field of sides $a-2b$ and $3a-4b$. (*Chap.* 7)

8. (*a*) Add together

$3a^2+4ab+b^2$, $ab-b^2-2a^2$, $5a^2-7b^2$.

(*b*) Subtract x^2-3x+9 from $2x^3-3x^2+4x+1$.

(*c*) A body is oscillating in a straight path about a fixed point. Its velocity at a given instant is given by

$$v = 8\sqrt{(1-k^2)} - 6k,$$

where $k = \frac{12}{13}$. When v is positive, the motion is towards the left, and when v is negative, the motion is towards the right. Find the actual velocity of the body from the formula. (*Chap.* 6)

9. (*a*) A rectangular metal plate is a ft. b in. long, and c ft. d in. wide. From this plate, p circles, each of radius q in., are punched.

Write an expression for the area in square feet, of the remaining surface. {Take area of circle $= \pi \times (\text{radius})^2$.}

(*b*) Evaluate t from the formula

$$t = \frac{44}{7}\sqrt{\left(\frac{a^2+b^2}{2gh}\right)},$$

where $a = 5$, $b = 12$, $g = 32$, and $h = 16$. (*Chap.* 5)

10. The coefficient of linear expansion (α) of a rod when heated is given by

$$\alpha = \frac{\text{Increase in length}}{\text{Original length} \times \text{rise in temperature}}.$$

A rod 50 cm. long, when heated from a temperature of 15° C. to a temperature of 90° C., expands 0·045 cm. Find the coefficient of linear expansion, and express in terms of 10^{-5}.

What would be the value of α if the rod were originally 50 ft. long and its expansion for the same rise of temperature of 75° C. were 0·045 ft.? (*Chap.* 2)

Answers to Examples for Chapter 8

1. (*a*) $\frac{1}{x^{\text{[illegible]}}}$, $\frac{1}{x^2}$, x^3, $\frac{1}{x^{\text{[illegible]}}}$. (*b*) $x^{\text{[illegible]}}$, $x^{-\text{[illegible]}}$, $x^{\text{[illegible]}}$, $x^{-\text{[illegible]}}$.

(*c*) $\sqrt{x^3}$, $\frac{1}{\sqrt{x}}$, $\frac{1}{\sqrt[3]{x^5}}$, $\sqrt[4]{x^7}$. (*d*) ± 2, $\pm \frac{1}{3}$, ± 9, ± 8.

2. (*a*) $x^{\text{[illegible]}}$, $x^{\text{[illegible]}}$, $x^{\text{[illegible]}}$, $x^{-\text{[illegible]}}$. (*b*) $a^{\text{[illegible]}}$, $b^{-\text{[illegible]}}$, c^2, d^{-4}.

(*c*) 10,000, $\frac{1}{4}$, ± 3, 1.

3. (*a*) $10^{1\cdot1761}$, $10^{0\cdot2219}$, $10^{2\cdot6532}$. (*b*) $10^{2\cdot2219}$, $10^{1\cdot6990}$.

(*c*) 5000, 9, 25.

4. (*a*) 0·4972, 2·4972, 2·8973, 0·8955, 1·0013, 6·0633.

(*b*) 3·648, 36·48, 590·7, 97·93, 104,000.

(*c*) 10·4771, 7·6232, 8·8513.

5. (*a*) 1·6232, 2·6232, 0·6232. (*b*) $10^{1\cdot8315} = 67\cdot85$.

6. (*a*) 0·3979, 0·5228, 0·6532, 1·9084.

(*b*) $10^{0\cdot6902}$, $10^{2\cdot0706}$. (*c*) 4·6902, 221·3.

7. (*a*) $28x^2 - 13xy - 6y^2$, $a^3 - 6a^2b + 10ab^2 - 3b^3$,

$-4x^3 - 12x^2 + 4x$.

(*b*) $-\dfrac{a}{12bc^4}$, x^6. (*c*) $3a^2 - 10ab + 8b^2$.

8. (*a*) $6a^2 + 5ab - 7b^2$. (*b*) $2x^3 - 4x^2 + 7x - 8$.

(*c*) $2\frac{6}{13}$ to right if positive root taken.

9. (*a*) $\left\{\left(a+\dfrac{b}{12}\right)\left(c+\dfrac{d}{12}\right)-\dfrac{\pi pq^2}{144}\right\}$ sq.ft. (*b*) $\frac{143}{56}$.

10. $\alpha = 1\cdot2 \times 10^{-5}$. Same result.

CHAPTER 9

USE OF LOGARITHMS (POSITIVE CHARACTERISTICS)

To multiply numbers using logarithms

Add the logarithms of the numbers. Since these are powers of 10, the sum of the logarithms is the power of 10 corresponding to the answer.

Example. Find the volume of a rectangular block 4·12 ft. by 7·93 ft. by 15·94 ft.

Volume

= length × breadth × thickness
= 15·94 × 7·93 × 4·12 cu.ft.
= 520·8 cu.ft.

Numbers	Logs	
15·94	1·2025	Addition
7·93	0·8993	
4·12	0·6149	
520·8	2·7167	

The right-hand column shows that the answer is $10^{2\cdot7167}$ by addition. It remains to find the value of $10^{2\cdot7167}$. As we know, this may be found from the table of logarithms, but it is less trouble to use the table of 'antilogarithms'. By means of this table we can find a number if we know its logarithm. It is used in the same way as the table of logarithms provided we remember that *only the mantissa* is found in the table. Thus in finding the number whose logarithm is 2·7167, we see from the characteristic 2 that the required number contains three figures before the decimal point, and then ignore it until the table has been used. Look in the left-hand column of the antilogarithm table for 71; then look in the corresponding main column headed 6. This gives 5200. The difference column headed 7 gives 8, so the number we require contains the digits 5208. The characteristic 2 fixes the decimal point between the 0 and the 8.

Rules for the use of the table of antilogarithms

(1) Look up the *mantissa* of the known logarithm (not the characteristic) in the table.

(2) Use the characteristic of the known logarithm merely to fix the decimal point in the answer.

From now on the table of antilogarithms will be used in the final stage of problems. The word 'antilogarithm' should not be used in the statements of a problem. An antilogarithm is only a number which has been obtained in the form of a power of 10.

To divide numbers using logarithms. Subtract the logarithm of the divisor from the logarithm of the dividend. As the logs are powers of 10, the subtraction of logs is the subtraction of indices.

Example. 149·2 ÷ 28·93.

Numbers	Logs	
149·2	2·1738	Subtraction
28·93	1·4614	
5·157	0·7124	

$$\frac{149{\cdot}2}{28{\cdot}93} = \underline{5{\cdot}157.}$$

The answer 5·157 is obtained as 5157 from the figures 7124 in the antilogarithm table. The point is fixed by the characteristic, 0.

To raise numbers to powers, using logarithms. Multiply the logarithm of the number by the power to which the number is to be raised.

Example. $(76{\cdot}97)^4$.

Numbers	Logs	
76·97	1·8863	Multiplication
	4	
35100000	7·5452	

$$\underline{(76{\cdot}97)^4 = 35100000.}$$

To find roots of numbers, using logarithms. Divide the logarithm of the number by the number which expresses the root, e.g. divide by 2 for square root, 3 for cube root, etc. If we think of expressing roots as fractional indices, the process becomes that of raising numbers to powers, for if we consider a root such as $\sqrt{58{\cdot}63}$ as $(58{\cdot}63)^{\frac{1}{2}}$, then the logarithm of 58·63 is multiplied by $\frac{1}{2}$ (i.e. divided by 2) to obtain the answer.

Example. $\sqrt{(58{\cdot}63)}$.

Numbers	Logs	
58·63	1·7681	(1)
$(58{\cdot}63)^{\frac{1}{2}}$	0·88405	(2)
7·658	0·8841	(3)

$$\underline{\sqrt{(58{\cdot}63)} = 7{\cdot}658.}$$

Line (1) in the log column is the log of 58·63. Line (2) is the previous log divided by 2, while line (3) is line (2) written correct to the fourth decimal place in order to use four-figure antilog tables.

To raise a number to a fractional power, using logarithms. Cases where the fractional power has unity for the numerator are merely roots, and have been dealt with above. Consider $(48\cdot37)^{\frac{3}{2}}$. This may be written $\sqrt{(48\cdot37)^3}$, or $(\sqrt{48\cdot37})^3$. To evaluate this by means of logarithms means *either* multiplying the logarithm of 48·37 by 3, then halving the result, *or* halving the logarithm of 48·37 and multiplying the result by 3.

To raise a number to a fractional power, multiply the logarithm of the number by the numerator of the power, and divide the result by the denominator of the power, *or*, divide the logarithm of the number by the denominator of the power, and multiply the result by the numerator of the power.

Example. $(809\cdot3)^{\frac{5}{3}}$.

	Numbers	Logs	
	809·3	2·9081	
	$(809\cdot3)^5$	14·5405	Multiplying by 5
	$(809\cdot3)^{\frac{5}{3}}$	4·84683	Dividing by 3
$\underline{(809\cdot3)^{\frac{5}{3}} = 70{,}280}$.	70,280	4·8468	To four decimal places

Problems involving several operations

Example 1. The number of gallons of water flowing through a pipe per minute is given by the formula

$$n = \sqrt{\frac{243d^5H}{L}}.$$

Find n when $d = 4\cdot25$, $H = 42\cdot5$, $L = 1760$:

$$n = \sqrt{\frac{243d^5H}{L}} = \sqrt{\frac{243 \times (4\cdot25)^5 \times 42\cdot5}{1760}} \quad \text{(substituting values).}$$

[The operations to be performed are:

(1) Add the logs of 243 and 42·5 to *five times* the log of 4·25.

(2) Subtract from this total the log of 1760.

(3) Halve the result (for square root).

(4) Find the number which has this logarithm from the table of antilogs.]

	Numbers	Logs	
	4·25	0·6284	
	$(4\cdot25)^5$	3·1420	Adding
	243	2·3856	
	42·5	1·6284	
		7·1560	Subtracting
	1760	3·2455	
		3·9105	
		1·95525	Dividing by 2
$n = 90\cdot22$ gallons.	90·22	1·9553	

Example 2. The radius of a sphere is given by

$$r = \sqrt[3]{\frac{3V}{4\pi}}.$$

Find the radius when $V = 86\cdot72$, $\pi = 3\cdot142$.

$$r = \sqrt[3]{\frac{3V}{4\pi}} = \sqrt[3]{\frac{3 \times 86\cdot72}{4 \times 3\cdot142}}.$$

[The operations to be performed are:

(1) Add the logs of 3 and of 86·72.

(2) Add the logs of 4 and of 3·142 and subtract from the sum in (1).

(3) Divide the result by 3 to extract the cube root.

(4) Find the number which has this logarithm.]

	Numbers	Logs	
	3	0·4771	Adding
	86·72	1·9381	
		2·4152	(A)
	4	0·6021	Adding
	3·142	0·4972	
		1·0993	(B)
		1·3159	Subtracting (B) from (A)
$r = 2\cdot746$.	2·746	0·4386	Dividing by 3

Alternative arrangement. Problems of this type may be set out in a more concise form as follows:

Example 1. $$n = \sqrt{\frac{243d^5H}{L}} \text{ gallons.}$$

$$n = \sqrt{\frac{243 \times (4{\cdot}25)^5 \times 42{\cdot}5}{1760}} \quad \text{(By substitution from given facts).}$$

Therefore $\log n = \frac{1}{2}(\log 243 + 5\log 4{\cdot}25 + \log 42{\cdot}5 - \log 1760)$.

[Referring to the operations enumerated when this example was set down in its first form, we see that the right-hand side of this last mathematical statement contains processes (1), (2) and (3). Since process (4) is not yet performed the result will be a logarithm, namely, $\log n$.]

$$\begin{aligned}\log n &= \tfrac{1}{2}(2{\cdot}3856 + 5 \times 0{\cdot}6284 + 1{\cdot}6284 - 3{\cdot}2455)\\ &= \tfrac{1}{2}(2{\cdot}3856 + 3{\cdot}1420 + 1{\cdot}6284 - 3{\cdot}2455)\\ &= \tfrac{1}{2}(7{\cdot}1560 - 3{\cdot}2455)\\ &= \tfrac{1}{2} \times 3{\cdot}9105\\ &\doteqdot 1{\cdot}9553.\end{aligned}$$

Therefore $\underline{n = 90{\cdot}22 \text{ gallons}}$ (from table of antilogarithms).

Example 2. $$r = \sqrt[3]{\frac{3V}{4\pi}} = \sqrt[3]{\frac{3 \times 86{\cdot}72}{4 \times 3{\cdot}142}}.$$

$\therefore$ (therefore) $\log r = \frac{1}{3}(\log 3 + \log 86{\cdot}72 - \log 4 - \log 3{\cdot}142)$
[Operations (1), (2) and (3)]

$$\begin{aligned} &= \tfrac{1}{3}(0{\cdot}4771 + 1{\cdot}9381 - 0{\cdot}6021 - 0{\cdot}4972)\\ &= \tfrac{1}{3}(2{\cdot}4152 - 1{\cdot}0993)\\ &= \tfrac{1}{3} \times 1{\cdot}3159\\ &\doteqdot 0{\cdot}4386.\end{aligned}$$

$$\therefore \underline{r = 2{\cdot}746.}$$

Formulae with additions and subtractions. The main point to remember is that if a formula contains additions or subtractions, these must be performed with *numbers*, not logarithms, for the addition of logs means the multiplication of numbers.

Example. The period of oscillation of a compound pendulum is given by

$$t = 2\pi\sqrt{\left(\frac{K^2 + h^2}{hg}\right)}.$$

Find t if $K = 18{\cdot}2$, $h = 20{\cdot}9$, $g = 32{\cdot}2$, $\pi = 3{\cdot}142$:

$$t = 2\pi\sqrt{\left(\frac{K^2+h^2}{hg}\right)} = 2\times 3{\cdot}142\sqrt{\left\{\frac{(18{\cdot}2)^2+(20{\cdot}9)^2}{20{\cdot}9\times 32{\cdot}2}\right\}}.$$

[The operations to be performed are:

(1) Square 18·2, square 20·9, and add the results. The squaring may be performed by logs, but the added results must be *numbers*.

(2) From the *log* of the sum just obtained, subtract the sum of the logs of 20·9 and 32·2.

(3) Halve the result (square root).

(4) Add to (3) the logs of 2 and of 3·142 (or the log of 6·284).

(5) Use the table of antilogarithms for the final answer.]

$$\begin{aligned} t &= 2\times 3{\cdot}142\sqrt{\left\{\frac{(18{\cdot}2)^2+(20{\cdot}9)^2}{20{\cdot}9\times 32{\cdot}2}\right\}} \\ &= 2\times 3{\cdot}142\sqrt{\left(\frac{331{\cdot}3+436{\cdot}7}{20{\cdot}9\times 32{\cdot}2}\right)} \\ &= 2\times 3{\cdot}142\sqrt{\frac{768{\cdot}0}{20{\cdot}9\times 32{\cdot}2}} \\ &= 6{\cdot}284\sqrt{\frac{768{\cdot}0}{20{\cdot}9\times 32{\cdot}2}} \end{aligned}$$

$$\therefore \underline{t = 6{\cdot}714}.$$

Numbers	Logs	
18·2	1·2601	
331·3	2·5202	Multiplying by 2
20·9	1·3201	
436·7	2·6402	Multiplying by 2
768·0	2·8854	(A)
20·9	1·3201	Adding
32·2	1·5079	
	2·8280	(B)
	0·0574	Subtracting (B) from (A)
	0·0287	Dividing by 2
6·284	0·7983	Adding
6·714	0·8270	

The presence of added terms in formulae of this type makes it impossible to write them in the alternative form which was shown for the two preceding examples.

Logarithmic statements

Example. If $I = \dfrac{PLAN}{33{,}000}$, write a statement for log I.

The statement required is an expression showing the processes necessary to evaluate I by the use of logarithms, omitting the final step of using the antilog table.

To obtain log I, we must add the logs of P, L, A and N, and subtract the log of 33,000.

$$\begin{aligned} \therefore \log I &= \log P + \log L + \log A + \log N - \log 33{,}000 \\ &= \underline{\log P + \log L + \log A + \log N - 4{\cdot}5185.} \end{aligned}$$

Example. Write a statement for $\log d$ if $d = K\sqrt[5]{\dfrac{4n^2}{H}}$.

[Note that the processes which would be performed first with logs are placed in brackets.]

Thus $\underline{\log d = \log K + \tfrac{1}{5}(\log 4 + 2\log n - \log H).}$

The reverse process appears more difficult at first:

Example. If $\log I = \log B + 3\log D - 1{\cdot}0792$, write a formula for I.

[Note that addition signs in the log formula imply multiplication of numbers, the subtraction sign, division, while the 3 before $\log D$ shows that D is to be cubed. Also 1·0792 is a log, so its antilog must be written in the final answer.]

$$\begin{aligned} \log I &= \log B + 3\log D - 1{\cdot}0792 \\ &= \log B + \log (D^3) - \log 12. \end{aligned}$$

$$\therefore I = \frac{BD^3}{12}.$$

Example. If

$$\log t = 0{\cdot}3010 + \log \pi + \tfrac{1}{2}\{\log I - (\log m + \log g + \log h)\}$$

write a formula for t.

$$\begin{aligned} \log t &= 0{\cdot}3010 + \log \pi + \tfrac{1}{2}\{\log I - (\log m + \log g + \log h)\} \\ &= \log 2 + \log \pi + \tfrac{1}{2}\{\log I - \log m - \log g - \log h\}. \end{aligned}$$

$$\therefore \underline{t = 2\pi\sqrt{\frac{I}{mgh}}.}$$

Examples to Chapter 9

1. Evaluate by logarithms:

(*a*) $3{\cdot}718 \times 29{\cdot}65 \times 1{\cdot}003$.

(*b*) $\dfrac{7{\cdot}952 \times 4{\cdot}718}{14{\cdot}96}$.

(*c*) The indicated horse-power of an engine is given by the formula $I = \dfrac{PLAN}{33{,}000}$. Find I when $P = 80$, $L = 2{\cdot}8$, $A = 50{\cdot}15$, $N = 270$. (*Chap.* 9)

2. (*a*) $1{\cdot}731 \times 10^4 \times 32{\cdot}96$.

(*b*) $\dfrac{42{\cdot}1 \times (31{\cdot}9)^2}{72{\cdot}67}$.

(*c*) Evaluate r from the formula $r = \sqrt{\dfrac{3V}{\pi h}}$ when $V = 81{\cdot}5$, $\pi = 3{\cdot}142$, $h = 26{\cdot}43$. (*Chap.* 9)

3. (*a*) Evaluate $\dfrac{4{\cdot}28\sqrt[3]{14{\cdot}92}}{7{\cdot}961}$.

(*b*) The loss of head (h) for water flowing through circular pipes is given by $h = \dfrac{lv^2}{66{\cdot}7dg}$. Find h when $l = 85$, $v = 6{\cdot}18$, $d = 1{\cdot}02$, $g = 32{\cdot}2$.

(*c*) If $\log Q = \log k + \log a + \frac{1}{2}(\log c + \log g + \log h)$, write a formula for Q. (*Chap.* 9)

4. (*a*) From the formula $P = \dfrac{wv^2}{rg}$, write a formula for $\log P$.

(*b*) From the formula $\log v = \frac{1}{2}(\log P + \log r + \log g - \log w)$, write a formula for v.

(*c*) The formula $F = \dfrac{M}{(d^2 + l^2)^{\frac{3}{2}}}$ gives the magnetic field (F) at a distance d from the centre of a magnet of length $2l$. Find F when $M = 1820$, $d = 10{\cdot}9$, and $l = 3{\cdot}21$. (*Chap.* 9)

5. (*a*) The moment of inertia of a rectangle may be obtained from the formula $I = \frac{bd^3}{12}$. Write a statement for log I.

(*b*) Find the 'radius of gyration' (K) of a circular disc rotating about a point $\frac{4}{5}$ of its radius from the centre from the formula

$$K = \sqrt{\left\{\frac{R^2}{2} + \left(\frac{4R}{5}\right)^2\right\}} \quad \text{when } R = 6{\cdot}52. \qquad (\textit{Chap. } 9)$$

6. The formula $\Delta = \sqrt{\{s(s-a)(s-b)(s-c)\}}$ gives the area of a triangle when the lengths of the three sides are known. Find the area of a triangle whose sides are $a = 3{\cdot}71$, $b = 4{\cdot}42$, $c = 6{\cdot}13$, where $s = \frac{a+b+c}{2}$. (*Chap.* 9)

7. (*a*) From $2x^2 - 3x + 4$ subtract $x^3 - 3x^2 + 1$.

(*b*) Two trains are travelling with constant acceleration in the same direction. After t sec. their distance apart (s) is given by

$$s = (V - v)t + \tfrac{1}{2}(F - f)t^2.$$

Find s when $V = 72$, $v = 44$, $F = -3$, $f = 2$, $t = 10$. (*Chap.* 6)

8. (*a*) Write an expression for the volume of a cube of edge $(a - b)$.

(*b*) Simplify $\sqrt{a^4b^3c} \div \sqrt[3]{ac^3d^2}$, and show that if $a = 64$, $b = 4$, $c = 16$, $d = 5\sqrt{5}$, the answer is $\frac{\sqrt{2}}{5}$. (*Chap.* 7)

9. The capital for a business was subscribed completely by four partners A, B, C and D, A giving $\frac{1}{3}$, B giving $\frac{1}{4}$, while the ratio of the shares for C and D was 12 to 13. A profit of £1950 is to be shared according to capital invested. Find the share for each partner. (*Chaps.* 1 *and* 3)

10. (*a*) A firm buys n new machines at £x each, m machines at £y each, sells goods for £z, and its costs for the period (excluding the purchase of machinery) amount to £p. Write an expression for the percentage profit on costs and outlay.

(*b*) Evaluate d from the formula

$$d = \frac{1}{3}\sqrt[5]{\frac{n^2L}{H}} \quad \text{if } n = 54,\ L = 729,\ H = 36. \qquad (\textit{Chap. } 5)$$

Answers to Examples for Chapter 9

1. (*a*) 110·6. (*b*) 2·508. (*c*) 91·91.

2. (*a*) 570,600. (*b*) 589·6. (*c*) 1·716.

3. (*a*) 1·323. (*b*) 1·482. (*c*) $Q = ka\sqrt{cgh}$.

4. (*a*) $\operatorname{Log} P = \log w + 2\log v - \log r - \log g$. (*b*) $v = \sqrt{\dfrac{Prg}{w}}$.
(*c*) 1·241.

5. (*a*) $\operatorname{Log} I = \log b + 3\log d - 1{\cdot}0792$. (*b*) 6·96.

6. 8·130.

7. (*a*) $-x^3 + 5x^2 - 3x + 3$. (*b*) 30.

8. (*a*) $a^3 - 3a^2b + 3ab^2 - b^3$. (*b*) $\dfrac{b^{\frac{3}{2}}}{a^{\frac{1}{15}}c^{\frac{1}{2}}d^{\frac{3}{5}}}$.

9. £650, £487. 10*s*., £390, £422. 10*s*.

10. (*a*) $\dfrac{(z - nx - my - p)}{nx + my + p}100$. (*b*) 3.

CHAPTER 10

USE OF LOGARITHMS (NEGATIVE CHARACTERISTICS)

The previous two chapters have dealt with logarithms of numbers greater than unity. We shall now use logarithms of numbers less than unity.

Take the number 0·4612, which may be written $4{\cdot}612 \times 10^{-1}$. And since $4{\cdot}612 = 10^{0{\cdot}6639}$ (i.e. $\log 4{\cdot}612 = 0{\cdot}6639$), thus

$$0{\cdot}4612 = 10^{-1+0{\cdot}6639}.$$

Now the index $-1 + 0{\cdot}6639$ could be written as the algebraic sum of -1 and $0{\cdot}6639$, viz. $-0{\cdot}3361$, so that $0{\cdot}4612 = 10^{-0{\cdot}3361}$, or $\log 0{\cdot}4612 = -0{\cdot}3361$. But it is not convenient to add characteristic to mantissa at this stage, so we shall agree that the mantissa of any logarithm shall be written positive, and if the characteristic happens to be negative, as it will be for numbers less than unity, it shall be written separately. In order to distinguish a negative characteristic from a positive mantissa, the negative sign is written over the characteristic.

Thus $\log 0{\cdot}4612$ is written as $\bar{1}{\cdot}6639$ (read 'bar one point six six three nine').

All numbers between 0 and 1 have negative characteristics. Thus:

$$0{\cdot}3194 = 3{\cdot}194 \times 10^{-1}, \quad \log 0{\cdot}3194 = \bar{1}{\cdot}5044,$$

$$0{\cdot}003194 = 3{\cdot}194 \times 10^{-3}, \quad \log 0{\cdot}003194 = \bar{3}{\cdot}5044,$$

$$0{\cdot}07962 = 7{\cdot}962 \times 10^{-2}, \quad \log 0{\cdot}07962 = \bar{2}{\cdot}9010,$$

$$0{\cdot}000003142 = 3{\cdot}142 \times 10^{-6}, \quad \log 0{\cdot}000003142 = \bar{6}{\cdot}4972.$$

Otherwise: 0·3194 lies between 0·1 and 1, i.e. between 10^{-1} and 10^{0}, as the characteristic of $\log 0{\cdot}3194$ is $\bar{1}$.

0·003194 lies between 0·001 and 0·01, i.e. between 10^{-3} and 10^{-2}, so the characteristic of $\log 0{\cdot}003194$ is $\bar{3}$.

Also 0·07962 lies between 0·01 and 0·1, i.e. between 10^{-2} and 10^{-1}, so the characteristic of $\log 0{\cdot}07962$ is $\bar{2}$.

The following rule will be helpful:

To find the characteristic of the logarithm of a number less than unity, count the noughts immediately following the decimal point, add one to the total, and write a negative sign over that result. Thus if one nought follows the point, the characteristic is $\bar{2}$, if two noughts, the characteristic is $\bar{3}$, and so on.

The student must now bear in mind that the rules for directed numbers apply to all operations with characteristics.

Multiplication of numbers including logarithms with negative characteristics

Example. 0·1564 × 51·62.

Numbers	Logs
0·1564	$\bar{1}$·1942
51·62	1·7128
8·072	0·9070

0·1564 × 51·62 = 8·072.

Fig. 16 represents the mantissae of both logarithms by positive distances, while one characteristic is negative and the other positive. Addition of mantissae gives 0·9070, while addition of characteristics as directed numbers gives zero.

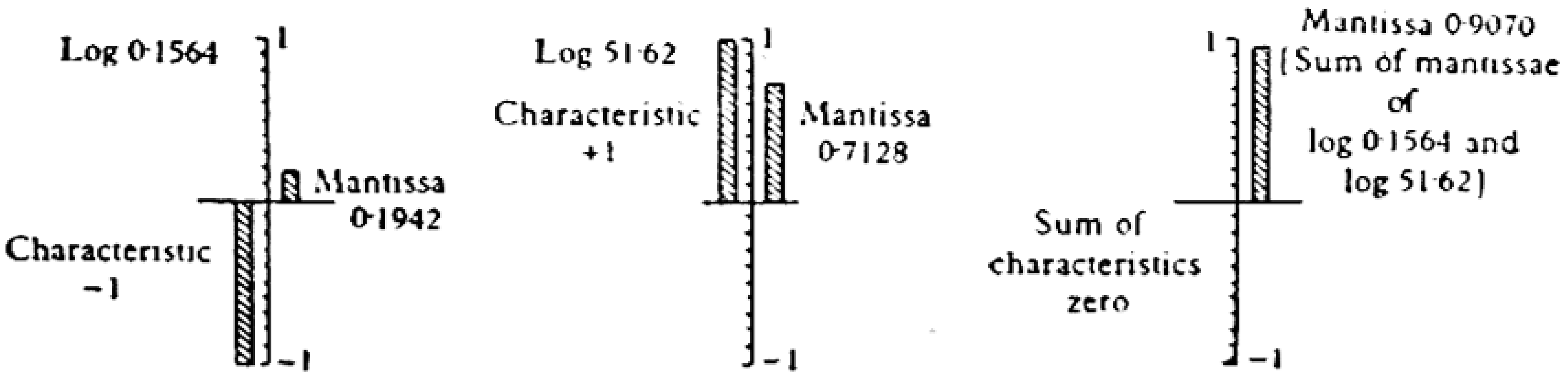


Fig. 16

Example. 72·84 × 0·08561.

Numbers	Logs
72·84	1·8623
0·08561	$\bar{2}$·9326
6·236	0·7949

72·84 × 0·08561 = 6·236.

Note that addition of the mantissae gives 1·7949, the figure 1 preceding the decimal point forms part of the characteristic of the result, so the final characteristic is obtained by adding this 1 to the $\bar{2}$ and the 1 already in that column. The result is 0·7949.

Fig. 17 illustrates the process of addition. Note that the sum of the original characteristics in (C) is $\bar{1}$, but the $+1$ which comes from the addition of the mantissae makes the total zero.

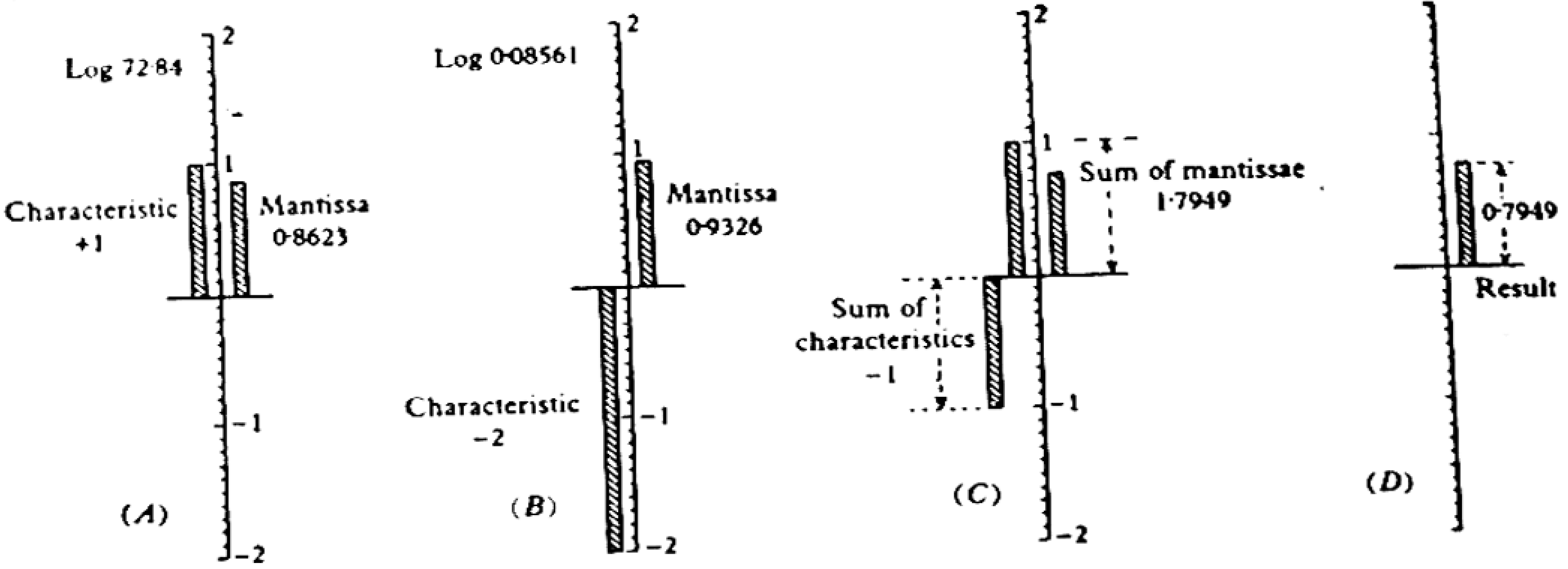


Fig. 17

Example. 61·64 × 0·3842 × 0·004172.

Numbers	Logs
61·64	1·7899
0·3842	$\bar{1}\cdot5845$
0·004172	$\bar{3}\cdot6203$
0·09879	$\bar{2}\cdot9947$

61·64 × 0·3842 × 0·004172 = 0·09879.

Division of numbers including logarithms with negative characteristics

Example. 0·7171 ÷ 0·03642.

Numbers	Logs
0·7171	$\bar{1}\cdot8556$
0·03642	$\bar{2}\cdot5613$
19·69	1·2943

$$\frac{0\cdot7171}{0\cdot03642} = 19\cdot69.$$

Subtracting a smaller mantissa from a greater presents no difficulties. The characteristics are subtracted by the rule for subtraction of directed numbers. See fig. 18.

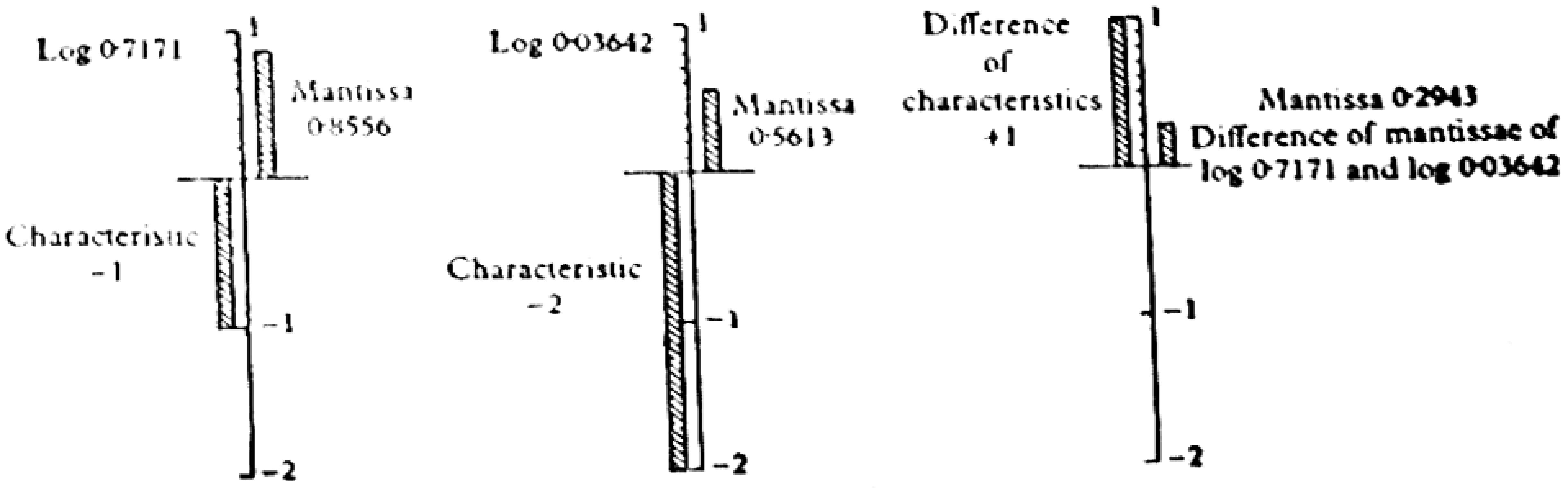

Log 0·7171
1
Mantissa 0·8556
Characteristic -1
-1
-2
Log 0·03642
1
Mantissa 0·5613
Characteristic -2
-1
-2
Difference of characteristics +1
1
Mantissa 0·2943 Difference of mantissae of log 0·7171 and log 0·03642
-1
-2

Fig. 18

Example. 723·9 ÷ 0·08654.

Numbers	Logs
723·9	2·8596
0·08654	$\bar{2}$·9372
8364	3·9224

$$\frac{723\cdot 9}{0\cdot 08654} = 8364.$$

When subtracting a larger mantissa from a smaller, as in this example, think as follows. Consider the logarithm of 723·9 to be 1 + 1·8596, then, subtracting 0·9372 from 1·8596 leaves 0·9224. Next subtract $\bar{2}$ from the remaining 1 of the characteristic of the log of 723·9. This gives +3. Fig. 19 illustrates these processes.

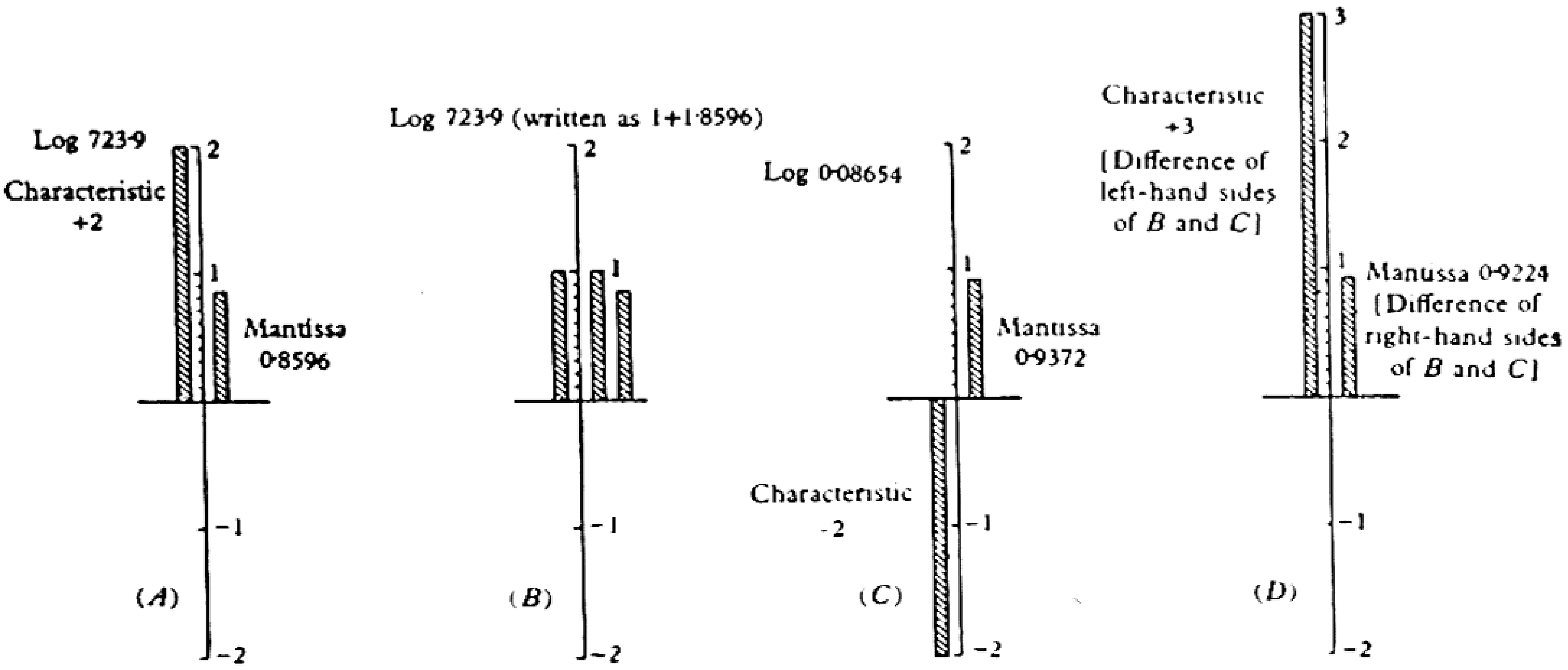

Log 723·9
Characteristic +2
Mantissa 0·8596
2
1
−1
−2
(A)
Log 723·9 (written as 1+1·8596)
2
1
−1
−2
(B)
Log 0·08654
2
1
Mantissa 0·9372
Characteristic −2
−1
−2
(C)
Characteristic +3 [Difference of left-hand sides of B and C]
3
2
1
Mantissa 0·9224 [Difference of right-hand sides of B and C]
−1
−2
(D)

Fig. 19

Example. 0·2140 ÷ 56·93.

Numbers	Logs
0·2140	$\bar{1}$·3304
56·93	1·7553
0·003759	$\bar{3}$·5751

0·2140 ÷ 56·93 = 0·003759.

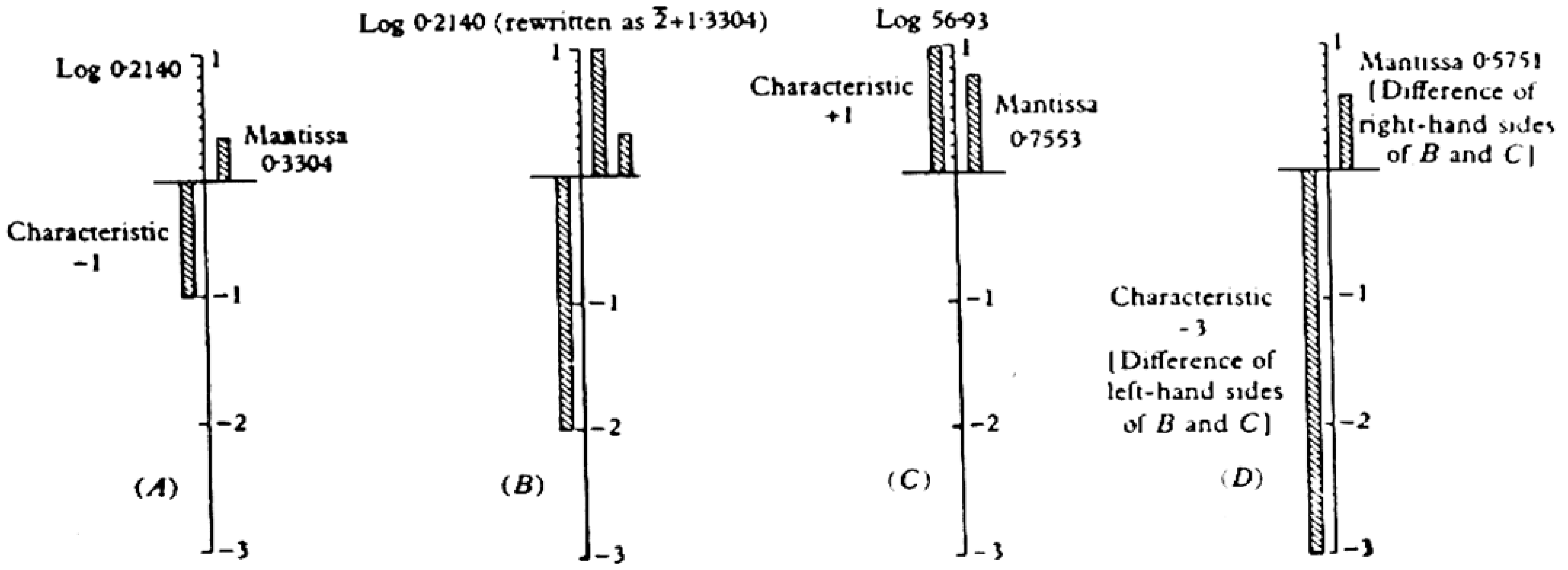

Log 0·2140
1
Mantissa 0·3304
Characteristic −1
−1
−2
−3
(A)
Log 0·2140 (rewritten as $\bar{2}$+1·3304)
1
−1
−2
−3
(B)
Log 56·93
1
Characteristic +1
Mantissa 0·7553
−1
−2
−3
(C)
1
Mantissa 0·5751 [Difference of right-hand sides of B and C]
Characteristic −3 [Difference of left-hand sides of B and C]
−1
−2
−3
(D)

Fig. 20

Again, the subtraction of a larger from a smaller mantissa necessitates our choosing 1·3304 for the positive part of the logarithm of 0·2140. The complete logarithm will be considered, therefore, as $\overline{2} + 1·3304$. Subtraction of 1·7553 from $\overline{2} + 1·3304$ gives 0·5751 for the positive part of the answer, while 1 in the divisor, subtracted from $\overline{2}$ in the log of the dividend gives $\overline{3}$. The answer is $\overline{3}$·5751. See fig. 20.

Powers of numbers including logarithms with negative characteristics

We know that the method of raising a number to a power by means of logarithms is to multiply the logarithm of the number by the power. When the multiplication of the mantissa produces an answer greater than 1, the integral part of this answer must be added algebraically to the product of the characteristic and the power.

Example. $(0·5149)^3$.

	Numbers	Logs	
	0·5149	$\overline{1}$·7118	
$(0·5149)^3 = 0·1366$.	0·1366	$\overline{1}$·1354	Multiplying by 3

Multiplying 0·7118 by 3 gives 2·1354, and multiplying $\overline{1}$ by 3 gives $\overline{3}$. The sum of $\overline{3}$ and 2 is $\overline{1}$, so the final logarithm is $\overline{1}$·1354 (see fig. 21).

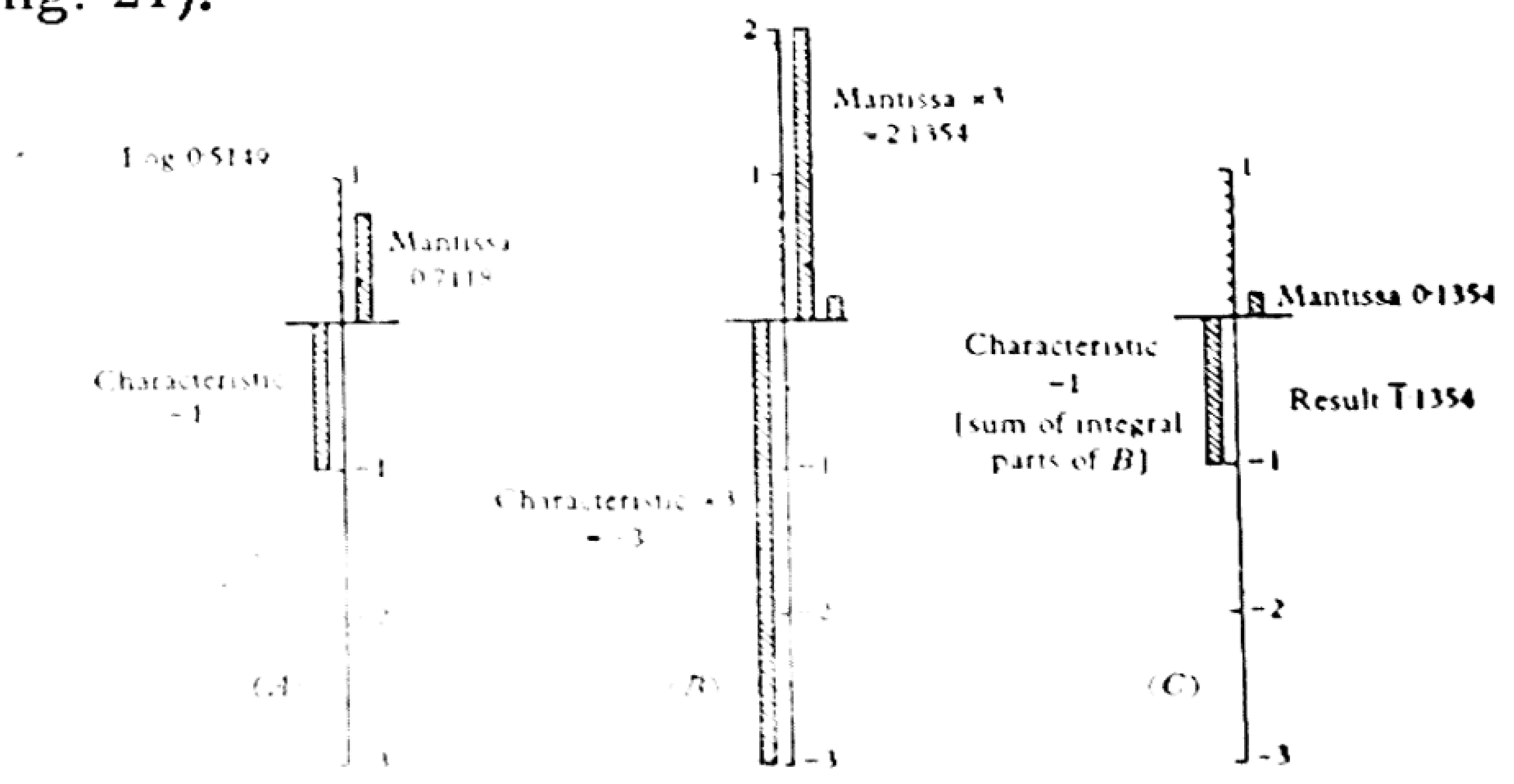


Fig. 21

Roots of numbers including logarithms with negative characteristics

Since we have decided to segregate the negative part from the positive part of a logarithm, we must make sure that the divisor (i.e. the number which expresses the root) will divide into the negative characteristic an *exact* number of times. Suppose we are finding a square root, which involves dividing the logarithm by 2, then no difficulty arises if the characteristic is $\overline{2}$, $\overline{4}$, $\overline{6}$ or any even negative number, for these are divisible by 2 without remainder, but if the characteristic is $\overline{1}$ it must be written (or thought of) as $\overline{2} + 1$, so that the value of the logarithm is unaltered. Similarly, if the characteristic is $\overline{3}$, it must be written as $\overline{4} + 1$.

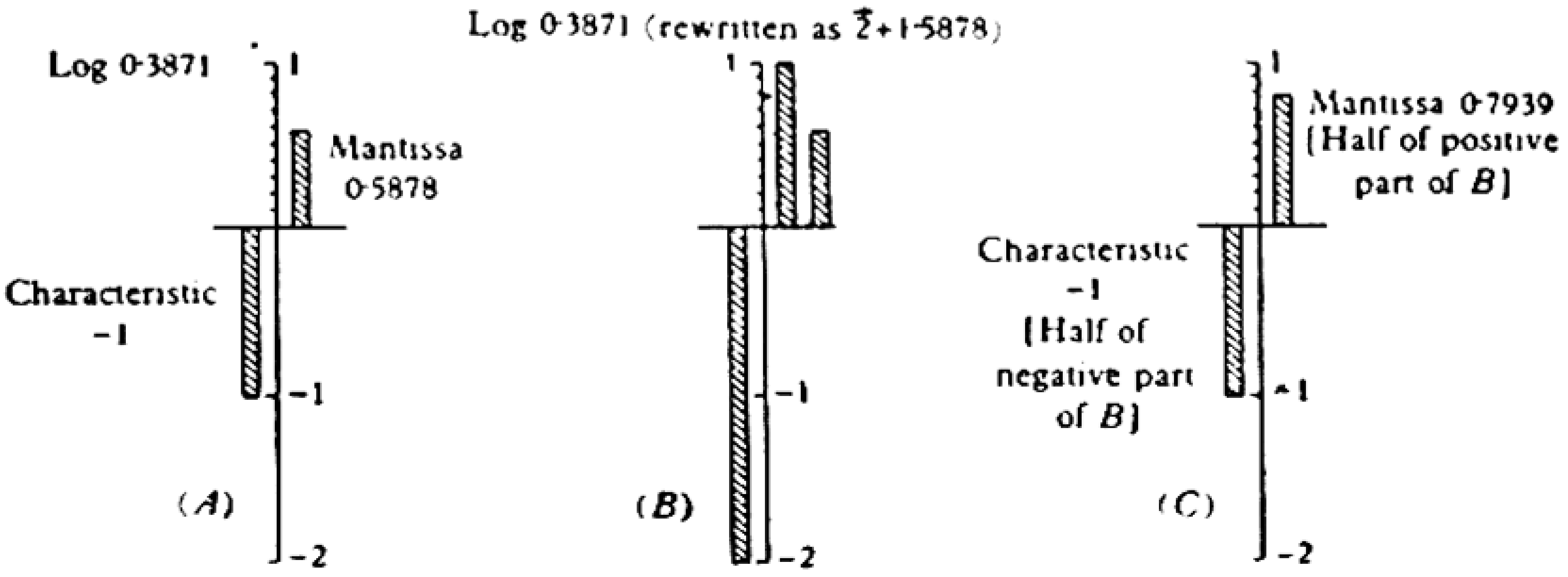


Fig. 22

When extracting a cube root, which involves dividing the logarithm by 3, a characteristic of $\overline{1}$ must be written as $\overline{3} + 2$, and a characteristic of $\overline{2}$ written as $\overline{3} + 1$. Similarly, $\overline{4}$ and $\overline{5}$ are written as $\overline{6} + 2$ and $\overline{6} + 1$ respectively in order that the negative part is exactly divisible by 3, and yet the value of the logarithm is unchanged by the initial transformation.

The rule is that a negative characteristic must be reduced (increased numerically but in the negative direction) to the nearest negative integer which is exactly divisible by the number expressing the root, and a positive correction must be added to restore the original value of the logarithm.

Example. $\sqrt{0{\cdot}3871}$.

	Numbers	Logs	
	0·3871	$\overline{1}$·5878	
		$\overline{2} + 1$·5878	(Rewritten)
$\sqrt{0{\cdot}3871} = 0{\cdot}6222$.	0·6222	$\overline{1}$ ·7939	Dividing by 2

Fig. 22 illustrates these processes.

Example. $\sqrt[3]{0\cdot6547}$.

	Numbers	Logs	
	0·6547	$\bar{1}\cdot8161$	
		$\bar{3}+2\cdot8161$	(Rewritten)
$\sqrt[3]{0\cdot6547}=0\cdot8684$.	0·8684	$\bar{1}$ ·9387	Dividing by 3

Miscellaneous worked examples.

Example. The formula $P_1 = P_2\left(\frac{V_2}{V_1}\right)^{\frac{5}{4}}$ gives the pressure of a gas when expanded adiabatically. Find P_1 if $P_2 = 14\cdot7$, $V_2 = 15\cdot2$ and $V_1 = 18\cdot9$.

$$P_1 = P_2\left(\frac{V_2}{V_1}\right)^{\frac{5}{4}} = 14\cdot7\left(\frac{15\cdot2}{18\cdot9}\right)^{\frac{5}{4}} \quad \text{(by substitution).}$$

[The processes involved are:

(1) Subtract log 18·9 from log 15·2.
(2) Multiply the result of subtraction by 5, and divide by 4.
(3) Add log 14·7.
(4) Find the antilogarithm of the final logarithm.]

	Numbers	Logs	
	15·2	1·1818	
	18·9	1·2765	
		$\bar{1}\cdot9053$	(By subtraction)
		$\bar{1}\cdot5265$	Multiplying by 5
		$\bar{4}+3\cdot5265$	Rewritten to divide by 4
		$\bar{1}$ ·8816	Dividing by 4
	14·7	1 ·1673	
$P_1 = 11\cdot19$	11·19	1 ·0489	(By addition)

Alternatively:

$$\begin{aligned}
\log P_1 &= \log P_2 + \tfrac{5}{4}(\log V_2 - \log V_1)\\
&= \log 14\cdot7 + \tfrac{5}{4}(\log 15\cdot2 - \log 18\cdot9)\\
&= 1\cdot1673 + \tfrac{5}{4}(1\cdot1818 - 1\cdot2765)\\
&= 1\cdot1673 + \tfrac{5}{4}\times\bar{1}\cdot9053\\
&= 1\cdot1673 + \tfrac{1}{4}\times\bar{1}\cdot5265\\
&= 1\cdot1673 + \tfrac{1}{4}(\bar{4} + 3\cdot5265)\\
&= 1\cdot1673 + \bar{1}\cdot8816\\
&= 1\cdot0489.
\end{aligned}$$

$$\therefore\ \underline{P_1 = 11\cdot19.}$$

Example. The period of an oscillating body is given by

$$t = 2\pi\sqrt{\left(\frac{b^2 + 3a^2}{2ag}\right)}.$$

Find t if $b = 1{\cdot}63$, $a = 0{\cdot}92$, $\pi = 3{\cdot}142$, $g = 32{\cdot}2$.

$$\begin{aligned}
t &= 2\pi\sqrt{\left(\frac{b^2 + 3a^2}{2ag}\right)} \\
&= 2\pi\sqrt{\left\{\frac{(1{\cdot}63)^2 + 3(0{\cdot}92)^2}{2 \times 0{\cdot}92 \times 32{\cdot}2}\right\}} \\
&= 6{\cdot}284\sqrt{\left(\frac{2{\cdot}657 + 2{\cdot}539}{1{\cdot}84 \times 32{\cdot}2}\right)} \\
&= 6{\cdot}284\sqrt{\left(\frac{5{\cdot}196}{1{\cdot}84 \times 32{\cdot}2}\right)} \\
&= \underline{1{\cdot}861}.
\end{aligned}$$

Numbers	Logs	
1·63	0·2122	
2·657	0·4244	Multiplying by 2
0·92	$\bar{1}$·9638	
	$\bar{1}$·9276	Multiplying by 2
3	0·4771	Adding log 3
2·539	0·4047	
5·196	0·7157	A
1·84	0·2648	
32·2	1·5079	
	1·7727	B (by addition)
	$\bar{2}$·9430	Subtracting B from A
	$\bar{1}$·4715	Dividing by 2
6·284	0·7983	
1·861	0·2698	(By addition)

Example. The formula $T_1 = T_2 e^{\mu\theta}$ occurs in the theory of belts. If $T_2 = 4{\cdot}93$, $e = 2{\cdot}718$, $\mu = \frac{1}{3}$ and $\theta = 5$, find T_1.

$$\begin{aligned} T_1 &= T_2 e^{\mu\theta} \\ &= 4{\cdot}93 \times (2{\cdot}718)^{5 \times \frac{1}{3}} \\ &= 4{\cdot}93 \times (2{\cdot}718)^{\frac{5}{3}} \\ &= \underline{26{\cdot}10} \end{aligned}$$

Numbers	Logs	
2·718	0·4343	
	2·1715	Multiplying by 5
	0·7238	Dividing by 3
4·93	0·6928	
26·10	1·4166	

Alternatively:

$$\begin{aligned} T_1 &= T_2 e^{\mu\theta} \\ &= 4{\cdot}93 \times (2{\cdot}718)^{\frac{5}{3}}. \\ \therefore \log T_1 &= \log 4{\cdot}93 + \tfrac{5}{3} \log 2{\cdot}718 \\ &= 0{\cdot}6928 + \tfrac{5}{3}(0{\cdot}4343) \\ &= 0{\cdot}6928 + \tfrac{1}{3}(2{\cdot}1715) \\ &= 0{\cdot}6928 + 0{\cdot}7238 \\ &= 1{\cdot}4166. \\ \therefore \underline{T_1} &\underline{= 26{\cdot}10.} \end{aligned}$$

The logarithms used in this book are four-figure tables, giving the answer to four significant figures. This accuracy is sufficient for the problems we shall meet, but when greater accuracy is required, such as in navigation, then seven-figure tables are used.

Examples for Chapter 10

1. Evaluate the following by means of logarithms:

(*a*) $0{\cdot}0461 \times 2{\cdot}9760 \times 0{\cdot}0058$.

(*b*) $\dfrac{1{\cdot}963 \times 5{\cdot}24}{26{\cdot}93}$.

(*c*) $\sqrt{0{\cdot}1965}$.

(*d*) $\sqrt[3]{0{\cdot}01965}$.

(*e*) $(0{\cdot}0428)^3$. **(*Chap.* 10)**

2. (*a*) Evaluate $\dfrac{0{\cdot}036 \times (0{\cdot}152)^3}{\sqrt{(27{\cdot}9 \times 1{\cdot}34)}}$.

(*b*) If $\log 2 = 0{\cdot}3010$, and $\log 3 = 0{\cdot}4771$, write $\log \frac{2}{3}$, $\log 0{\cdot}15$, $\log 0{\cdot}12$ without using tables. **(*Chap.* 10)**

3. (*a*) The deflection of a helical spring is given by

$$D = \frac{64WR^2n}{Cd^4}.$$

Evaluate D if $W = 28$, $R = 0{\cdot}75$, $n = 15$, $d = 0{\cdot}13$, $C = 2{\cdot}1 \times 10^7$.

(*b*) If $\log y = \bar{1}{\cdot}5228 + \log w + 4 \log l - \log E - \log I$, write a formula for y. (*Chap.* 10)

4. The formula $t = \dfrac{2A\sqrt{H}}{Ca\sqrt{(2g)}}$ occurs in hydraulics. Write a formula for $\log t$, and evaluate t when $A = 12$, $H = 15{\cdot}1$, $C = 0{\cdot}53$, $a = 0{\cdot}25$, $g = 32{\cdot}2$. (*Chap.* 10)

5. The time of one oscillation of a pendulum is given by

$$t = 2\pi\sqrt{\left(\frac{0{\cdot}17l^2 + 0{\cdot}075R^2}{0{\cdot}65gl}\right)}.$$

Find t when $l = 75{\cdot}5$, $R = 2{\cdot}8$, $g = 981$, $\pi = 3{\cdot}142$. (*Chap.* 10)

6. (*a*) If $\log m = \frac{1}{2}(0{\cdot}3010 + \log g) + \log L + \frac{3}{2}\log H$, write a formula for m.

b) Evaluate D from the formula

$$D = 3{\cdot}33(L - 0{\cdot}2H)\,H^{\frac{3}{2}},$$

where $L = 45$, $H = 0{\cdot}91$. (*Chap.* 10)

7. The following figures represent the pressures at equidistant points during the stroke of a piston:

2·1, 2·0, 1·8, 1·6, 1·3, 1·1, 0·9, 0·6, 0·4, 0·2.

Find the average pressure if 1 unit on the scale represents a pressure of 80 lb. to the square inch.

The work done in foot-pounds during one stroke is given by average pressure × area of piston × length of stroke. Find the work done when the area of the piston is 108, and the stroke is 2. (*Chap.* 4)

8. (*a*) Evaluate, by factors, $\sqrt{(4356)}$, $\sqrt{(4225)}$.

(*b*) Show, without division, that $\sqrt{(2450)} = 35\sqrt{2}$.

(*c*) Evaluate $\sqrt{(5^2 + 12^2)}$, $\sqrt{(25^2 - 7^2)}$.

(*d*) From the results of (*c*) write $\sqrt{0{\cdot}0169}$ and $\sqrt{0{\cdot}0576}$. (*Chap.* 4)

9. (*a*) Evaluate: (i) $\dfrac{4^{\frac{3}{2}}}{27^{\frac{2}{3}}}$, (ii) $\sqrt{(121)^4 \times 10^{-2}}$.

(*b*) $x^2y^{\frac{1}{2}}z^3 \times x^{-1}y^{-\frac{1}{2}}z^4$.

(*c*) $(4x+3y+z)(3x-2y)$. (*Chaps.* 7 *and* 8)

10. (*a*) Define a logarithm to any base, and hence show that log 1 = 0 whatever base is used.

(*b*) Write 8 as a power of 2, and hence write the value of the logarithm of 8 to the base 2. Similarly, write values for the logarithm of 8 to the base 64, and the logarithm of 27 to the base 3.

(*c*) If log 9 = 0·9542, and log 7 = 0·8451, write 63 and $1\frac{2}{7}$ as powers of 10. (*Chap.* 8)

Answers to Examples for Chapter 10

1. (*a*) 0·0007958. (*b*) 0·3819. (*c*) 0·4433. (*d*) 0·2699.
(*e*) 0·00007838.

2. (*a*) 0·00002066. (*b*) $\bar{1}\cdot 8239$, $\log 1\cdot 5 = \log \frac{3}{2} = 0\cdot 1761$, $\therefore \log 0\cdot 15 = \bar{1}\cdot 1761$, $\log 12 = 1\cdot 0791$, $\therefore \log 0\cdot 12 = \bar{1}\cdot 0791$.

3. (*a*) 2·523. (*b*) $y = \dfrac{0\cdot 3333wl^4}{EI}$.

4. $\log t = 0\cdot 3010 + \log A + \frac{1}{2}\log H - \log C - \log a - \frac{1}{2}(0\cdot 3010 + \log g)$; 87·70.

5. 0·8917.

6. (*a*) $m = \sqrt{(2g)}\, LH^{\frac{3}{2}}$. (*b*) 129·5.

7. 96 lb. per sq.in.; 20,736 ft. lb.

8. (*a*) 66, 65. (*c*) 13, 24. (*d*) 0·13, 0·24.

9. (*a*) (i) $\frac{8}{9}$, (ii) 146·41. (*b*) xz^7.
(*c*) $12x^2 + xy - 6y^2 + 3xz - 2yz$.

10. (*b*) 3, $\frac{1}{2}$, 3. (*c*) $10^{1\cdot 7993}$, $10^{0\cdot 1091}$.

Revision Examples on Chapters 1–10

1. (*a*) Evaluate $\frac{1}{2}$ of $3\frac{1}{4} - \frac{1}{3}$ of $\frac{4}{5} + \frac{2}{3}(\frac{7}{8} - \frac{2}{3})$.

(*b*) Evaluate $\sqrt{(a^2+b^2+c^2)}$, where $a = 1\frac{1}{8}$, $b = \frac{1}{4}$, $c = \frac{3}{4}$.

(*c*) A firm distributes its weekly goods equally between two wholesalers, each of whom distributes his share equally among five retailers. When each wholesaler decides to add one retailer to his list, the retailers are each reduced by 9 articles. Find the weekly output of the firm. (*Chaps.* 1 *and* 5)

2. The radius of a sphere is measured as 1·1 in., but it is known that a 2 % error may have been made in the measurement. Find the largest and smallest answers that could be obtained for the volume of the sphere, if the volume is given by $\frac{4\pi r^3}{3}$, and $\pi = 3{\cdot}142$ (r = radius). (*Chaps.* 3 *and* 9)

3. (*a*) Find $\sqrt{5}$ correct to five significant figures.

(*b*) The following expression gives an approximation for the same result. Evaluate this to five significant figures:

$$2(1 + \tfrac{1}{8} - \tfrac{1}{128} + \tfrac{1}{1024}).$$ (*Chap.* 4)

4. (*a*) If 1 lb. wt. = 453·6 g. wt., and 1 g. wt. is equal to a force of 981 dynes, express a weight of n lb. in dynes.

(*b*) The number of transverse oscillations per second of a wire is given by

$$n = \frac{1}{2l}\sqrt{\frac{T}{m}},$$

where T is expressed in dynes. Find n to the nearest integer when $l = 50$, $T = 2$ lb. wt., $m = 0{\cdot}05$. (*Chaps.* 5 *and* 10)

5. (*a*) The value of a given mass of gas is directly proportional to $(t + 273)$, where t is the temperature in degrees Centigrade. If the volume is 54 when the temperature is 15° C., find the volume when the temperature is −13° C. (° C. means 'degrees Centigrade'.)

(*b*) The volume of a mass of gas is inversely proportional to the pressure if the temperature remains constant (Boyle's Law). In a certain experiment the pressure is 14 lb. per sq.in. when the volume is 150 cu.in. Find the volume when the pressure is 20 lb. per sq.in. (*Chaps.* 3 *and* 6)

6. A train whose total weight is W tons is travelling on a level track at a constant speed of v m.p.h. The locomotive is exerting a force of P lb. wt. for every ton of train. The horse-power developed by the locomotive is equal to the force in lb. wt. multiplied by the number of feet travelled per second, divided by 550. Find an expression for the horse-power developed by the locomotive, and evaluate this in the case when $P = 50$, $W = 200$, $v = 30$. (*Chap.* 5)

7. (*a*) From the sum of $4ab - 3b^2 - 4a^2$ and $9ab + 8b^2 - 7a^2$ subtract the sum of $2a^2 - 3b^2 + 8ab$ and $5a^2 - 6b^2$.

(*b*) A shell fired at a certain elevation from the ground with velocity V is y ft. above the ground at a distance x ft. measured horizontally from the gun, where $y = \frac{x}{2} - \frac{5gx^2}{8V^2}$. Taking $V = 1000$, $g = 32$, show that the range is 25,000 ft. (*Chap.* 6)

8. (*a*) $4xyz \times \frac{1}{3}x^2y^3z^4$.

(*b*) $\dfrac{x^2z^3\sqrt{y}}{4} \div \dfrac{yz^2\sqrt{x}}{3}$.

(*c*) $(3x - 4y)(5y - 3x)$.

(*d*) The velocity in any position for a certain oscillating body is given by $v = n\sqrt{(a^2 - x^2)}$, where x is a directed number expressing the distance of the body from its equilibrium position. Find v when $n = 5$, $a = 2\frac{1}{2}$, $x = -\frac{3}{2}$. (*Chap.* 7)

9. (*a*) Express as powers of 4: 1, $\frac{1}{4}$, 8, 32, $\frac{1}{8}$.

(*b*) Given $\log 5 = 0{\cdot}6990$, write 25, 125, 50 as powers of 10.

(*c*) $\text{Log } 12 = 1{\cdot}0791$. Find log 3, without tables, if

$\log 2 = 0{\cdot}3010$. (*Chap.* 8)

10. (*a*) Evaluate $\dfrac{4{\cdot}23\sqrt{(51{\cdot}28)}}{15{\cdot}95}$.

(*b*) The number of calories of heat produced in a wire is given by $H = \dfrac{I^2Rt}{4{\cdot}2}$, where I is the current in amperes, R is the resistance of the wire in ohms, and t is the time in seconds. Find the number of calories produced when $I = 1{\cdot}1$ amp., $R = 4{\cdot}5$ ohms, and $t = 20$ min. (*Chap.* 9)

11. (*a*) The formula $H = \dfrac{2\pi na^2i}{r^3}$ gives the strength of magnetic field produced by a wire carrying a current. Write a formula for $\log H$.

(*b*) Evaluate P from the formula $P = P_0\left(\dfrac{V_0}{V}\right)^{\gamma}$, where $P_0 = 20{\cdot}2$, $V_0 = 56{\cdot}3$, $V = 62{\cdot}1$, and $\gamma = \frac{7}{5}$. (*Chap.* 10)

12. The formula $t = 4\pi\sqrt{\left\{\dfrac{(M+3m)\,l}{3g(M+2m)}\right\}}$ gives the period of a compound pendulum. Find t when $\pi = 3{\cdot}142$, $M = 2{\cdot}45$, $m = 1{\cdot}32$, $l = 1{\cdot}25$. (*Chap.* 10)

Answers to Revision Examples on Chapters 1–10

1. (*a*) $\frac{539}{360}$. (*b*) $1\frac{3}{8}$. (*c*) 540.

2. 5·919 and 5·248.

3. (*a*) 2·2361. (*b*) 2·2363.

4. (*a*) 445000*n* dynes. (*b*) $n = 42$.

5. (*a*) $48\frac{3}{4}$. (*b*) 105 cu.in.

6. $\dfrac{PWv}{375}$, 800.

7. (*a*) $5ab + 14b^2 - 18a^2$.

8. (*a*) $\dfrac{4x^3y^4z^5}{3}$. (*b*) $\dfrac{3x^{\frac{3}{2}}z}{4y^{\frac{1}{2}}}$. (*c*) $27xy - 20y^2 - 9x^2$. (*d*) $v = 10$.

9. (*a*) 4°, 4^{-1}, $4^{\frac{1}{2}}$, $4^{\frac{3}{2}}$, $4^{-\frac{3}{2}}$. (*b*) $10^{1 \cdot 3980}$, $10^{2 \cdot 0970}$, $10^{1 \cdot 6990}$.
(*c*) 0·4771.

10. (*a*) 1·899. (*b*) 1556.

11. (*a*) $\operatorname{Log} H = 0 \cdot 3010 + \log \pi + \log n + 2 \log a + \log i - 3 \log r$.
(*b*) 17·61.

12. 1·604.

CHAPTER 11

ALGEBRAIC DIVISION. IDENTITIES

Algebraic division

The rules of signs and indices for division have already been covered (Chapter 7). The present chapter aims to extend the work on division to include cases of division of one expression by another.

Example 1. $(2x^3-13x^2-48x+27)\div(x-9)$.

$$
\begin{array}{ll}
x-9)2x^3-13x^2-48x+27(2x^2+5x-3 & \qquad (1)\\
\qquad\quad \underline{2x^3-18x^2} & \qquad (2)\\
\qquad\qquad\qquad 5x^2-48x & \qquad (3)\\
\qquad\qquad\qquad \underline{5x^2-45x} & \qquad (4)\\
\qquad\qquad\qquad\qquad -3x+27 & \qquad (5)\\
\qquad\qquad\qquad\qquad \underline{-3x+27} & \\
\qquad\qquad\qquad\qquad \underline{\underline{\quad\cdot\qquad\cdot\quad}} &
\end{array}
$$

Answer: $2x^2+5x-3$.

(1) x will divide $2x^2$ times into $2x^3$.

(2) Multiply divisor by $2x^2$ and write in proper columns.

(3) Subtract and bring down next term of dividend.

(4) x will divide $5x$ times into $5x^2$.

(5) x will divide -3 times into $-3x$.

The rules which have been observed are:

(1) Write the divisor and dividend in descending powers of x, that is, the terms in higher powers of x are on the left of terms in lower powers of x.

(2) Divide the first term of the divisor into the first term of the dividend, and write the answer in the quotient. (In the above example, this answer is $2x^2$.)

(3) Multiply the whole of the divisor by the answer just obtained, and write the result of this multiplication under the appropriate terms in the dividend.

(4) Subtract from the dividend and bring down another term (or terms). (In the above example the term brought down is $-48x$.)

(5) Divide the first term of the divisor into the first term of the line just obtained by subtraction. (In the above example, that is, divide x into $5x^2$.) Write the answer in the quotient and multiply the whole of the divisor by this last term written in the quotient. Subtract as before.

Example 2. $(x^3 - 1) \div (x - 1)$.

Both expressions are already written in descending powers of x, but as there are no terms in x^2 and x in the dividend, we must leave room when setting down for such terms to appear in the course of the division.

$$
\begin{array}{r l}
x - 1) & x^3 \qquad\qquad\quad - 1(x^2 + x + 1 \\
 & \underline{x^3 - x^2} \\
 & \qquad\; x^2 \\
 & \qquad\; \underline{x^2 - x} \\
 & \qquad\qquad + x - 1 \\
 & \qquad\qquad\;\; \underline{x - 1} \\
 & \qquad\qquad\;\; \cdot \quad \cdot
\end{array}
$$

$$\text{Answer: } \frac{x^3 - 1}{x - 1} = x^2 + x + 1.$$

Example 3. $(6a^3 - a^2b + 3ab^2 - 2b^3) \div (2a - b)$.

There are no powers of x, but there are descending powers of a and b, and we may choose which we like to be written in the required order. Let us choose descending powers of a, then the dividend and divisor are correctly arranged already.

$$
\begin{array}{r l}
2a - b) & 6a^3 - a^2b + 3ab^2 - 2b^3(3a^2 + ab + 2b^2 \\
 & \underline{6a^3 - 3a^2b} \\
 & \qquad\;\; 2a^2b + 3ab^2 \\
 & \qquad\;\; \underline{2a^2b - ab^2} \\
 & \qquad\qquad\qquad 4ab^2 - 2b^3 \\
 & \qquad\qquad\qquad \underline{4ab^2 - 2b^3} \\
 & \qquad\qquad\qquad\;\; \cdot \qquad \cdot
\end{array}
$$

$$\text{Answer: } \frac{6a^3 - a^2b + 3ab^2 - 2b^3}{2a - b} = 3a^2 + ab + 2b^2.$$

The three examples just completed have given these results:

(1) $\dfrac{2x^3 - 13x^2 - 48x + 27}{x - 9} = 2x^2 + 5x - 3.$

(2) $\dfrac{x^3-1}{x-1} = x^2+x+1.$

(3) $\dfrac{6a^3-a^2b+3ab^2-2b^3}{2a-b} = 3a^2+ab+2b^2.$

Compare these with $\frac{12}{3} = 4$, from which, obviously, $4 \times 3 = 12$, i.e. 4 and 3 are *factors* of 12.

Hence $x-9$ and $2x^2+5x-3$ are *factors* of

$$2x^3-13x^2-48x+27,$$

i.e. $$(x-9)(2x^2+5x-3) = 2x^3-13x^2-48x+27.$$

Similarly, $$(x-1)(x^2+x+1) = x^3-1,$$

and $$(2a-b)(3a^2+ab+2b^2) = 6a^3-a^2b+3ab^2-2b^3.$$

Example 4. Show that $(2q-3p)$ and $(p-q)$ are factors of $23p^2q-13pq^2-12p^3+2q^3$, and find the third factor.

If $(2q-3p)$ and $(p-q)$ are factors of the given expression they will both divide into it an exact number of times. We could therefore divide $23p^2q-13pq^2-12p^3+2q^3$ first by $(2q-3p)$ and then divide the quotient by $(p-q)$, but it will be quicker to multiply $(2q-3p)$ by $(p-q)$ and divide this product into the expression

$$\begin{aligned}(2q-3p)(p-q) &= 2pq-2q^2-3p^2+3pq \\ &= -3p^2+5pq-2q^2. \quad \text{(This is to be the divisor.)}\end{aligned}$$

Writing divisor and dividend in descending powers of p, we have

$$\begin{array}{r l}
-3p^2+5pq-2q^2) & -12p^3+23p^2q-13pq^2+2q^3(4p-q \\
 & \underline{-12p^3+20p^2q-8pq^2} \\
 & \qquad\qquad 3p^2q-5pq^2+2q^3 \\
 & \qquad\qquad \underline{3p^2q-5pq^2+2q^3} \\
 & \qquad\qquad \underline{\underline{\;\cdot \qquad \cdot \qquad \cdot\;}}
\end{array}$$

Hence the third factor is $4p-q$, so the factors of

$$23p^2q-13pq^2-12p^3+2q^3 \quad \text{are} \quad (2q-3p)(p-q)(4p-q).$$

Algebraic identities

Certain basic algebraic facts which should be understood and memorised will now be proved.

An identity is a statement of equality which is true under all conditions. For instance, $x(2+x)$ is equal to $2x+x^2$ whatever value is given to x, so $x(2+x) = 2x+x^2$ is an identity. On the other hand, $2x+1 = 3$ is not an identity because it is true only when x has the value 1. It is called an equation. [Some books distinguish an identity from an equation by writing the sign '$\equiv$' between the two statements; thus $x(2+x)\equiv 2x+x^2$.]

The difference of two squares. $(a-b)(a+b) = a^2-b^2$ by multiplication. Any expression of the type a^2-b^2 (the difference of two squares) may be expressed as the product of two factors consisting of (the square root of the first minus the square root of the second) multiplied by (the square root of the first plus the square root of the second).

Consider fig. 23, which illustrates the identity geometrically.

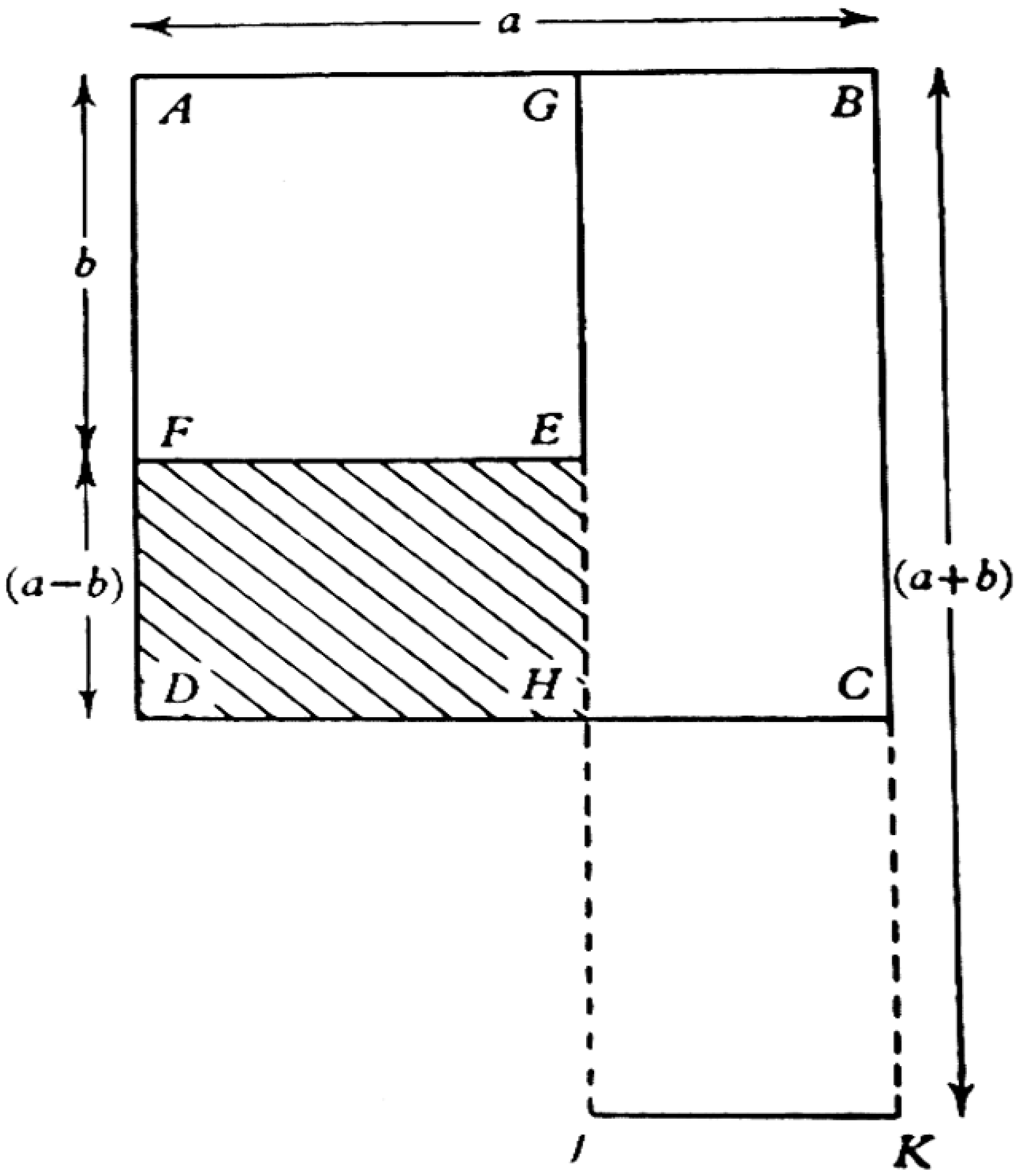


Fig. 23

$ABCD$ is a square of side a, hence area of $ABCD = a^2$.
$AGEF$ is a square of side b, hence area of $AGEF = b^2$.

By subtraction of these areas it follows that a^2-b^2 = figure $GEFDCB$ which is L-shaped. Imagine the rectangle $FEHD$ to be cut off and attached in the position $HCKJ$. Then the long rectangle $GBKJ$ represents the area left when b^2 is subtracted from a^2.

Now $GB = (a-b)$ and $BK = (a+b)$. Hence $GBKJ$ has area $(a-b)(a+b)$. Thus $\underline{a^2-b^2 = (a-b)(a+b).}$

Using this identity:

$$x^2-1 = (x-1)(x+1) \quad \text{since the square root of 1 is 1,}$$

$$x^4-y^2 = (x^2-y)(x^2+y),$$

$$16p^2-4 = (4p-2)(4p+2).$$

Certain problems in arithmetic and mensuration may be simplified by the use of the identity, as follows:

$$\begin{aligned}(1{\cdot}25)^2-(1{\cdot}05)^2 &= (1{\cdot}25-1{\cdot}05)(1{\cdot}25+1{\cdot}05)\\ &= 0{\cdot}2\times 2{\cdot}30\\ &= 0{\cdot}46,\end{aligned}$$

and

$$\begin{aligned}695^2-585^2 &= (695-585)(695+585)\\ &= 110\times 1280\\ &= 140{,}800.\end{aligned}$$

The square of the sum of two terms and the square of the difference of two terms:

$$(a+b)^2 = (a+b)(a+b) = a^2+2ab+b^2,$$

$$(a-b)^2 = (a-b)(a-b) = a^2-2ab+b^2.$$

The form of the answer obtained by squaring should be memorised. Notice that the square of a binomial expression (i.e. an expression containing two terms) contains three terms, which are:

(1) the square of the first term, (2) the product of the two terms in the bracket multiplied by 2, (3) the square of the second term in the bracket.

Hence

$$\begin{aligned}(3x-2y)^2 &= (3x)^2+2(3x)(-2y)+(-2y)^2\\ &= 9x^2-12xy+4y^2,\end{aligned}$$

$$\begin{aligned}\left(5-\frac{z}{4}\right)^2 &= (5)^2+2(5)\left(-\frac{z}{4}\right)+\left(-\frac{z}{4}\right)^2\\ &= 25-\frac{5z}{2}+\frac{z^2}{16},\end{aligned}$$

$$\begin{aligned}\left(x-\frac{1}{x}\right)^2 &= x^2+2(x)\left(-\frac{1}{x}\right)+\left(-\frac{1}{x}\right)^2\\ &= x^2-2+\frac{1}{x^2}.\end{aligned}$$

Fig. 24 shows the square of side $(a+b)$ to consist of a square of area a^2, a square of area b^2, and two rectangles of area ab.

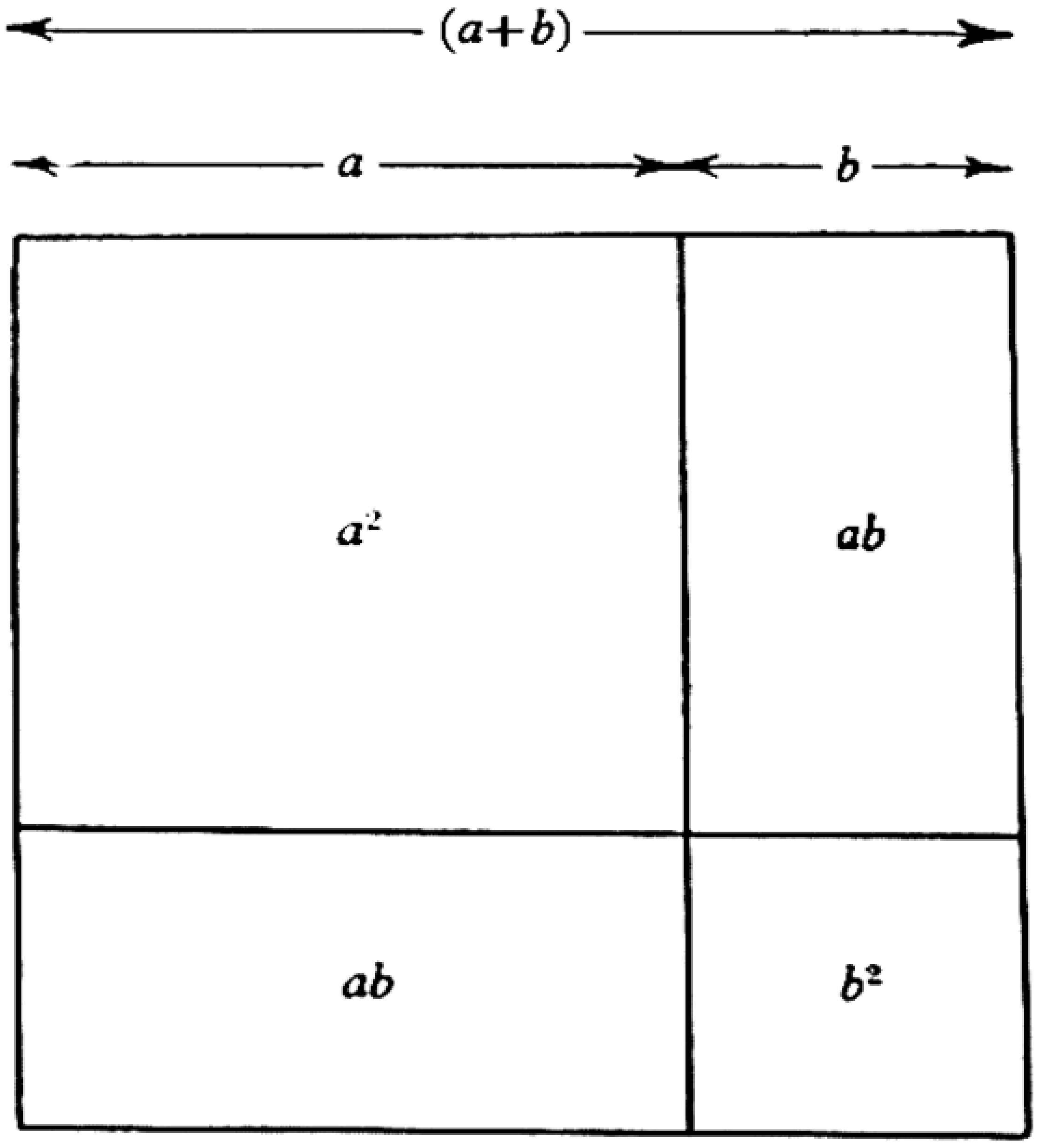


Fig. 24

Fig. 25 represents the square of the difference of two terms. The square $ABCD$ represents $(a-b)^2$. Now a^2 is the square $AEGJ$, b^2 is the square $CFGH$. Thus a^2+b^2 is the sum of two

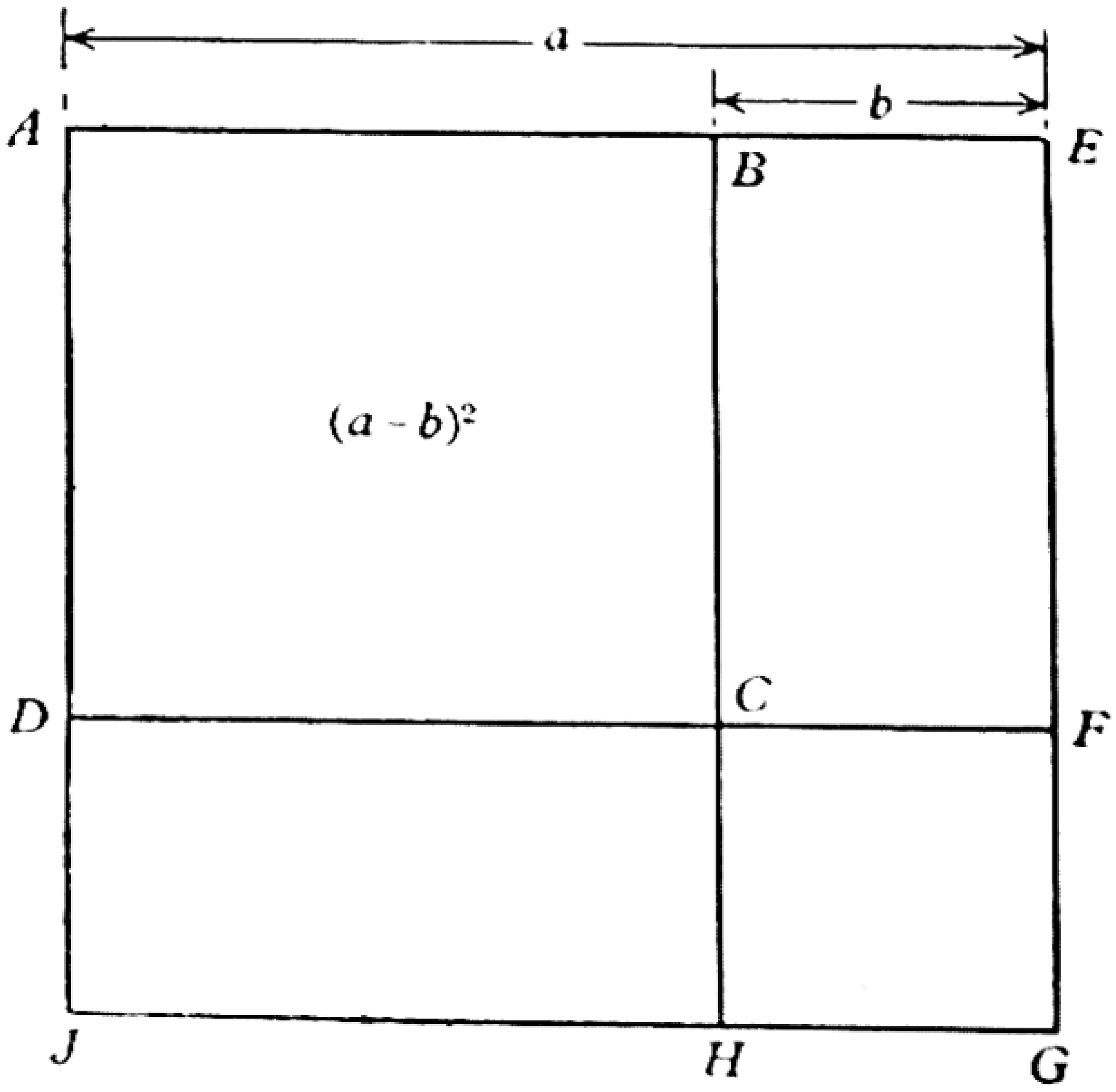


Fig. 25

squares equal in area to $AEGJ$ and $CFGH$ respectively. From this total area must be subtracted the rectangle $BEGH$, plus the rectangle $DCHJ$, plus a square equal to $CFGH$. It will be seen that this total quantity to be subtracted is equal to two rectangles each the same area as $BEGH$ (i.e. $2ab$). Hence

$$(a-b)^2 = a^2+b^2-2ab.$$

Many simple algebraic products may be represented geometrically. Take, for instance, $(a+b)(c+d)$. This is shown in fig. 26 to be a rectangle whose sides are $(a+b)$ and $(c+d)$. The area consists of the areas of four rectangles $ac+bc+ad+bd$. Algebraic multiplication gives the same result:

$$(a+b)(c+d) = ac+bc+ad+bd.$$

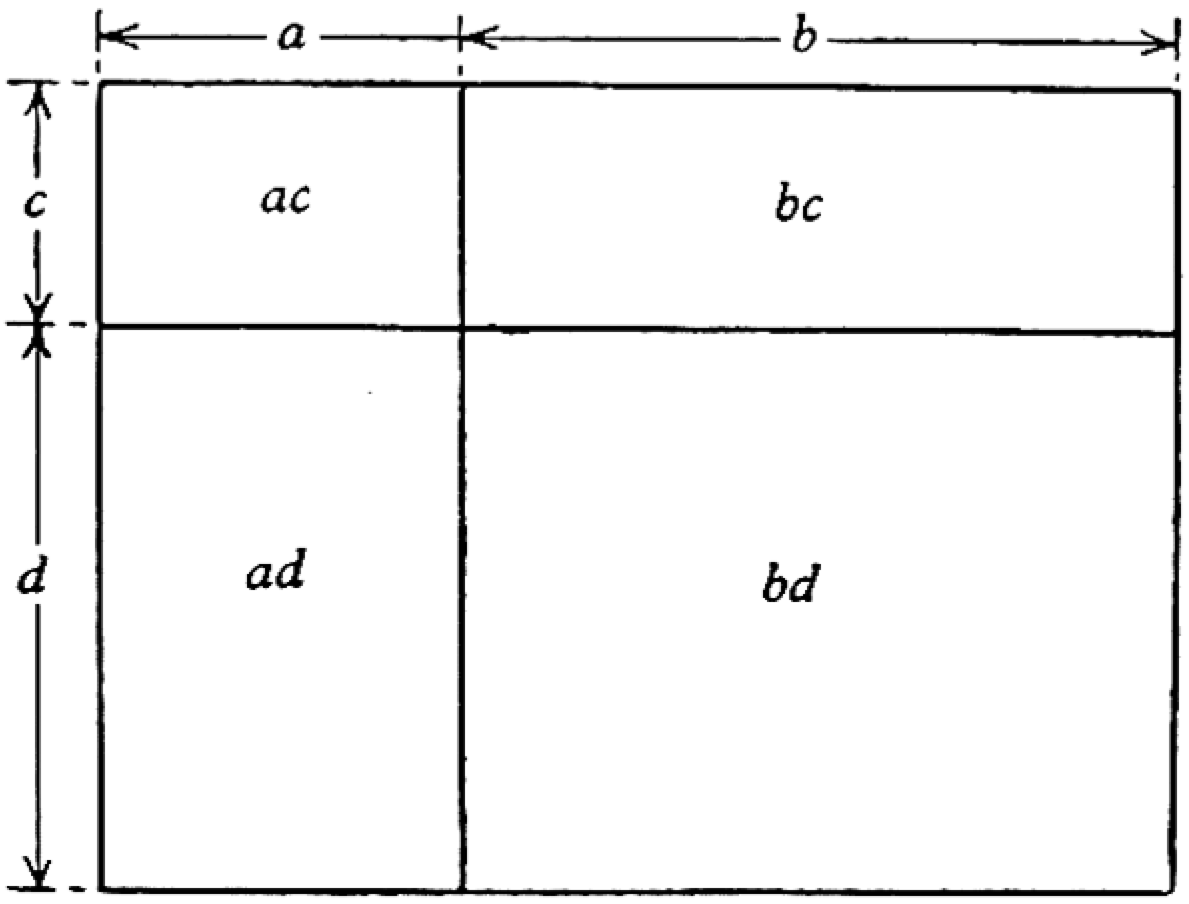


Fig. 26

The sum of two cubes. $(a+b)(a^2-ab+b^2) = a^3+b^3$ by multiplication. These factors should be memorised.

Thus $\qquad 8a^3+27b^3 = (2a+3b)(4a^2-6ab+9b^2)$

and $\qquad 64p^3+1 = (4p+1)(16p^2-4p+1)$.

The difference of two cubes. $(a-b)(a^2+ab+b^2) = a^3-b^3$ by multiplication.

Thus $\qquad 8a^3-27b^3 = (2a-3b)(4a^2+6ab+9b^2)$

and $\qquad \left(1-\frac{x^3}{8}\right) = \left(1-\frac{x}{2}\right)\left(1+\frac{x}{2}+\frac{x^2}{4}\right)$.

Note that there are no ordinary factors for the sum of two squares, i.e. a^2+b^2 cannot be factorised in the same way as the previous identities.

Examples for Chapter 11

1. (*a*) Divide $2x^3 - x^2 - 7x + 6$ by (i) $2x - 3$, (ii) $x + 2$, (iii) $x - 1$.

(*b*) Write the factors of $2x^3 - x^2 - 7x + 6$. (*Chap.* 11)

2. (*a*) Divide $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$, and divide the quotient by $a + b$. Hence write the factors of

$$a^3 + 3a^2b + 3ab^2 + b^3.$$

(*b*) Using the result of (*a*) write expressions for $(x + y)^3$ and $(x - y)^3$. (*Chap.* 11)

3. (*a*) Evaluate: $(1{\cdot}01)^2 - (0{\cdot}91)^2$, $546^2 - 537^2$.

(*b*) The area of an annulus, or ring, is given by

$$A = \pi(R^2 - r^2),$$

where R and r are the outer and inner radii. Find the area of an annulus whose radii are 8·5 and 1·5 in., taking $\pi = \frac{22}{7}$. (*Chap.* 11)

4. (*a*) If the length of the longest side of a right-angled triangle is a, and the length of another side is b, then the length of the third side is $\sqrt{(a^2 - b^2)}$. Find the length of the third side in the following cases: (i) $a = 5$, $b = 3$, (ii) $a = 13$, $b = 5$, (iii) $a = 25$, $b = 7$.

(*b*) The distance travelled by a body projected vertically upwards is given by $s = \dfrac{u^2 - v^2}{2g}$. Find s when $u = 80$, $v = 16$, $g = 32$. (*Chap.* 11)

5. (*a*) Write the factors of $a^2 + 2ab + b^2$, and of $a^2 - 2ab + b^2$. Hence factorise $(a^2 + 2ab + b^2) - c^2$, and $x^2 + 2xy + y^2 - z^2$.

(*b*) Divide $2p^3 + 3p^2q - q^3$ by $2p - q$, and hence write the factors of $2p^3 + 3p^2q - q^3$. (*Chap.* 11)

6. (*a*) Multiply $(a + b)$ by $(c - d)$ and illustrate the answer by a rectangular diagram.

(*b*) Write the factors of $27a^3 - 1$, $8b^3 + 1$, $1728 - 64x^3$.

(*c*) Write as the product of two factors $(a + c)^2 - (b - d)^2$. (*Chap.* 11)

7. (*a*) Evaluate $\dfrac{101{\cdot}3 \times \sqrt{7{\cdot}65}}{(7{\cdot}35)^2}$.

(*b*) The magnetic field along the axis of a circular coil is given by

$$H = \frac{2\pi n a^2 i}{(x^2+a^2)^{\frac{3}{2}}}.$$

Evaluate H when $\pi = 3{\cdot}142, n = 12, a = 4{\cdot}5, i = 0{\cdot}2, x = 11{\cdot}2.$
(*Chaps.* 9 *and* 10)

8. On the Centigrade scale of temperature the melting-point of ice is read as 0° C., and the boiling-point of water as 100° C. The same two temperatures are read as 32° F. and 212° F. on the Fahrenheit scale of temperature. Find the ratio between 1 degree on the Centigrade scale, and 1 degree on the Fahrenheit scale.

Evolve a formula to convert a temperature on the Centigrade scale to a temperature on the Fahrenheit scale, and use it to write 50° C. in degrees Fahrenheit. (*Chap.* 5)

9. The area of a square is 650·862144 square metres. Find the length of one side. (*Chap.* 4)

Convert this length to feet, correct to one decimal place, given that 1 inch = 2·54 cm. (*Chap.* 2)

10. The 'root-mean square value' of a set of numbers is obtained by squaring each number, finding the average value of the squares, and extracting the square root of this average. Find the root-mean square value of the following numbers:

$$\tfrac{1}{2},\quad \frac{\sqrt{3}}{2},\quad 1,\quad \frac{\sqrt{3}}{2},\quad \tfrac{1}{2},\quad 0,\quad -\tfrac{1}{2},\quad \frac{-\sqrt{3}}{2},\quad -1,\quad \frac{-\sqrt{3}}{2},\quad -\tfrac{1}{2},\quad 0.$$

What is the average value of the numbers? (*Chap.* 4)

Answers to Examples for Chapter 11

1. (*a*) (i) x^2+x-2, (ii) $2x^2-5x+3$, (iii) $2x^2+x-6$.
(*b*) $(2x-3)(x+2)(x-1)$.

2. (*a*) $(a+b)(a+b)(a+b)$.
(*b*) $x^3+3x^2y+3xy^2+y^3$; $x^3-3x^2y+3xy^2-y^3$.

3. (*a*) 0·192, 9747. (*b*) 220 sq.in.

4. (*a*) 4, 12, 24. (*b*) $s = 96$.

5. (*a*) $(a+b-c)(a+b+c)$, $(x+y-z)(x+y+z)$.
   (*b*) $(2p-q)(p+q)(p+q)$.

6. (*a*) $ac+bc-ad-bd$.
   (*b*) $(3a-1)(9a^2+3a+1)$, $(2b+1)(4b^2-2b+1)$, $(12-4x)(144+48x+16x^2)$.
   (*c*) $(a+c+b-d)(a+c-b+d)$.

7. (*a*) 5·187. (*b*) 0·1737.

8. $\frac{9}{5}$; $F = \dfrac{9c}{5} + 32$; 122° F.

9. 25·512 metres; 83·7 ft.

10. $\dfrac{1}{\sqrt{2}}$; 0.

CHAPTER 12

ALGEBRAIC FACTORS

Just as 2 and 3 are factors of 6, so $x-1$ and $2x+3$ are factors of $2x^2+x-3$, and x and $x-1$ are factors of x^2-x. The factors of an algebraic expression are the terms or expressions which will multiply together to give the original expression. We have been introduced to factors in the preceding chapter. This chapter will explain the methods of factorising certain types of expressions.

Single-term common factor

Consider the expression $4x^3+3x^2-2x$. Every term contains the letter x; therefore the expression may be written as

$$x(4x^2+3x-2).$$

Multiplication of the terms in the bracket by x will result in $4x^3+3x^2-2x$; thus the factors of $4x^3+3x^2-2x$ are x and $4x^2+3x-2$. Whatever is common to all terms in an expression is a factor of the expression and may be written outside a bracket, inside which is the quotient when the original expression is divided term by term by that common factor.

Thus

$$15x^4-5x^2 = 5x^2(3x^2-1),$$

$$a^4b^2-3a^3b^2+5ab^4 = ab^2(a^2b-3a^2+5b^2),$$

$$-21xy^2z-9x^2y^4z-12xy^3z^2 = -3xy^2z(7+3xy^2+4yz).$$

In some cases, when a common factor has been placed outside the bracket, it will be seen that the bracket itself may be factorised further as one of the types illustrated in Chapter 11, viz. the difference of two squares, the square of the sum or difference of two terms, or the sum or difference of two cubes. Consider the following examples:

$$3x^3-3xy^2 = 3x(x^2-y^2) = 3x(x-y)(x+y)$$

[Difference of two squares],

$$45a^3b-5ab = 5ab(9a^2-1) = 5ab(3a-1)(3a+1)$$

[Difference of two squares],

$$x^4z+2x^3yz+x^2y^2z = x^2z(x^2+2xy+y^2) = xy(x+y)(x+y)$$

[Square of sum of two terms],

$$4a^4b^2d^2-4a^3bcd^2+a^2c^2d^2 = a^2d^2(4a^2b^2-4abc+c^2)$$
$$= a^2d^2(2ab-c)(2ab-c)$$

[Square of difference of two terms],

$$p^4q^2r^2+pq^5r^2 = pq^2r^2(p^3+q^3) = pq^2r^2(p+q)(p^2-pq+q^2)$$

[Sum of two cubes],

$$128m^4n^2-2mn^2 = 2mn^2(64m^3-1) = 2mn^2(4m-1)(16m^2+4m+1)$$

[Difference of two cubes].

Factorisation of three-term expressions

Chapter 7 showed the method of multiplying two binomial expressions, and in some cases a three-term expression (trinomial) resulted. The problem here is to find the two binomial expressions which will multiply together to produce the given trinomial expression.

Consider

$$(x+4)(x+3) = x^2+4x+3x+12 = x^2+7x+12.$$

Starting with $x^2+7x+12$ we require to find its factors as two binomials. It is obvious that each bracket must begin with an x in order to produce the term x^2 on multiplication. Also the other two terms must be factors of 12. There are, however, several pairs of factors of 12, but we are restricted to the pair which, when added, give 7 (from $4x+3x$). Hence we know by inspection of $x^2+7x+12$ that one term in each bracket is x, while the other two terms are 4 and 3, so the factors of $x^2+7x+12$ are $(x+4)$ and $(x+3)$.

Consider next x^2-5x+6. Again each bracket contains an x, while factors of $+6$ which add to -5 must be chosen. These cannot be $+6$ and -1, or -6 and $+1$, because both of these pairs are factors of -6. Our factors must be -3 and -2. Thus

$$x^2-5x+6 = (x-3)(x-2).$$

Take next, x^2-5x-6. Each bracket contains an x. Factors of -6 which add to -5, are -6 and $+1$. Hence

$$x^2-5x-6 = (x-6)(x+1).$$

Similarly x^2+5x-6 has factors $(x+6)(x-1)$, and $a^2+5ab-6b^2$ has factors $(a+6b)(a-b)$.

All of these cases are comparatively simple because one of the terms of the second degree (i.e. a term in x^2 or a^2) has a coefficient of unity. When this is the case we know that one term is common to both brackets, while the other terms are factors of the other term of the second degree. An error, especially in a sign, is common to beginners, so it is advisable to check by multiplication all results of factorisation.

Three-term expressions. Coefficients not unity

Take

$$(2x-3)(x+5) = 2x^2-3x+10x-15 = 2x^2+7x-15.$$

Let us attempt to work the problem in reverse as we did with the previous type. Starting with the term $2x^2$, it is obvious that one bracket contains a term x, and the other a term $2x$. The other two terms are factors of -15, but they do not add to $+7$. We know, in fact, that the required factors are -3 and $+5$, but the total $+7$ is obtained by multiplying one of these factors of -15 by 2 and the other by 1. Although the factors can be obtained mentally, and will be after some practice, it is helpful to set down all the possibilities as below until the correct pairs of factors are obtained. It will help if the factors of the first term are repeated by inversion, but not those of the other term:

Factors of $2x^2$		**Factors of -15**			
$2x$	$x \leftarrow$	$-5 \rightarrow$	$+5$	-15	$+15$
x	$2x \leftarrow$	$+3 \rightarrow$	-3	$+1$	-1
(1)	(2)	(3)	(4)	(5)	(6)

Whichever of columns (1) and (2) is chosen represents the first term in each bracket, and whichever of columns (3), (4), (5) or (6) is chosen represents the second term in each bracket. Now look above to see how to obtain the middle term $+7x$. It occurs from the addition of $-3x$ and $+10x$. Now $-3x$ arises from the multiplication of the *second* term in the *first* bracket by the *first* term in the *second* bracket, while $+10x$ arises from the multiplication of the *first* term in the *first* bracket by the *second* term in the *second* bracket. Hence the *top* term in either column (1) or column (2) must multiply the *bottom* term in either (3), (4), (5) or (6), while the *bottom* term in (1) or (2) must multiply the *top* term in (3), (4), (5) or (6), so that addition of the two results will give $+7x$. Trying out all pairs, we see that the appropriate columns are (2) and (4). The top line gives one bracket, and the bottom line gives the other, the required factors being

$$(x+5)(2x-3).$$

Example. Factorise $3x^2+11x-4$.

Factors of $3x^2$ Factors of -4

$$\begin{array}{lllll} 3x & x \leftarrow & 2 & -4 & \rightarrow +4 \\ x & 3x \leftarrow & -2 & +1 & \rightarrow -1 \end{array}$$

Hence $\underline{3x^2+11x-4 = (x+4)(3x-1)}$.

Example. Factorise $4x^2+2x-12$.

Factors of $4x^2$ Factors of -12

$$\begin{array}{llllllllll} 4x & x & 2x \leftarrow & -12 & -1 & -6 & -2 & -4 & \rightarrow -3 \\ x & 4x & 2x \leftarrow & +1 & +12 & +2 & +6 & +3 & \rightarrow +4 \end{array}$$

Hence $\underline{4x^2+2x-12 = (2x-3)(2x+4)}$.

Example. Factorise $11x^2-86xy+63y^2$.

Factors of $11x^2$ Factors of $63y^2$

$$\begin{array}{llllll} 11x & x \leftarrow & 63y & 7y & -63y & \rightarrow -7y \\ x & 11x \leftarrow & y & 9y & -y & \rightarrow -9y \end{array}$$

Hence $\underline{11x^2-86xy+63y^2 = (x-7y)(11x-9y)}$.

Common factor and trinomial

Whenever a common factor has been extracted and the other factor is a trinomial, the latter should be tested for binomial factors if it does not obviously belong to one of the standard types shown in the previous chapter.

Example. Factorise $20x^3+26x^2-6x$.

$$20x^3+26x^2-6x = 2x(10x^2+13x-3).$$

Testing $10x^2+13x-3$ for factors:

Factors of $10x^2$ Factors of -3

$$\begin{array}{llllll} 10x & x & 5x \leftarrow & 2x & \rightarrow -1 & -3 \\ x & 10x & 2x \leftarrow & 5x & \rightarrow +3 & +1 \end{array}$$

Thus the factors of $10x^2+13x-3$ are $(5x-1)(2x+3)$.

Hence $\underline{20x^3+26x^2-6x = 2x(5x-1)(2x+3)}$.

Four-term expressions. Common factors

Referring again to Chapter 7, we saw that when two binomial expressions containing different letters were multiplied, the result was an expression containing four terms. We wish now to reverse the process to find the two binomial factors of a given four-term expression.

Example. Factorise $ab+2a+2+b$.

Examine the terms *in pairs* for a common factor.

Now $ab+2a$ has a common factor a, so $ab+2a = a(b+2)$ and $2+b$ has only 1 in common, so $2+b = 1(b+2)$.

We may now write $ab+2a+2+b$ as $\underbrace{a(b+2)}_{\text{A}} + \underbrace{1(b+2)}_{\text{B}}$

Consider A and B as two terms having a common factor $(b+2)$.

Thus $a(b+2)+1(b+2) = (b+2)(a+1)$.

We could have selected different pairs in the first place, as follows:

$$\begin{aligned} ab+2a+2+b &= (ab+b)+(2a+2) \\ &= b(a+1)+2(a+1) \\ &= (a+1)(b+2) \quad \text{as before.} \end{aligned}$$

The method is thus:

(1) Choose two pairs of terms such that each pair will have the *same* binomial factor.

(2) This binomial factor is one bracket of the answer, while the other bracket contains the terms which formerly stood outside the common bracket.

Example. Factorise $a^2b+acd+2ab^2+2bcd$.

$$\begin{aligned} a^2b+acd+2ab^2+2bcd &= (a^2b+acd)+(2ab^2+2bcd) \\ &= a(ab+cd)+2b(ab+cd) \\ &= (ab+cd)(a+2b). \end{aligned}$$

$$\therefore a^2b+acd+2ab^2+2bcd = (ab+cd)(a+2b).$$

[The student should try to obtain the same result by selecting $(a^2b+2ab^2)+(acd+2bcd)$.]

When some of the terms have negative signs, care should be taken to ensure that the binomial factors obtained are exactly the same.

Example. Factorise $xz-3yz-4wx+12wy$.

$$xz-3yz-4wx+12wy=(xz-3yz)-(4wx-12wy).$$

[Notice that in placing $-4wx+12wy$ inside a bracket we have left the negative sign outside the bracket and *hence* changed the sign of $12wy$.]

$$\begin{aligned}(xz-3yz)-(4wx-12wy)&=z(x-3y)-4w(x-3y)\\&=(x-3y)(z-4w).\end{aligned}$$

$$\text{Hence }\underline{xz-3yz-4wx+12wy=(x-3y)(z-4w)}.$$

Example. Factorise $lmn^2p-2np^2-3lmnq+6pq$.

$$\begin{aligned}lmn^2p-2np^2-3lmnq+6pq&=(lmn^2p-2np^2)-(3lmnq-6pq)\\&=np(lmn-2p)-3q(lmn-2p)\\&=(lmn-2p)(np-3q).\end{aligned}$$

$$\therefore\ \underline{lmn^2p-2np^2-3lmnq+6pq=(lmn-2p)(np-3q)}.$$

We have seen that there are two ways in which pairs may be chosen, but there is also a wrong choice which will not give common brackets. Suppose we had chosen

$$(lmn^2p+6pq)-(2np^2+3lmnq)=p(lmn^2+6q)-n(2p^2+3lmq),$$

then the fact that the brackets are not the same would demand a different choice.

It is possible that four-term expressions may yield more than two brackets finally by further factorisation of one or both brackets obtained by the four-term method.

Example. Factorise $a^2c-a^2d-b^2c+b^2d$.

$$\begin{aligned}a^2c-a^2d-b^2c+b^2d&=(a^2c-a^2d)-(b^2c-b^2d)\\&=a^2(c-d)-b^2(c-d)\\&=(c-d)(a^2-b^2)\\&=\underline{(c-d)(a-b)(a+b)}\quad\text{[Difference of two squares].}\end{aligned}$$

Example. Factorise $w^2x^2y^2+z^2-w^2y^2-x^2z^2$.

$$\begin{aligned}w^2x^2y^2+z^2-w^2y^2-x^2z^2&=(w^2x^2y^2-w^2y^2)+(z^2-x^2z^2)\\&=w^2y^2(x^2-1)+z^2(1-x^2)\\&=w^2y^2(x^2-1)-z^2(x^2-1)\\&=(x^2-1)(w^2y^2-z^2)\\&=\underline{(x-1)(x+1)(wy-z)(wy+z)}.\end{aligned}$$

Examples for Chapter 12

1. Factorise the following expressions:

(i) $3x^2y - xy^2 + 2xy$. (ii) $9a^3b - 12a^2b^2 + 6a^2bc$.
(iii) $-lm^2n - 2l^2mn - 3lmn^2$.
(iv) $2a^2bcd - 2ab^2cd - 2abc^2d + 6abcd^2$.
(v) $6x^2y + 12xy - 3x$. (vi) $3a^3 + 6a^2b + 3ab^2$.
(vii) $20x^2y^2 + 20xy^2 + 5y^2$. (viii) $4x^3y + 4y^4$.
(ix) $16p^2q^2l^2 - 8p^2q^2lm + p^2q^2m^2$. (x) $16ax^3 - 16ay^3z^3$.
(xi) $a^3b^3cd - abc^3d^3$. (xii) $16x^2yz - 81yz$. (*Chap.* 12)

2. Factorise:

(i) $x^2 + 5x + 6$. (ii) $x^2 - 2x - 8$.
(iii) $a^2 - ab - 6b^2$. (iv) $x^2y^2 + xy - 20$.
(v) $l^2 - 14m^2 + 5lm$. (vi) $20 - x - x^2$.
(vii) $6 - 5ab + a^2b^2$. (viii) $a^2b^2c^2 + 6abcd - 16d^2$.
(ix) $1 - 13c + 36c^2$. (x) $w^2x^2 - 132y^2z^2 + wxyz$.
(*Chap.* 12)

3. Factorise·

(i) $xz - yz + wx - yw$. (ii) $2ax + 8bx + 3ay + 12by$.
(iii) $a^2bc - ab^2d - ac^2d + bcd^2$. (iv) $x^2y^2 + y^2 + 2x^2 + 2$.
(v) $x - 5xy + 2z - 10yz$. (vi) $p^2 + pq^2 - pq^3 - q^5$.
(vii) $a^2bc + 3ac + 2ab + 6$. (viii) $11x^2 - 9xy + 11x - 9y$.
(ix) $15 - \frac{3}{x} + \frac{5}{x^2} - \frac{1}{x^3}$.
(x) $a^2b^2e^2f^2 + c^2e^2f^2 - a^2b^2d^2 - c^2d^2$. (*Chap.* 12)

4. Factorise:

(i) $2p^2 - 17p + 21$. (ii) $2x^2 + 7xy - 22y^2$.
(iii) $4x^2 + 13xy + 3y^2$. (iv) $13p^2q^2 + 9pq - 4$.
(v) $11ab - 3b^2 - 6a^2$. (vi) $72x^2 - 2 + 7x$.
(vii) $5p^2 + 2pq - 3q^2$. (viii) $121a^2 - 88ab + 7b^2$.
(ix) $7x^2y^2z^2 - 6wxyz - w^2$. (x) $6r^2s^2 - 35qrs + 11q^2$.
(*Chap.* 12)

5. Resolve into simplest factors:

(i) $ap^2 + 2apq + aq^2$.
(ii) $a^4 - b^4$.
(iii) $6a^2b - 15ab^2 - 9b^3$.
(iv) $5a^4 + 5ab^3$.
(v) $ax^2 - 4a + bx^2 - 4b$.
(vi) $4a^2 - 9a^2x^2 - 4b^2 + 9b^2x^2$.
(*Chap.* 12)

6. Resolve into simplest factors:

(i) $24a^3b^2 + 32a^2bc + 6a^2bd + 8acd$.
(ii) $4x^2y^4z^2 - 32x^5yz^2$.
(iii) $24a^3b^3 + 14a^2b^2c - 24abc^2$.
(iv) $a^2x^3y^3 + a^2z^3 - b^2x^3y^3 - b^2z^3$.
(v) $6p^2qr - 4pq^2r - 9p^2qs + 6pq^2s$.
(vi) $x^6 - z^6$. (*Chap.* 12)

7. Squares of side x in. are cut from each of the corners of a rectangular sheet of metal a in. by b in. The remaining piece is then folded to form an open rectangular box. Write an expression for the volume of the box so formed, and calculate the volume when $a = 8$, $b = 6$, $x = 1\frac{1}{2}$. (*Chap.* 11)

8. (*a*) Divide $9cx^3 + (6c - 9d)x^2 + (c - 6d)x - d$ by $cx - d$, and hence find the factors of $9cx^3 + (6c - 9d)x^2 + (c - 6d)x - d$.

(*b*) Show that $(x - 1)$ and $(x - 2)$ are factors of both the following expressions, and find the third factor in each case:

(i) $x^3 - 7x + 6$, (ii) $x^3 - 6x^2 + 11x - 6$. (*Chap.* 11)

9. A formula for the tensions in a belt may be expressed as follows:

$$\log T_1 = \log T_2 + 0{\cdot}4343\mu\theta.$$

Show that $T_1 = T_2e^{\mu\theta}$ (where $e = 2{\cdot}718$).

Evaluate T_1 when $T_2 = 7{\cdot}5$, $\mu = \frac{3}{4}$ and $\theta = 2$. (*Chap.* 9)

10. The electric current (I amperes) flowing through a coil is directly proportional to the potential difference (V volts) between the ends of the coil. When $V = 6{\cdot}6$ volts and $I = 2{\cdot}5$ amp., what is the highest potential difference that should be applied to the coil if the highest safe current is $6{\cdot}3$ amp.?

The ratio V/I is called the resistance of the coil in ohms. Calculate it. (*Chap.* 3)

Answers to Examples for Chapter 12

1. (i) $xy(3x-y+2)$. (ii) $3a^2b(3a-4b+2c)$.
(iii) $-lmn(m+2l+3n)$. (iv) $2abcd(a-b-c+3d)$.
(v) $3x(2xy+4y-1)$. (vi) $3a(a+b)(a+b)$.
(vii) $5y^2(2x+1)(2x+1)$. (viii) $4y(x+y)(x^2-xy+y^2)$.
(ix) $p^2q^2(4l-m)(4l-m)$.
(x) $16a(x-yz)(x^2+xyz+y^2z^2)$.
(xi) $abcd(ab-cd)(ab+cd)$. (xii) $yz(4x-9)(4x+9)$.

2. (i) $(x+3)(x+2)$. (ii) $(x-4)(x+2)$.
(iii) $(a-3b)(a+2b)$. (iv) $(xy+5)(xy-4)$.
(v) $(l+7m)(l-2m)$. (vi) $(4-x)(5+x)$.
(vii) $(3-ab)(2-ab)$. (viii) $(abc-2d)(abc+8d)$.
(ix) $(1-4c)(1-9c)$. (x) $(wx+12yz)(wx-11yz)$.

3. (i) $(x-y)(z+w)$. (ii) $(2x+3y)(a+4b)$.
(iii) $(ac-bd)(ab-cd)$. (iv) $(x^2+1)(y^2+2)$.
(v) $(1-5y)(x+2z)$. (vi) $(p+q^2)(p-q^3)$.
(vii) $(ab+3)(ac+2)$. (viii) $(11x-9y)(x+1)$.
(ix) $\left(5-\frac{1}{x}\right)\left(3+\frac{1}{x^2}\right)$. (x) $(a^2b^2+c^2)(e^2f^2-d^2)$.

4. (i) $(2p-3)(p-7)$. (ii) $(2x+11y)(x-2y)$.
(iii) $(4x+y)(x+3y)$. (iv) $(13pq-4)(pq+1)$.
(v) $(2a-3b)(b-3a)$. (vi) $(8x-1)(9x+2)$.
(vii) $(5p-3q)(p+q)$. (viii) $(11a-7b)(11a-b)$.
(ix) $(7xyz+w)(xyz-w)$. (x) $(3rs-q)(2rs-11q)$.

5. (i) $a(p+q)(p+q)$. (ii) $(a^2+b^2)(a-b)(a+b)$.
(iii) $3b(a-3b)(2a+b)$. (iv) $5a(a+b)(a^2-ab+b^2)$.
(v) $(a+b)(x-2)(x+2)$.
(vi) $(a-b)(a+b)(2-3x)(2+3x)$.

6. (i) $2a(3ab+4c)(4ab+d)$.
   (ii) $4x^2yz^2(y-2x)(y^2+2xy+4x^2)$.
   (iii) $2ab(3ab+4c)(4ab-3c)$.
   (iv) $(xy+z)(x^2y^2-xyz+z^2)(a-b)(a+b)$.
   (v) $pq(3p-2q)(2r-3s)$.
   (vi) $(x-z)(x+z)(x^2-xz+z^2)(x^2+xz+z^2)$.

7. Volume $= abx-2bx^2-2ax^2+4x^3$; $22\frac{1}{2}$ cu.in.

8. (*a*) $(cx-d)(3x+1)(3x+1)$. (*b*) (i) $(x+3)$, (ii) $(x-3)$.

9. 33·62.

10. 16·63 volts; 2·64 ohms.

CHAPTER 13

FURTHER FACTORS. FRACTIONAL EXPRESSIONS

Factorisation of certain polynomials

The name 'binomial' has been given to expressions containing two terms, and 'trinomial' to expressions of three terms. When the number of terms exceeds three, the expression is called a polynomial, and is referred to as a polynomial of four terms, of five terms, and so on. It is difficult to know whether a polynomial will resolve into simple factors unless it is of a type studied in the previous chapter, but there is a method which may be applied when all the factors except two are binomials of the simplest type.

Let us work in reverse, as previously, to discover the rule.

Consider $(x-1)(x-2)(x-3)$. By multiplication of the first two brackets we have $(x^2-3x+2)(x-3)$, and by further multiplication we have $x^3-6x^2+11x-6$ for our polynomial.

Now write $+1$ for x and the polynomial becomes

$$1-6+11-6=0 \quad \text{(i.e. it vanishes).}$$

Now write $+2$ for x, and the polynomial becomes

$$8-24+22-6=0 \quad \text{(Again it vanishes).}$$

Similarly, writing $+3$ for x, we have

$$27-54+33-6=0.$$

The factors, we know, are $(x-1)(x-2)(x-3)$, and when we write 1, 2 or 3 for x in the polynomial, it becomes zero.

Next, consider the polynomial $x^3-x^2-10x-8$, produced by multiplying the factors $(x+1)(x-4)(x+2)$.

Write -1 for x, and the polynomial becomes

$$-1-1+10-8=0.$$

Write $+4$ for x, and we have $64-16-40-8=0$.
Write -2 for x, and we have $-8-4+20-8=0$.

Generalising from these two examples, then, we see that if $(x-h)$ is a factor of a polynomial (where h is any number), the polynomial vanishes when h is written for x. Note that if the polynomial vanishes when $-h$ is written for every x, then $x+h$ is a factor.

When one factor has been discovered by this method, divide this factor into the original expression, and factorise the quotient. We shall restrict ourselves in this book to cases of not more than three binomial factors, so the quotient, on division by the first factor, will contain an expression which, if resolvable into further factors, can be factorised by one of the methods already covered. It is advisable to try for factors in the order $(x-1)$, $(x+1)$, $(x-2)$, $(x+2)$ and so on, until one is found.

Example. Factorise $3x^3+28x^2+7x-18$.

Trying $(x-1)$ as a factor, i.e. writing $+1$ for x in the expression, we have

$3+28+7-18 = 20$ [As this is not zero, so $(x-1)$ is not a factor of the expression].

Trying $(x+1)$ as a factor, i.e. writing -1 for x in the expression, we have

$$3(-1)^3+28(-1)^2+7(-1)-18$$
$$= -3+28-7-18 = 0.$$

Therefore $(x+1)$ is a factor.

Now divide $3x^3+28x^2+7x-18$ by $(x+1)$.

$$\begin{array}{r}
x+1)\,3x^3+28x^2+7x-18\,(3x^2+25x-18 \\
\underline{3x^3+3x^2} \\
25x^2+7x \\
\underline{25x^2+25x} \\
-18x-18 \\
\underline{-18x-18} \\
\underline{\underline{\cdot\quad\cdot}}
\end{array}$$

The quotient, $3x^2+25x-18$, will factorise into $(3x-2)(x+9)$ by the method of Chapter 12; hence the complete factors of

$$3x^3+28x^2+7x-18$$

are $$\underline{(x+1)(3x-2)(x+9).}$$

Example. Factorise $12a^3 - 16a^2b - ab^2 + 5b^3$.

Since the polynomial contains two letters, try the simplest factor which contains both letters, i.e. $(a-b)$.

Trying $(a-b)$ as a factor, i.e. writing $+b$ for a, we have

$$12b^3 - 16b^3 - b^3 + 5b^3 = 0.$$

Hence $(a-b)$ is a factor.

$$\begin{array}{l}
a-b)12a^3 - 16a^2b - ab^2 + 5b^3(12a^2 - 4ab - 5b^2 \\
\qquad\;\; \underline{12a^3 - 12a^2b} \\
\qquad\qquad\qquad - 4a^2b - ab^2 \\
\qquad\qquad\qquad \underline{- 4a^2b + 4ab^2} \\
\qquad\qquad\qquad\qquad - 5ab^2 + 5b^3 \\
\qquad\qquad\qquad\qquad \underline{- 5ab^2 + 5b^3} \\
\qquad\qquad\qquad\qquad\quad \cdot \qquad \cdot
\end{array}$$

Now $12a^2 - 4ab - 5b^2 = (2a+b)(6a-5b)$.

Therefore

$$\underline{12a^3 - 16a^2b - ab^2 + 5b^3 = (a-b)(2a+b)(6a-5b).}$$

Simplification of algebraic fractions

Addition and subtraction of fractional expressions follow the same general rule as the corresponding operations in arithmetic. In all cases the L.C.M. of the several denominators is used for the common denominator.

Case 1. *Denominators numerical*

Example. Simplify $\dfrac{x-3}{3} + \dfrac{2x-3}{4} - \dfrac{4x+5}{6}$.

The common denominator is 12, so the expressions may be written

$$\frac{4(x-3) + 3(2x-3) - 2(4x+5)}{12}$$

$$= \frac{4x - 12 + 6x - 9 - 8x - 10}{12} \quad \text{(Removing brackets)}$$

$$= \frac{2x-31}{12} \quad \text{(By addition).}$$

Case 2. *Some denominators binomials*

Example. Simplify $\dfrac{4x}{x-2} + \dfrac{2(x-1)}{5x} - \dfrac{x^2}{5(x-2)}$.

Notice that the L.C.M. of the denominators is $5x(x-2)$, this being the least expression into which all the separate denominators will divide exactly.

$$\frac{4x}{x-2}+\frac{2(x-1)}{5x}-\frac{x^2}{5(x-2)}=\frac{5x\times 4x+2(x-2)(x-1)-x^2(x)}{5x(x-2)}$$
$$=\frac{20x^2+2x^2-6x+4-x^3}{5x(x-2)}$$
$$=\frac{4-6x+22x^2-x^3}{5x(x-2)}.$$

Case 3. *Denominators which will factorise*

Example. Simplify $\dfrac{3}{10x+15}-\dfrac{2(3-x)}{2x^2+x-3}+\dfrac{x}{4x-4}$.

Factorise the denominators first:

$$\frac{3}{10x+15}-\frac{2(3-x)}{2x^2+x-3}+\frac{x}{4x-4}$$
$$=\frac{3}{5(2x+3)}-\frac{2(3-x)}{(2x+3)(x-1)}+\frac{x}{4(x-1)}$$
$$=\frac{12(x-1)-40(3-x)+5x(2x+3)}{20(2x+3)(x-1)}$$
$$=\frac{12x-12-120+40x+10x^2+15x}{20(2x+3)(x-1)}$$
$$=\frac{10x^2+67x-132}{20(2x+3)(x-1)}.$$

Example. Simplify $\dfrac{1}{x}+\dfrac{x}{x-1}+\dfrac{1}{x-2}+\dfrac{2(x^2-2x-1)}{x(x-1)(x-2)}$.

$$\frac{1}{x}+\frac{x}{x-1}+\frac{1}{x-2}+\frac{2(x^2-2x-1)}{x(x-1)(x-2)}$$
$$=\frac{(x-1)(x-2)+x^2(x-2)+(x-1)x+2x^2-4x-2}{x(x-1)(x-2)}$$
$$=\frac{x^2-3x+2+x^3-2x^2+x^2-x+2x^2-4x-2}{x(x-1)(x-2)}$$
$$=\frac{x^3+2x^2-8x}{x(x-1)(x-2)}$$
$$=\frac{x(x+4)(x-2)}{x(x-1)(x-2)}\quad\text{(Since the numerator will factorise)}$$
$$=\frac{x+4}{x-1}.$$

Cases may arise, as above, where cancelling is possible in the final stage, but cancelling may only be performed when numerator and denominator are expressed as factors, it may *not* be done with added or subtracted terms.

Example. Simplify

$$\frac{\sqrt{(x^2+3)}}{x-3}+\frac{x}{\sqrt{(x^2+3)}}-\frac{9+x}{(x-3)\sqrt{(x^2+3)}}.$$

$$\begin{aligned}\frac{\sqrt{(x^2+3)}}{x-3}+\frac{x}{\sqrt{(x^2+3)}}-\frac{9+x}{(x-3)\sqrt{(x^2+3)}} &= \frac{\sqrt{(x^2+3)}\sqrt{(x^2+3)}+x(x-3)-(9+x)}{(x-3)\sqrt{(x^2+3)}}\\ &= \frac{x^2+3+x^2-3x-9-x}{(x-3)\sqrt{(x^2+3)}}\\ &= \frac{2x^2-4x-6}{(x-3)\sqrt{(x^2+3)}}\\ &= \frac{2(x-3)(x+1)}{(x-3)\sqrt{(x^2+3)}}=\frac{2(x+1)}{\sqrt{(x^2+3)}}.\end{aligned}$$

Examples for Chapter 13

1. (*a*) Factorise completely $3x^3-x^2-10x+8$.

(*b*) Factorise completely $2x^4+12x^3+24x^2+16x$. (*Chap.* 13)

2. (*a*) Show that $(a-3b)$ is a factor of

$$10a^3-33a^2b+8ab^2+3b^3,$$

and find the other factors.

(*b*) Given that $l^2-8lm+16m^2$ and $l^3-5l^2m-2lm^2+24m^3$ have a factor in common, factorise the latter. (*Chap.* 13)

3. (*a*) Show that $(b+1)$ and $(a-b)$ are both factors of

$$b^2(4d-c)+b(ac-c-4ad+4d)+ac-4ad.$$

(*b*) If $2x^3-5x^2-4x+3$ is an expression representing the volume of a rectangular block, write expressions for the areas of the six faces. (*Chap.* 13)

4. (*a*) Simplify $\frac{x}{2}-\frac{4(x-2)}{5}$.

(*b*) Simplify $\frac{5}{x-4}+\frac{2x}{3(x-2)}+\frac{2}{x(x-4)}$. (*Chap.* 13)

5. (*a*) Simplify $\frac{x-2}{3(x-1)}-\frac{2(x-1)}{3(x-2)}$.

(*b*) Factorise x^3-2x^2-x+2 and hence simplify

$$\frac{1}{x^2-3x+2}+\frac{2}{x^2-1}+\frac{6}{x^3-2x^2-x+2}.$$ (*Chap.* 13)

6. (*a*) Simplify $\frac{x}{3x-2}-\frac{x+2}{3x}+\frac{1}{4}$.

(*b*) Simplify $\frac{3}{2x-1}-\frac{1}{x+2}+\frac{x-8}{2x^2+3x-2}$.

(*c*) Simplify $\frac{\sqrt{(2x-5)}}{3}+\frac{3x+2}{\sqrt{(2x-5)}}$. (*Chap.* 13)

7. Factorise:

(i) $x^2+2x-15$. (ii) $4x^2-38x+18$.

(iii) $3a^2-3ab-2ac+2bc$. (iv) $a^2c-b^2c-a^2d+b^2d$.

(v) $a^2+2ab+b^2-d^2$. (vi) $a^2x^3y^2-a^2y^5$. (*Chap.* 12)

8. (*a*) Write log 0·5, log 0·25, log 8 without tables, if

$$\log 20 = 1{\cdot}3010.$$

(*b*) Evaluate $0{\cdot}8686\left(\frac{1}{3}+\frac{1}{3\times 3^3}+\frac{1}{5\times 3^5}\right)$, and show that this is an approximate value for log 2. (*Chaps.* 8 *and* 10)

9. The edges of a cube of edge a are increased by 2 %. Show that the new volume is given by

$$a^3(1+\tfrac{3}{50}+\tfrac{3}{2500}+\tfrac{1}{125000}).$$

Calculate the percentage increase in volume to four significant figures. (*Chaps.* 7 *and* 3)

10. (*a*) Write the first seven terms of the series

$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\dots$$

and find the sum of the terms.

(*b*) Check the answer from the formula

$$S_n = \frac{1-r^n}{2(1-r)},$$

where S_n denotes the sum of the first n terms (in this case $n = 7$), and r is the ratio of any term in the series to the preceding term.

Use the formula also to find the sum of the first nine terms.

(*Chaps.* 1 *and* 5)

Answers to Examples for Chapter 13

1. (*a*) $(x-1)(x+2)(3x-4)$. (*b*) $2x(x+2)^3$.

2. (*a*) $(a-3b)(5a+b)(2a-b)$. (*b*) $(l-4m)(l+2m)(l-3m)$.

3. (*b*) $x^2-2x-3,\ 2x^2-7x+3,\ 2x^2+x-1$.

4. (*a*) $\dfrac{16-3x}{10}$. (*b*) $\dfrac{2x^3+7x^2-24x-12}{3x(x-2)(x-4)}$.

5. (*a*) $\dfrac{2-x^2}{3(x-1)(x-2)}$. (*b*) $\dfrac{3}{(x-1)(x-2)}$.

6. (*a*) $\dfrac{9x^2-22x+16}{12x(3x-2)}$. (*b*) $\dfrac{1}{x+2}$. (*c*) $\dfrac{11x+1}{3\sqrt{(2x+5)}}$.

7. (i) $(x+5)(x-3)$. (ii) $(4x-2)(x-9)$.

(iii) $(a-b)(3a-2c)$. (iv) $(a-b)(a+b)(c-d)$.

(v) $(a+b-d)(a+b+d)$.

(vi) $a^2y^2(x-y)(x^2+xy+y^2)$.

8. (*a*) $\bar{1}\cdot6990,\ \bar{1}\cdot3980,\ 0\cdot9030$.

9. 6·122 %.

10. (*a*) $\tfrac{127}{128}$. (*b*) $\tfrac{511}{512}$.

CHAPTER 14

SIMPLE EQUATIONS

The preceding chapters on algebra have been somewhat abstract, but the fundamental operations have to be learned before algebra can be applied to the solution of concrete problems. To an engineering student, perhaps the most important use of algebra is the solution of equations. This consists in finding the numerical value (or values) of a letter (or letters) through the use of algebraic operations.

An equation is a statement of equality between two terms or two expressions, and may be thought of as a balance which must not be disturbed during any operations performed.

Fig. 27 shows a pair of scales with two weights balancing eight equal discs, that is,

weight of two weights = weight of eight discs.

Fig. 28 shows three similar weights balancing twelve equal discs. This is to be expected from the previous diagram. It is obvious that one weight is equal to four discs, so we have added equal weights to both sides and have not disturbed the balance thereby. We could consider instead that we had multiplied both the number of weights and the number of discs by $\frac{3}{2}$ without disturbing the balance.

Hence, we may *add* equal quantities to both sides of an equation without disturbing the balance, and we may also *multiply* both sides of an equation by the same quantity without disturbance.

Thus, if

$$x = 4,$$

$$\therefore\quad x+7 = 4+7 \quad \text{(Adding 7 to both sides)}$$

and

$$5x = 5\times 4 \quad \text{(Multiplying both sides by 5).}$$

Fig. 29 shows one weight balancing four discs. Thus, we have either subtracted equal quantities from both sides, or divided both sides by the same number. We see from these illustrations that additions or subtractions may be made to or from both sides of an equation, or multiplications and divisions performed equally on both of them. Whatever is done to one side must be done to the other.

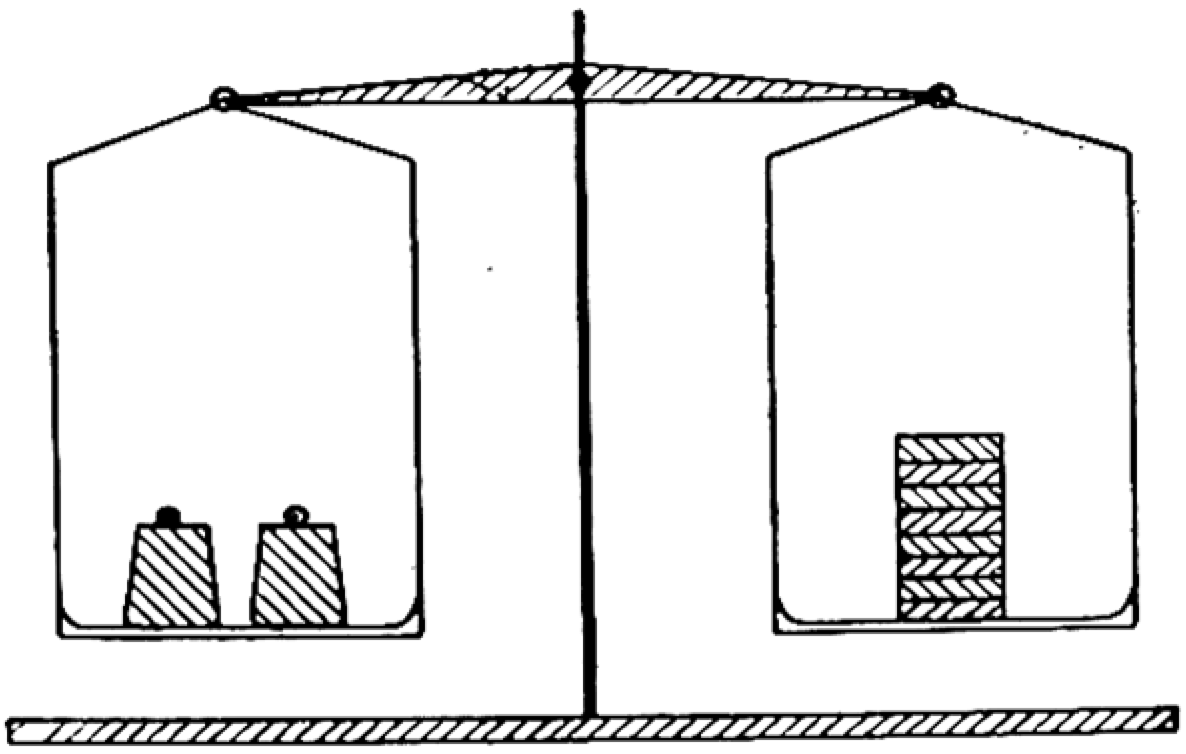


Fig. 27

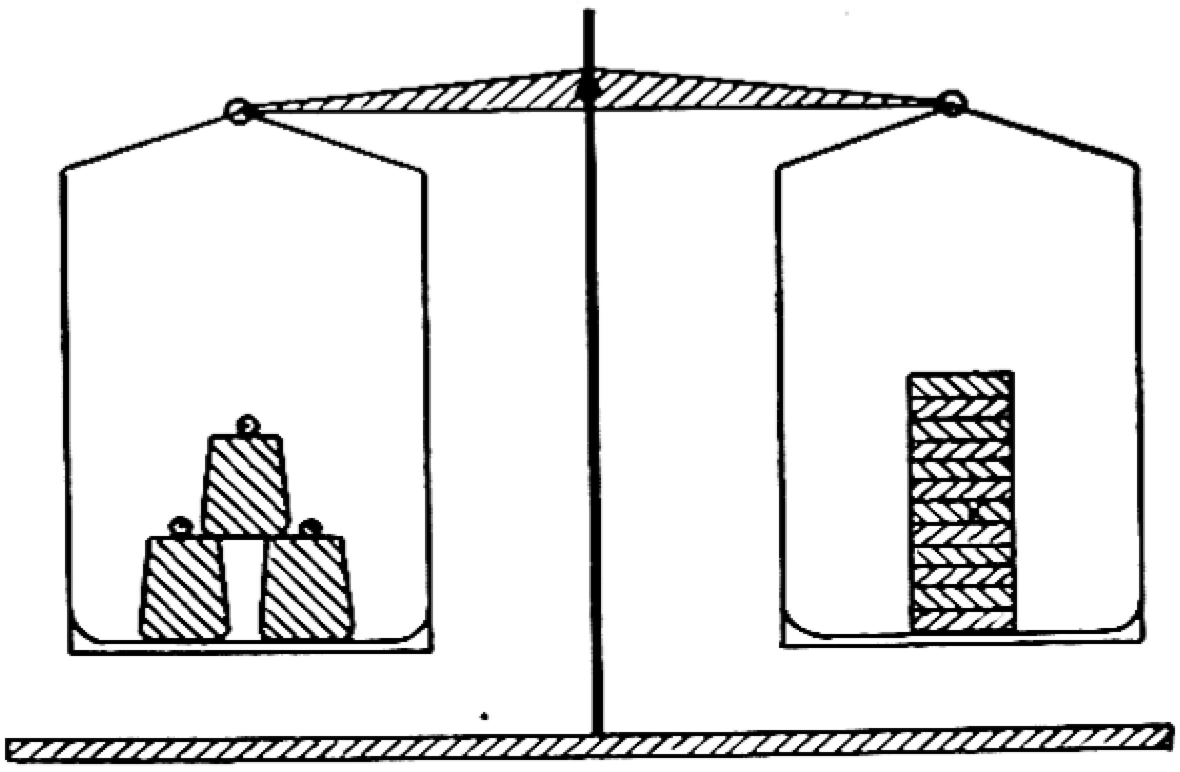


Fig. 28

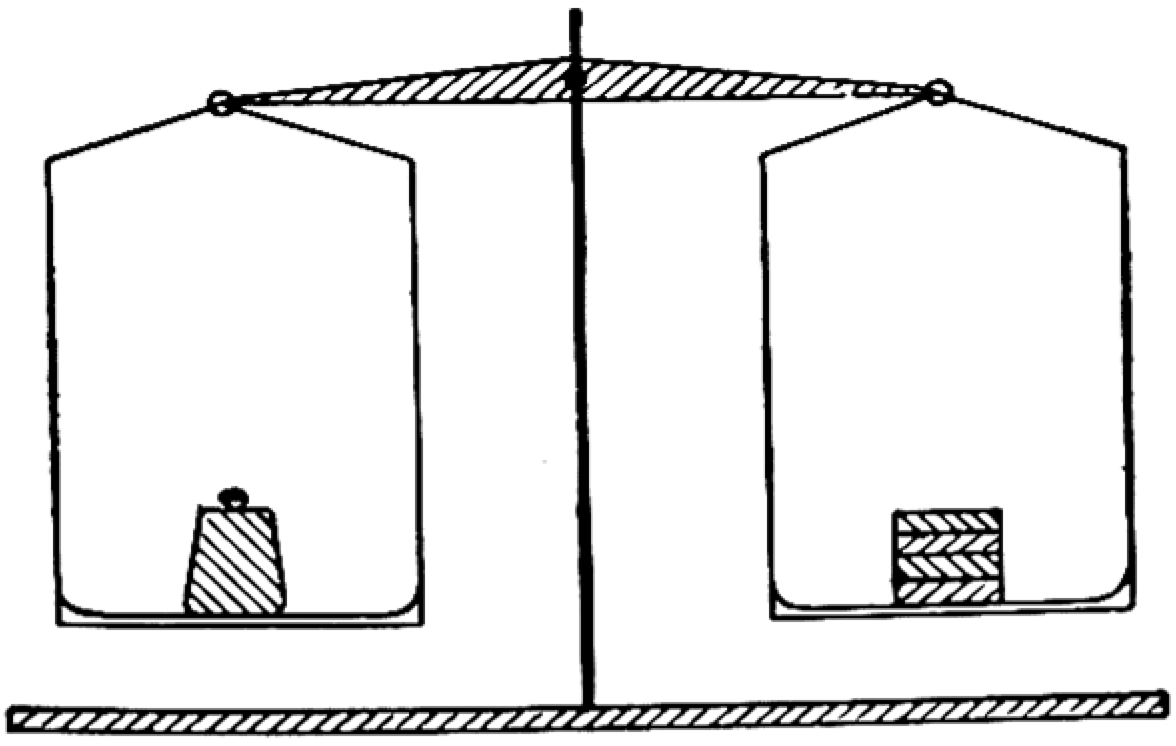


Fig. 29

Case 1. *Simple separation of knowns and unknowns*

Example. Solve for x: $3x-2+5x = 7x+1$.

If we look upon the constant terms as weights, and the terms in x as articles to be weighed, we can obtain no result until the terms in x (unknowns) are together on one side of the equation, and the constants (knowns) on the other side. Consider that all terms in x should be on the left, and all knowns on the right. We may remove terms by addition and subtraction, or multiplication and division, but all operations performed must be applied to both sides.

Add $+2$ to both sides. This will remove the -2 from the left-hand side. Next add $-7x$ to both sides to remove the $7x$ from the right-hand side, e.g.

$$3x-2+5x+2 = 7x+2+1 \quad \text{(Adding 2 to both sides)}$$

and

$$3x-2+5x+2-7x = 7x-7x+2+1 \quad \text{(Adding } -7x \text{ to both sides).}$$

$$\therefore\ 8x-7x = 2+1 \quad \text{(By addition),}$$

$$\underline{x = 3.}$$

Checking of results. It is advisable to check the solutions to all equations in the following manner. Write the result obtained for the unknown in the original equation, and if both sides balance numerically then the result obtained is correct.

To check the problem just completed, write $+3$ for every x in the original equation, viz.

$$3x-2+5x = 7x+1 \quad \text{becomes}$$

$$9-2+15 = 21+1$$

$$22 = 22.$$

Hence $x = 3$ is the correct answer.

Example. Solve for y: $15y = 30+3y$.

Subtract $3y$ from both sides to remove the term in y from the right-hand side.

$$15y = 30+3y,$$

$$15y-3y = 30+3y-3y \quad \text{(Subtracting } 3y \text{ from both sides).}$$

$$\left.\begin{array}{ll} 12y = 30. & \text{(A)} \\ \therefore\ y = \frac{30}{12} = 2\frac{1}{2}. & \text{(B)} \end{array}\right\} \text{See below.}$$

Check: Write $2\frac{1}{2}$ for y in $15y = 30+3y$

$$37\tfrac{1}{2} = 30+7\tfrac{1}{2}.$$

Hence $y = 2\frac{1}{2}$ is correct.

The rules observed in these two examples are common to all equations.

(1) Separate unknown terms to one side, and known terms to the other side by addition or subtraction.

(2) Reduce each side to a single term or expression by addition (A), and then divide the known term or expression by the coefficient of the unknown term (B).

Example. $6x+5-2x = 4x-x+4.$

$$\therefore\ 4x+5 = 3x+4.$$

To remove $3x$ from the right-hand side subtract $3x$ from both sides, and to remove 5 from the left-hand side subtract 5 from both sides.

$$\therefore\ 4x-3x = 4-5,$$

$$\therefore\ x = -1.$$

Check: $6x+5-2x = 4x-x+4.$

Substituting -1 for x:

$$6(-1)+5-2(-1) = 4(-1)-(-1)+4$$

$$-6+5+2 = -4+1+4$$

$$1 = 1.$$

Hence $x = -1$ is correct.

Consider the two lines $4x+5 = 3x+4$ and $4x-3x = 4-5$ in the problem above. The method of obtaining the second of these lines from the first has been explained in the text, but it looks as if the $3x$ has been taken across to the left-hand side and its sign changed, and similarly the $+5$ taken over to the right-hand side and its sign changed. Certain terms are bound to reappear on the opposite side of an equation with their signs changed whenever we add or subtract the same term on both sides. This fact gives rise to a useful rule which will save time. Any term may be changed over to the opposite side of the sign of equality if its sign is changed when it is written on the opposite side.

Case 2. Simplification preceding separation

When brackets are present, they may be removed before the equation is reduced to the previous type.

Example. Solve for x: $4(3-x)+5(2+x) = 7(1-4x)$

$$4(3-x)+5(2+x) = 7(1-4x).$$

$$\therefore\ 12-4x+10+5x = 7-28x \quad \text{(Removing brackets),}$$

$$x+22 = 7-28x,$$

$$28x+x = 7-22 \quad \text{(Changing } -28x \text{ and } +22 \text{ to opposite sides).}$$

$$\therefore\ 29x = -15.$$

$$\therefore\ x = -\tfrac{15}{29} \quad \text{(Dividing both sides by 29).}$$

[The student should check this and the other examples to follow.]

Example. Solve for y:

$$15(y-5)+3(y-2) = 10(2y-1)+4(y-3).$$

$$15(y-5)+3(y-2) = 10(2y-1)+4(y-3),$$

$$15y-75+3y-6 = 20y-10+4y-12,$$

$$18y-81 = 24y-22,$$

$$18y-24y = 81-22,$$

$$-6y = 59.$$

$$\therefore\ y = -\tfrac{59}{6}$$

$$\underline{= -9\tfrac{5}{6}.}$$

Example. Solve for z: $4\{z+2(z+3)\}-5(2z-3) = 39.$

$$4\{z+2(z+3)\}-5(2z-3) = 39,$$

$$4\{z+2z+6\}-10z+15 = 39,$$

$$4z+8z+24-10z+15 = 39,$$

$$2z = 39-15-24,$$

$$= 39-39.$$

$$\therefore\ 2z = 0.$$

$$\therefore\ z = 0 \quad \text{(Dividing both sides by 2 gives } z = \tfrac{0}{2} = 0\text{).}$$

Case 3. *Fractional expressions included*

We have seen that simplification of fractional expressions requires a common denominator, but when equations contain fractional expressions a common denominator must not be written because an equation written over a common denominator is meaningless. Instead of this we find the L.C.M. of the denominators of the several expressions, and then *multiply* every term on both sides of the equation by this L.C.M. By this means all fractions are removed, and the equation is unaltered because the multiplication has applied to both sides alike.

Example. Solve for x:

$$\frac{4x-2}{3}+\frac{2x-3}{4}=5x.$$

If these expressions were not part of an equation, the common denominator would be 12. We therefore multiply throughout by 12.

$$\frac{4x-2}{3}+\frac{2x-3}{4}=5x.$$

$$4(4x-2)+3(2x-3) = 60x \quad \text{(Multiplying throughout by 12).}$$

$$\therefore\ 16x-8+6x-9 = 60x,$$

$$22x-17 = 60x,$$

$$22x-60x = 17,$$

$$-38x = 17.$$

$$\therefore\ \underline{x = -\tfrac{17}{38}}.$$

Example. Solve for x: $\dfrac{3x}{x-2}+\dfrac{4}{x-3}=3.$

$$\frac{3x}{x-2}+\frac{4}{x-3}=3.$$

$$\therefore\ \frac{3x\cancel{(x-2)}\,(x-3)}{\cancel{(x-2)}}+\frac{4(x-2)\,\cancel{(x-3)}}{\cancel{(x-3)}} = 3(x-2)\,(x-3)$$

[Multiplying by $(x-2)(x-3)$],

i.e. $$3x(x-3)+4(x-2) = 3(x-2)\,(x-3).$$

$$\therefore\ 3x^2-9x+4x-8 = 3x^2-15x+18,$$

$$\cancel{3x^2}-\cancel{3x^2}-9x+4x+15x = 18+8,$$

$$10x = 26.$$

$$\therefore\ \underline{x = \tfrac{26}{10} = \tfrac{13}{5} = 2\tfrac{3}{5}}.$$

Examples to Chapter 14

1. Solve the following equations for x or y:

(*a*) $3x = 4 - 7x$. (*b*) $2x + 3 - 5x = 6x + 4 - 2x - 1$.

(*c*) $5y + 3 = 7 + 2y$. (*d*) $4x - 1{\cdot}4 = 3{\cdot}4 - 2x$.

(*e*) $y + 2\frac{1}{2} = 6y + 1\frac{1}{3}$. (*f*) $x - 5 + 5x - 4 = 2x + 3 - 7x - 18$.

(*g*) $5x + 6 + 2x = 10x + 8 - 2x$.

(*h*) $3y + 7{\cdot}52 = 2{\cdot}8 + 5y + 3{\cdot}67$. (*i*) $4 - y = 3 + 5y$.

(*j*) $13 - 112x - 18 + 15x - 8 = 91x - 13$. **(*Chap.* 14)**

2. Solve for x, y, or z:

(*a*) $3(4 - 2x) + 6 = 5(x + 3) - 8$.

(*b*) $2 - 3(1 + x) + 5(2 - x) = 2(x + 5)$.

(*c*) $4y - 3(2\frac{1}{2} + 2y) = 7(1 + 1\frac{1}{2}y) - 2$.

(*d*) $6 - 3\{x - 2(x + 1)\} = 3(x - 2) + 4x$.

(*e*) $5(3y - 2{\cdot}3) - 2(y + 1{\cdot}6) = 11{\cdot}4 - 0{\cdot}05y$.

(*f*) $z - 0{\cdot}3(z + 1) = 2z + 5(1 - z)$.

(*g*) $\dfrac{z}{4} + 3 - 2(z + \frac{1}{2}) = 9$.

(*h*) $3(x - 2) + 4(x - 2) = 5(x - 2) + 9$. **(*Chap.* 14)**

3. Solve for x or y:

(*a*) $\dfrac{x-2}{3} + \dfrac{x-4}{4} = \dfrac{x-2}{2}$. (*b*) $\dfrac{x-1}{x} + \dfrac{3}{4} = \dfrac{5-3x}{x}$.

(*c*) $\dfrac{5-3x}{7} - \dfrac{2+4x}{9} = 5\frac{46}{63}$. (*d*) $\dfrac{\{y + 3(y-1)\}}{y} = 3\frac{1}{2}$.

(*e*) $\dfrac{5(y-2)}{3} + \dfrac{4(3-y)}{7} = 6\frac{5}{21}$. (*f*) $\dfrac{3 - \frac{x}{4}}{5} = \dfrac{\frac{3}{8} + 2x}{3}$. **(*Chap.* 14)**

4. Solve the following equations:

(*a*) $\dfrac{x-2}{x-1} = \dfrac{x+4}{x-5}$. (*b*) $\dfrac{3(x-1)}{x} - \dfrac{3x-2}{x-1} = \dfrac{5}{x(x-1)}$.

(*c*) $\{x - 5(x-1)\} = \dfrac{1 - 40x^2}{10x}$. (*d*) $\dfrac{4}{x-1} - \dfrac{10}{2x-5} + \dfrac{1}{x} = 0$.

(*Chap.* 14)

5. (*a*) Solve for x: $\dfrac{2x\{3-2(x-1)\}}{x-2} = 1-4x$.

(*b*) Solve for a: $4\{2a(3-2a)-5(1-a^2)+a(1-a)\}+6 = 0$.

(*c*) Solve for z: $\dfrac{z}{2}+\dfrac{2z}{3} = 5z+23$. (*Chap.* 14)

6. (*a*) Show that $x = 2a$ satisfies the equation

$$5x-3a = 4(x-3a)+\frac{11x}{2}.$$

(*b*) Show that $x = \dfrac{a}{2}$ satisfies the equation

$$5(1-x)+3(2+5x) = 11+5a.$$

(*c*) Solve for x: $\dfrac{1+x}{1-x}+\dfrac{2+x}{2-x}+2 = 0$. (*Chap.* 14)

7. (*a*) Factorise completely $3a^3-7a^2-18a-8$.

(*b*) Simplify $\dfrac{a\{a^2+2(a+1)\}}{a-2}+\dfrac{2a(a^2-1)}{a^2-3a+2}$. (*Chap.* 13)

8. Factorise the following:

(*a*) $15a^2+a-6$. (*b*) $12xy^2-15x^2y+3x^3$.

(*c*) $5x^2y^2z^2-45x^2y^4z^4$. (*d*) $1-1728b^3$.

(*e*) $4y-2xy-20x+10x^2$. (*f*) $2a^2-a^2y+b^2y-2b^2$.

(*Chap.* 12)

9. (*a*) Express with positive indices,

$$\sqrt[3]{x^2},\quad y^{-\frac{3}{4}},\quad (\sqrt{z})^5,\quad \sqrt{x}\times\sqrt{xy}.$$ (*Chap.* 8)

(*b*) The moment of inertia of a beam is given by

$$I = bd\left(\frac{d^2}{12}+h^2\right).$$

Evaluate I when $b = 10{\cdot}3$, $d = 3{\cdot}6$, $h = 6{\cdot}35$. (*Chap.* 9)

10. The heat produced in a coil of wire by the passage of an electric current is proportional to the square of the voltage. A voltage of 4 volts produced 480 calories of heat in a measured time. Find the quantity of heat produced in the same coil, in the same time, by voltages of 6 and of 8 volts. (*Chap.* 3)

Answers to Examples for Chapter 14

1. (*a*) $x = \frac{2}{5}$. (*b*) $x = 0$. (*c*) $y = \frac{4}{3}$. (*d*) $x = 0{\cdot}8$.
(*e*) $y = \frac{7}{30}$. (*f*) $x = -\frac{6}{11}$. (*g*) $x = -2$. (*h*) $y = 0{\cdot}525$.
(*i*) $y = \frac{1}{6}$. (*j*) $x = 0$.

2. (*a*) $x = 1$. (*b*) $x = -\frac{1}{10}$. (*c*) $y = -1$. (*d*) $x = 4\frac{1}{2}$.
(*e*) $y = 2$. (*f*) $z = 1{\cdot}\dot{4}3\dot{2}$. (*g*) $z = -4$. (*h*) $x = 6\frac{1}{2}$.

3. (*a*) $x = 8$. (*b*) $x = \frac{24}{19}$. (*c*) $x = -6$. (*d*) $y = 6$.
(*e*) $y = \frac{165}{23}$. (*f*) $x = \frac{57}{86}$.

4. (*a*) $x = 1{\cdot}4$. (*b*) $x = -\frac{1}{2}$. (*c*) $x = \frac{1}{50}$. (*d*) $x = \frac{5}{17}$.

5. (*a*) $x = -2$. (*b*) $a = \frac{1}{2}$. (*c*) $z = -6$.

6. (*c*) $x = \frac{4}{3}$.

7. (*a*) $(a+1)(3a+2)(a-4)$.
(*b*) $\dfrac{a(a^3+3a^2-4)}{(a-2)(a-1)} = \dfrac{a(a-1)(a+2)(a+2)}{(a-1)(a-2)} = \dfrac{a(a^2+4a+4)}{a-2}$.

8. (*a*) $(5a-3)(3a+2)$. (*b*) $3x(4y-x)(y-x)$.
(*c*) $5x^2y^2z^2(1-3yz)(1+3yz)$.
(*d*) $(1-12b)(1+12b+144b^2)$. (*e*) $(4-2x)(y-5x)$.
(*f*) $(a-b)(a+b)(2-y)$.

9. (*a*) $x^{\frac{3}{2}}$, $\dfrac{1}{y^{\frac{1}{2}}}$, $z^{\frac{3}{2}}$, $xy^{\frac{1}{3}}$.
(*b*) 1536.

10. 1080 calories, 1920 calories.

CHAPTER 15

LITERAL EQUATIONS. PROBLEMS ON SIMPLE EQUATIONS

Literal equations

Equations containing more than one letter, one or more of which is considered as a 'known', are called literal equations. The answers for the unknown letter (or letters) will therefore contain a letter or letters. The method of solution is as before.

Example. Solve for x: $4x-2a = 3x-a$.

$$4x-2a = 3x-a,$$
$$4x-3x = 2a-a.$$
$$\therefore\ \underline{x = a.}$$

Example. Solve for x: $4(x+5b)-b = 5(2b+x)-4x$.

$$4(x+5b)-b = 5(2b+x)-4x,$$
$$4x+20b-b = 10b+5x-4x,$$
$$4x+19b = 10b+x,$$
$$4x-x = 10b-19b,$$
$$3x = -9b.$$
$$\therefore\ x = -\frac{9b}{3} \underline{= -3b.}$$

Example. Find the value of x in terms of l and m from

$$\frac{10x}{lm}-\frac{6}{l} = \frac{3(l+m)+x}{lm}.$$

$$\frac{10x}{lm}-\frac{6}{l} = \frac{3(l+m)+x}{lm},$$

$$10x-6m = 3(l+m)+x \quad \text{(Multiplying throughout by } lm\text{)},$$
$$10x-6m = 3l+3m+x,$$
$$10x-x = 3l+3m+6m,$$
$$9x = 3l+9m.$$
$$\therefore\ x = \frac{3l+9m}{9} = \underline{\frac{l}{3}+m.}$$

Problems on simple equations

The student will require to solve many equations in his engineering studies, but these will not necessarily occur in the form of ready-made equations. The facts of the problem must be written mathematically to express a balance between two expressions. It is usually the best policy to write x for the unknown quantity in the problem, and then build an equation in x from the given data.

Example. A number of articles were sold at $5d.$ each. If each article had been $1d.$ dearer, two less could have been sold for the same money. Find the number of articles sold in the first place.

Let $x =$ the number of articles sold in the first place.

Then the total money received from the sale $= 5x$ pence.

Total money received from the sale of two less articles at $6d. = 6(x-2)$ pence.

We are told that these two sums of money are equal, so

$$\begin{aligned} 5x &= 6(x-2), \\ 5x &= 6x-12, \\ 5x-6x &= -12, \\ -x &= -12, \\ \therefore\ x &= 12. \end{aligned}$$

Therefore 12 articles were sold in the first place.

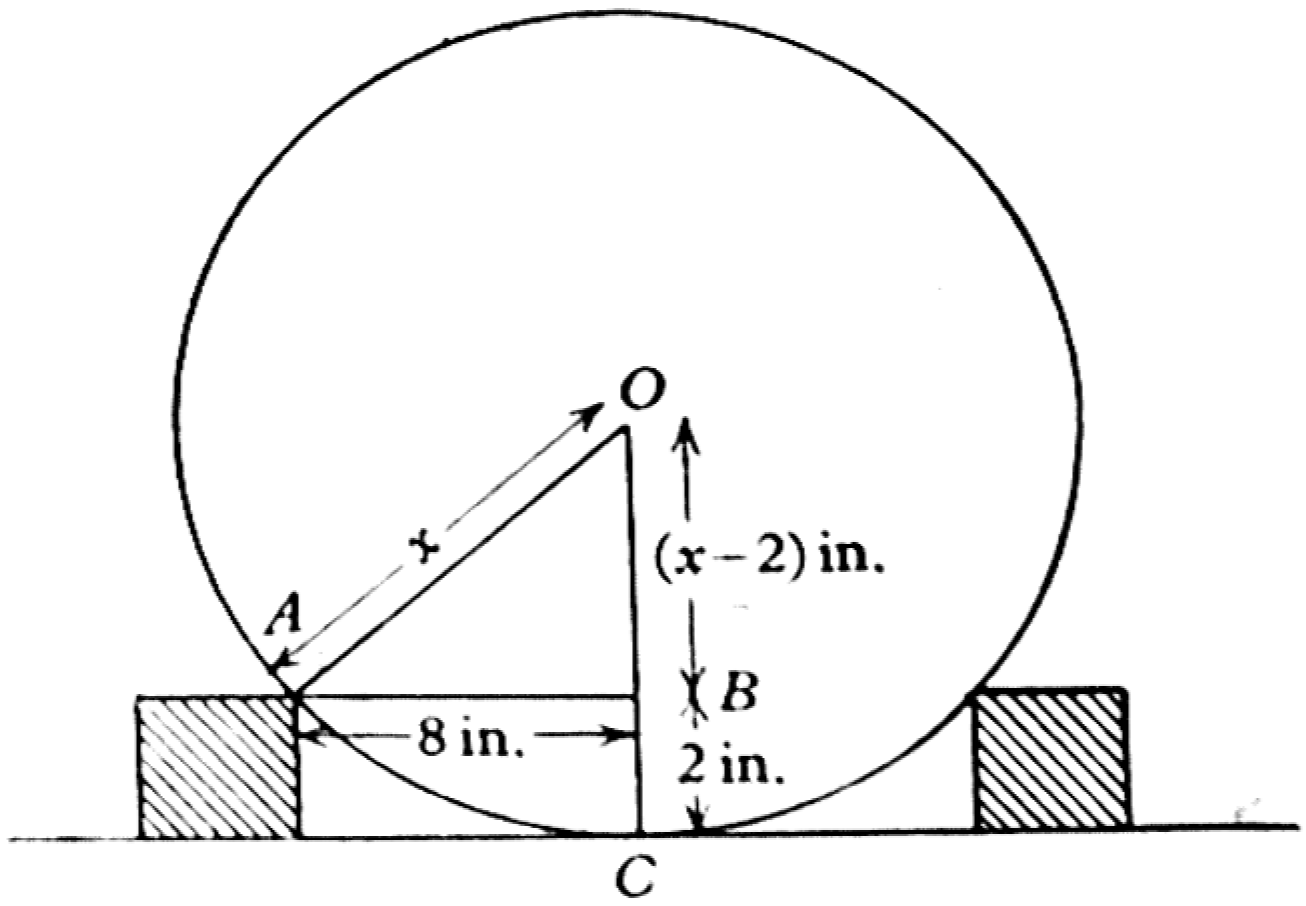


Fig. 30

Example. A steel tube lying horizontally is prevented from rolling by two rectangular blocks, each 2 in. thick, placed with one corner of each in contact with the circumference of the tube. If the blocks are 1 ft. 4 in. apart, find the radius of the tube.

See fig. 30. Consider the triangle OAB which is right-angled at B. Since OC is a radius of the circle, $OB = (x-2)$ in., where x is the radius in inches.

By the Theorem of Pythagoras (Chapter 30)

$$\begin{aligned} OA^2 &= OB^2 + AB^2 \\ \text{or} \qquad x^2 &= 8^2 + (x-2)^2, \\ \cancel{x^2} &= 8^2 + \cancel{x^2} - 4x + 4. \\ \therefore\ 4x &= 64 + 4 \\ &= 68. \\ \therefore\ x &= 17. \end{aligned}$$

Therefore the radius of the tube is 17 in.

Example. An alloy whose density is 8·2 g. per c.c. is composed of two metals whose densities are 7·0 g. per c.c. and 8·5 g. per c.c. How much of the heavier metal will require to be mixed with 50 c.c. of the lighter metal?

Weight of first metal = volume × density = 50 × 7·0 g.

Let volume of second metal be x c.c.

Then weight of second metal = $8{\cdot}5x$ g.

Now, total weight of sample of alloy = $(50+x)\,8{\cdot}2$ g., and since weight of constituents = weight of alloy,

$$\begin{aligned} \therefore\ 50 \times 7{\cdot}0 + 8{\cdot}5x &= (50+x)\,8{\cdot}2, \\ 350 + 8{\cdot}5x &= 8{\cdot}2 \times 50 + 8{\cdot}2x, \\ 0{\cdot}3x &= 60. \\ \therefore\ 2 &= \frac{60}{0{\cdot}3} = 200. \end{aligned}$$

Therefore volume of heavier metal = 200 c.c.

Example. A cyclist rides over a hill, a total journey of 9 miles. His average speed uphill is 8 m.p.h., and his average speed downhill is 20 m.p.h. If the journey occupies 45 min., find the distances he rode uphill and downhill.

Let x miles be the distance he rode uphill, then $(9-x)$ miles is the distance downhill. But distance = average speed × time taken. Therefore time taken $= \dfrac{\text{Distance}}{\text{Average speed}}$.

Thus, time taken uphill $= \frac{x}{8}$ hours and time taken downhill $= \frac{9-x}{20}$ hours.

Total time $= \frac{3}{4}$ hours.

$$\therefore\ \frac{x}{8}+\frac{9-x}{20}=\frac{3}{4}.$$

Multiplying by 40:

$$\begin{aligned} 5x+2(9-x) &= 30,\\ 5x+18-2x &= 30,\\ 3x &= 30-18\\ &= 12,\\ \therefore\ x &= 4. \end{aligned}$$

Therefore the distance he rides uphill is $\underline{\text{4 miles}}$, and the distance downhill must be $9-4 = \underline{\text{5 miles}}$.

Examples for Chapter 15

1. (*a*) Solve for x: $x-3p+5x-p = 7x-5p$.

(*b*) Solve for x: $3(x-5c)+4(x-2c) = 3(2x+c)$.

(*c*) Find the number which exceeds $\frac{3}{8}$ of itself by 15.

(*d*) When the reciprocal of a number is multiplied by 20, and then subtracted from 12, the result is 7. Find the number. (*Chap.* 15)

2. (*a*) Solve for y: $y-4a+2(y-b) = y+2a$.

(*b*) Solve for x: $abx-(a-b)^2-a^2b^2 = 4x-(a+b)^2$.

(*c*) A consignment of goods was sold at £25 for each article. When the price was lowered by £5, 40 more articles were included for the same total price. Find the number of articles in the original consignment. (*Chap.* 15)

3. (*a*) One wall of a workshop is 18 ft. high and 40 ft. long. Four windows, each 15 ft. high, are to be made in this wall so that the total area of window-space is equal to the area of brickwork. Find the width of each window.

(*b*) 120 c.c. of sulphuric acid of specific gravity 1·1 are mixed with sulphuric acid of specific gravity 1·36 in order to fill accumulators with acid of specific gravity 1·25. Find the volume of the heavier acid required. (*Chap.* 15)

4. (*a*) A cylindrical water-tank rests horizontally in a rectangular channel whose two sides are 3 ft. high, and whose width is 7 ft. The lowest part of the curved surface of the cylinder is then 1 ft. 6 in. from the bottom of the channel. Find the radius of the tank.

(*b*) A stopping train covers its journey at an average speed of 45 m.p.h., but when several stops are cancelled the average speed is increased by 5 m.p.h., and the time reduced by 24 min. Find the length of the journey. (*Chap.* 15)

5. (*a*) A block of wood 3 in. by 2 in. by 2 ft. has a density of $46\frac{1}{2}$ lb. per cu.ft. What volume of lead of density $706\frac{4}{5}$ lb. per cu.ft. must be attached to this block so that the combination will have the same density as water? (Take density of water as 62 lb. per cu.ft.)

(*b*) The first part of a motorist's journey is uphill, and his average speed over this road is 25 m.p.h. The next part is downhill and his average speed is 40 m.p.h. The total journey is 35 miles, and the time taken $61\frac{1}{2}$ min. Find the time taken to climb the hill. (*Chap.* 15)

6. (*a*) Solve for x: $3x-(a+b)+\dfrac{1}{(a+b)}=x\{4-(a+b)^2\}$.

(*b*) A cylindrical pillar whose radius is 1 ft. 6 in. is surmounted by a cone of the same radius. The total height of the pillar is 12 ft., and its volume is $80\frac{1}{7}$ cu.ft. Taking $\pi=\frac{22}{7}$, find the height of the cylindrical portion. (*Chap.* 15)

7. (*a*) Factorise the following expressions:

(i) $4a^2+12ab+9b^2$.

(ii) $4a^2+12ab+9b^2-16d^2$.

(iii) $9y-4x^2y-27x+12x^3$.

(iv) $\dfrac{8l^3}{m^3}-1$. (*Chap.* 12) (*b*) Simplify $\dfrac{a+3(a+b)}{2ab}+\dfrac{3a+2b}{b(a+b)}+\dfrac{b}{a(a+b)}$.

(*Chap.* 13)

8. (*a*) Evaluate $\dfrac{5{\cdot}03\times 4{\cdot}12\times 10^7}{\sqrt{0{\cdot}184}}$.

(*b*) The formula $s=\dfrac{n\sqrt{Q}}{H^{\frac{3}{4}}}$ is used in connection with centrifugal pumps. Evaluate s when $n = 550, Q = 2{\cdot}3, H = 15{\cdot}25$. (*Chap.* 10)

9. (*a*) Add $5a-3b+2c$ and $c-5a+2b$ and subtract the result from the sum of $a-c$ and $2b+c-a$. (*Chap.* 6)

(*b*) Evaluate $(a+2b)^3$, and hence show that

$$(a+2b)^3-(a-2b)^3 = 4b(3a^2+4b^2). \qquad (\textit{Chap. } 7)$$

10. (*a*) The effective resistance (R) of two resistances R_1 and R_2 wired in parallel is equal to their product divided by their sum. Express this as a formula and evaluate R when $R_1 = 4{\cdot}5$ ohms and $R_2 = 3{\cdot}75$ ohms.

(*b*) The current in amperes flowing in the circuit is given by voltage divided by effective resistance. Find the current in the previous case if the voltage is 12. (*Chap.* 5)

Answers to Examples for Chapter 15

1. (*a*) $x = p$. (*b*) $x = 26c$. (*c*) 25. (*d*) 4.

2. (*a*) $y = 3a+b$. (*b*) $x = ab$. (*c*) 160.

3. (*a*) 6 ft. (*b*) 164 c.c. (approx.).

4. (*a*) 4 ft. 10 in. (*b*) 180 miles.

5. (*a*) 0·002 cu.ft. (approx.). (*b*) 24 min.

6. (*a*) $x=\dfrac{1}{a+b}$. (*b*) 11 ft.

7. (*a*) (i) $(2a+3b)(2a+3b)$; (ii) $(2a+3b-4d)(2a+3b+4d)$;

(iii) $(y-3x)(3-2x)(3+2x)$;

(iv) $\left(\frac{2l}{m}-1\right)\left(\frac{4l^2}{m^2}+\frac{2l}{m}+1\right)$.

(*b*) $\frac{10a^2+11ab+5b^2}{2ab(a+b)}$.

8. (*a*) $4{\cdot}83\times10^8$. (*b*) 108·1.

9. (*a*) $3(b-c)$.

10. (*a*) 2·05 ohms (approx.). (*b*) 5·87 amp. (approx.).

CHAPTER 16

MANIPULATION OF FORMULAE

A formula is a generalised equation, and the rules which apply to equations also apply to formulae. Since formulae are used to obtain numerical answers it is assumed that figures will be available for all quantities mentioned except one. This quantity, represented by a symbol, is called the subject of the formula, and is arranged, by algebraic processes, to stand on its own on the left-hand side of the sign of equality.

Consider the formula for the volume of a cylinder, $V = \pi r^2 h$. Assuming that figures for π, r and h are known, then the volume V is obtained by multiplication. It may happen in problems that the volume is known, as well as the radius and the value of π, but the height is to be determined. In that case it is correct to write the formula with h as subject. In other circumstances, given V and h, then r must be written as the subject. This chapter will show the method of changing the subject of a formula.

Case 1. *Formulae involving multiplication and division only*

Consider the formula for the indicated horse-power of an engine,

$$I = \frac{PLAN}{33{,}000}.$$

P represents average pressure in the cylinder, L is the length of stroke, A is the area of cross-section of the cylinder, and N is the number of strokes of the piston per minute. As it stands, the formula is convenient for the calculation of I, when P, L, A and N are given.

To make P the subject: Arrange that the part of the formula which contains P is on the left-hand side:

$$\frac{PLAN}{33{,}000} = I.$$

In order that the left-hand side shall ultimately contain P standing alone, the present left-hand side must be *divided* by L, A and N, and *multiplied* by 33,000. These processes must, of course, be applied to both sides of the formula.

Therefore $$\frac{P\cancel{LAN}}{\cancel{33{,}000}} \times \frac{\cancel{33{,}000}}{\cancel{LAN}} = I \times \frac{33{,}000}{LAN}.$$

$$\therefore\ P = \frac{33{,}000I}{LAN}.$$

The same result may be found by the process of *cross-multiplication*, which will be found useful in many cases. When both sides of an equation are expressed as single fractions, then the product of the left numerator and the right denominator is equal to the product of the left denominator and the right numerator. For instance, if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$, i.e. $\frac{a}{b}> = <\frac{c}{d}$ represents the process. A warning should be given against misuse of this principle. Note that cross-multiplication may only be employed when each side is a single fraction, so the following formula, as it stands, could not be manipulated in this way: $\frac{a}{b} = \frac{c}{d}+e$. If, however, the right-hand side is made into a single fraction by placing both terms over the common denominator d, then the process may be used.

Then $$\frac{a}{b} = \frac{(c+de)}{d}.$$

$$\therefore\ ad = b(c+de) \quad \text{(By cross-multiplication).}$$

Referring to our previous formula $I = \frac{PLAN}{33{,}000}$, cross-multiplication gives $PLAN = 33{,}000I$. And starting with this arrangement, it is a simple process to make any of the symbols P, L, A or N into the subject by division.

Thus $$L = \frac{33{,}000I}{PAN}, \quad A = \frac{33{,}000I}{PLN}, \quad N = \frac{33{,}000I}{PLA}.$$

Example. The formula $\frac{P_1V_1}{\theta_1} = \frac{P_2V_2}{\theta_2}$ relates the pressure, volume and temperature of the same mass of gas under changing conditions. Rewrite the formula to express (*a*) P_1 in terms of the other quantities, (*b*) θ_1 in terms of the other quantities.

Cross-multiplication gives $P_1V_1\theta_2 = P_2V_2\theta_1$.

Thus $P_1 = \dfrac{P_2V_2\theta_1}{V_1\theta_2}$ (Dividing both sides by $V_1\theta_2$)

and $\theta_1 = \dfrac{P_1V_1\theta_2}{P_2V_2}$ (Dividing both sides by P_2V_2).

Case 2. *Formulae of the previous type including roots and powers*

When the letter or symbol which is to become the new subject is raised to a power in the original formula, it is convenient to remove the root or power in the final stage, but if the root or power applies to a group of letters in a bracket, the bracket must be removed early in the transformation.

Example. $I = \dfrac{bd^3}{12}$. Write d in terms of the other symbols.

Cross-multiplication gives $12I = bd^3$.

$$\therefore \frac{12I}{b} = d^3.$$

$$\therefore d = \sqrt[3]{\frac{12I}{b}}.$$

Example. $$t = 2\pi\sqrt{\frac{l}{g}}.$$

Write (1) l in terms of the other symbols, (2) g in terms of the other symbols.

$$t = 2\pi\sqrt{\frac{l}{g}}.$$

$\therefore t^2 = 4\pi^2\dfrac{l}{g}$ (Squaring both sides to remove the bracket).

$\therefore gt^2 = 4\pi^2 l$ (Cross-multiplication).

(1) $\therefore l = \dfrac{gt^2}{4\pi^2}$ (Dividing both sides by $4\pi^2$).

(2) Also $g = \dfrac{4\pi^2 l}{t^2}$ (Dividing both sides by t^2).

Example. Transform the formula $y = \dfrac{wl^4}{3EI}$ so that (1) l becomes the subject, (2) I becomes the subject.

$$y = \frac{wl^4}{3EI}.$$

$$\therefore\ 3yEI = wl^4 \quad \text{(By cross-multiplication).}$$

$$\therefore\ l^4 = \frac{3yEI}{w} \quad \text{(Dividing both sides by } w\text{).}$$

(1) $$\therefore\ l = \sqrt[4]{\frac{3yEI}{w}} \quad \text{(Extracting the fourth root of both sides).}$$

(2) Also $$I = \frac{wl^4}{3yE} \quad \text{(Dividing both sides by } 3Ey\text{).}$$

Example. Write H as the subject in the formula $s = \dfrac{n\sqrt{Q}}{H^{\frac{3}{4}}}$.

$$S = \frac{n\sqrt{Q}}{H^{\frac{3}{4}}}.$$

$$\therefore\ SH^{\frac{3}{4}} = n\sqrt{Q} \quad \text{(By cross-multiplication).}$$

$$\therefore\ H^{\frac{3}{4}} = \frac{n\sqrt{Q}}{s}.$$

$$\therefore\ H^3 = \left(\frac{n\sqrt{Q}}{s}\right)^4 \quad \text{(Raising both sides to the power 4)}$$

$$= \frac{n^4Q^2}{s^4}.$$

$$\therefore\ H = \frac{n^{\frac{4}{3}}Q^{\frac{2}{3}}}{s^{\frac{4}{3}}} \quad \text{(Extracting the cube root of both sides).}$$

Case 3. *Formulae containing new subject in a bracket*

When the new subject occurs as an added or subtracted term in a bracket, it is advisable to isolate the bracket before removing it.

Example. Transform the formula $H = \dfrac{(T_1 - T_2)\,2\pi Rn}{33{,}000}$ so that T_2 becomes the subject.

$$H = \frac{(T_1 - T_2)\,2\pi Rn}{33{,}000}.$$

$$\therefore\ (T_1 - T_2)\,2\pi Rn = 33{,}000H.$$

$$\therefore\ T_1 - T_2 = \frac{33{,}000H}{2\pi Rn} \quad \text{(Isolating bracket by dividing both sides by } 2\pi Rn\text{).}$$

$$\therefore \quad -T_2 = \frac{33{,}000H}{2\pi Rn} - T_1$$ (Subtracting T_1 from both sides).

$$\therefore \quad T_2 = T_1 - \frac{33{,}000H}{2\pi Rn}$$ (Multiplying both sides by -1).

Example. From the formula $t = 2\pi\sqrt{\left(\frac{K^2+h^2}{hg}\right)}$ transform the letters so that K becomes the subject.

$$t = 2\pi\sqrt{\left(\frac{K^2+h^2}{hg}\right)}.$$

$$\therefore \quad t^2 = 4\pi^2\frac{(K^2+h^2)}{hg}$$ (Squaring both sides).

$$\therefore \quad t^2hg = 4\pi^2(K^2+h^2)$$ (Cross-multiplying).

$$\therefore \quad K^2+h^2 = \frac{t^2hg}{4\pi^2}$$ (Isolating bracket).

$$\therefore \quad K^2 = \frac{t^2hg}{4\pi^2} - h^2$$ (Subtracting h^2 from both sides)

$$= \frac{t^2hg - 4\pi^2h^2}{4\pi^2}$$ (Placing over common denominator).

$$\therefore \quad K = \frac{1}{2\pi}\sqrt{t^2hg - 4\pi^2h^2}.$$

Example. Transform the formula $\frac{T}{I} = E\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$ so that R_1 becomes the subject.

$$\frac{T}{I} = E\left(\frac{1}{R_2} - \frac{1}{R_1}\right).$$

$$\therefore \quad T = EI\left(\frac{1}{R_2} - \frac{1}{R_1}\right).$$

$$\therefore \quad \frac{1}{R_2} - \frac{1}{R_1} = \frac{T}{EI}.$$

$$\therefore \quad -\frac{1}{R_1} = \frac{T}{EI} - \frac{1}{R_2}$$ (Subtracting $1/R_2$ from both sides).

$$\therefore \quad \frac{1}{R_1} = \frac{1}{R_2} - \frac{T}{EI}$$ (Multiplying both sides by -1).

R_1 is now the reciprocal of this formula, but the reciprocal of the right-hand side cannot be obtained until it is written as a single fraction.

$$\therefore \quad \frac{1}{R_1} = \frac{EI - R_2T}{R_2EI}.$$

$$\therefore \quad R_1 = \frac{R_2EI}{EI - R_2T}.$$

Case 4. *Formulae in which the new subject occurs in more than one term*

All the terms in which the new subject appears must be placed on one side of the formula as soon as possible, and then the required symbol may be extracted as a common factor.

Example. If $P = 2r + 2x + \pi r$, write r in terms of the other letters.

$$P = 2r + 2x + \pi r.$$

$$\therefore \quad 2r + \pi r = P - 2x \quad \text{(Subtracting } 2x \text{ from both sides).}$$

$$\therefore \quad r(2 + \pi) = P - 2x \quad \text{(Writing } r \text{ as a common factor).}$$

$$\therefore \quad r = \frac{P - 2x}{2 + \pi} \quad \text{(Dividing both sides by } 2 + \pi\text{).}$$

Example. If $\dfrac{Q}{P} = \dfrac{1 - \sin\phi}{1 + \sin\phi}$, write $\sin\phi$ in terms of the other letters.

$$\frac{Q}{P} = \frac{1 - \sin\phi}{1 + \sin\phi}.$$

$$\therefore \quad Q(1 + \sin\phi) = P(1 - \sin\phi) \quad \text{(By cross-multiplication).}$$

$$\therefore \quad Q + Q\sin\phi = P - P\sin\phi \quad \text{(Removing brackets).}$$

$$\therefore \quad Q\sin\phi + P\sin\phi = P - Q.$$

$$\therefore \quad \sin\phi(Q + P) = P - Q.$$

$$\therefore \quad \sin\phi = \frac{P - Q}{P + Q}.$$

Example. Express H in terms of E, v and g from the formula

$$E = \frac{H - \frac{v^2}{2g}}{H}.$$

$$E = \frac{H - \frac{v^2}{2g}}{H}.$$

$$\therefore \; EH = H - \frac{v^2}{2g}.$$

$$\therefore \; H - EH = \frac{v^2}{2g}.$$

$$\therefore \; H(1-E) = \frac{v^2}{2g}.$$

$$\therefore \; H = \frac{v^2}{2g(1-E)}.$$

Examples for Chapter 16

1. Transform the following formulae in the manner indicated:

(*a*) $P = \frac{Wv^2}{rg}$. Subject r.

(*b*) $H = \frac{VIt}{4{\cdot}2}$. Subject t.

(*c*) $v^2 = u^2 + 2fs$. Subject s.

(*d*) $I = \frac{WR^2}{2g}$. Subject g.

(*e*) $V = \frac{\pi R^2 h}{3}$. Subject h.

(*f*) $I = \frac{BD^3}{12}$. Subject B.

(*g*) $y = \frac{Ml^2}{8EI}$. Subject I.

(*h*) $P = \frac{4\pi^2 EI}{l^2}$. Subject I.

(*i*) $D = \frac{4\pi R^2 nq}{Cd}$. Subject C.

(*j*) $E = \frac{wav^2}{2g}$. Subject a. **(*Chap.* 16)**

2. Transform the following formulae in the manner indicated:

(*a*) $t = 2\pi\sqrt{\dfrac{I}{Mgh}}$. Subject h.

(*b*) $V = \dfrac{4\pi R^3}{3}$. Subject R.

(*c*) $I = \dfrac{WR^2}{2g}$. Subject R.

(*d*) $y = \dfrac{Wa^2b^2}{3EIl}$. Subject b.

(*e*) $Q = Kc\sqrt{h}$. Subject h.

(*f*) $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$. Subject R_1.

(*g*) $s = \sqrt{\left(\dfrac{b^2}{4} + q^2\right)}$. Subject b.

(*h*) $V_t = V_0(1+Kt)$. Subject K.

(*i*) $H = \dfrac{W^2}{2g}(r_1^2 - r_2^2)$. Subject r_2.

(*j*) $s = \dfrac{n\sqrt{Q}}{H^{\frac{3}{2}}}$. Subject Q. (*Chap.* 16)

3. Transform the following formulae as indicated:

(*a*) $t = \dfrac{2A\sqrt{H}}{Ca\sqrt{(2g)}}$. Subject H.

(*b*) $H = \dfrac{(T_1 - T_2)\,2\pi Rn}{33{,}000}$. Subject T_1.

(*c*) $\dfrac{P}{W} = \dfrac{\tan\theta + \mu}{1 - \mu\tan\theta}$. Subject $\tan\theta$.

(*d*) $T = \sqrt{\left(H + \dfrac{w^2l^2}{4}\right)}$. Subject l.

(*e*) $\dfrac{1}{m} = \dfrac{3K - 2c}{2c + 6K}$. Subject K.

(*f*) $c = \dfrac{K}{1 + \dfrac{a}{\sqrt{m}}}$. Subject m.

(*g*) $D = \dfrac{2C\sqrt{(2g)}\,LH^{\frac{3}{2}}}{3}$. Subject H.

(*h*) $d = \dfrac{1}{s} + \dfrac{P}{20}$. Subject P. (*Chap.* 16)

4. (*a*) Express the radius of a sphere in terms of its volume.

(*b*) Find the radius of the largest sphere which could be cast from a rectangular block of metal 8 in. by 4 in. by 3 in., assuming no loss in casting. (*Chap.* 16)

5. The number of gallons of water flowing through a pipe per minute is given by
$$n = \sqrt{\frac{(3d)^5 H}{L}}.$$
Express d in terms of the other quantities and find the value of d when $n = 81$, $H = 90$, $L = 810$. (*Chap.* 16)

6. The period of oscillation of a suspended magnet is given by $t = 2\pi\sqrt{\frac{I}{MH}}$. Express M in terms of the other symbols and evaluate M when $t = 2{\cdot}85$, $\pi = 3{\cdot}142$, $I = 18{\cdot}4$, $H = 0{\cdot}18$. (*Chap.* 16)

7. (*a*) Solve for x: $4(3-x^2)+2(5-2x) = 3(2-x^2)-x(x-4)$.

(*b*) Solve for x: $\frac{5x+2}{3} - \frac{3-x}{4} = x$. (*Chap.* 14)

(*c*) The sum of the length and girth of a rectangular parcel with square ends is 6 ft. If l is the length and x is one of the shorter edges, write an expression for the statement in the previous sentence. Solve for x when $l = 3$ ft. 4 in., and calculate the volume of the parcel in that case. (*Chap.* 15)

8. (*a*) Evaluate $\sqrt[5]{32} \times \frac{1}{3^2} \times \sqrt[4]{(81)^3} \times (4\sqrt{2})^2$. (*Chap.* 8)

(*b*) If
$$y_1 = 3x^3 - 4x^2 + 2x - 9, \text{ and } y_2 = -2x^3 + 3x^2 - 5x + 20,$$
evaluate y_1 and y_2 when $x = -2$, and write the value of $y_1^2 - y_2^2$. (*Chaps.* 7 *and* 11)

9. (*a*) Show, without multiplication, that $(a-b)$, $(b-c)$ and $(c-a)$ are factors of
$$bc(c-b) + ac(a-c) + ab(b-a).$$

(*b*) Simplify $\frac{1}{x-3} - \frac{2x}{x-2} + \frac{x}{x^2-5x+6}$. (*Chap.* 13)

10. (*a*) Calculate $\sqrt{5}$ correct to four significant figures.

(*b*) The square root of 5 is given approximately by $2(1+\frac{1}{8}-\frac{1}{128})$. Calculate this correct to four significant figures, and find the percentage error involved. (*Chaps.* 4 *and* 3)

Answers to Examples for Chapter 16

1. (*a*) $r = \dfrac{Wv^2}{Pg}$. (*b*) $t = \dfrac{4{\cdot}2H}{VI}$. (*c*) $s = \dfrac{v^2-u^2}{2f}$.

(*d*) $g = \dfrac{WR^2}{2I}$. (*e*) $h = \dfrac{3V}{\pi R^2}$. (*f*) $B = \dfrac{12I}{D^3}$.

(*g*) $I = \dfrac{Ml^2}{8Ey}$. (*h*) $I = \dfrac{Pl^2}{4\pi^2 E}$. (*i*) $C = \dfrac{4\pi R^2 nq}{Dd}$.

(*j*) $a = \dfrac{2Eg}{Wv^2}$.

2. (*a*) $h = \dfrac{4\pi^2 I}{Mgt^2}$. (*b*) $R = \sqrt[3]{\dfrac{3V}{4\pi}}$. (*c*) $R = \sqrt{\dfrac{2gI}{W}}$.

(*d*) $b = \dfrac{1}{a}\sqrt{\dfrac{3yEIl}{W}}$. (*e*) $h = \dfrac{Q^2}{K^2c^2}$. (*f*) $R_1 = \dfrac{RR_2}{R_2-R}$.

(*g*) $b = 2\sqrt{(s^2-q^2)}$. (*h*) $K = \dfrac{V_t-V_0}{V_0 t}$.

(*i*) $r_2 = \dfrac{1}{W}\sqrt{(r_1^2W^2-2gh)}$. (*j*) $Q = \dfrac{s^2H^{\frac{3}{2}}}{n^2}$.

3. (*a*) $H = \dfrac{t^2c^2a^2g}{2A^2}$. (*b*) $T_1 = \dfrac{33{,}000H+2\pi RnT_2}{2\pi Rn}$.

(*c*) $\tan\theta = \dfrac{P-W\mu}{W+P\mu}$. (*d*) $l = \dfrac{2}{W}\sqrt{(T^2-H)}$.

(*e*) $K = \dfrac{2c(1+m)}{3(m-2)}$. (*f*) $m = \dfrac{a^2c^2}{(K-c)^2}$.

(*g*) $H = \left(\dfrac{3d}{2cL}\right)^{\frac{2}{3}}\left(\dfrac{1}{2g}\right)^{\frac{1}{3}}$. (*h*) $P = 20\left(d-\dfrac{1}{s}\right)$.

4. (*a*) $R = \sqrt[3]{\dfrac{3V}{4\pi}}$. (*b*) 2·841 in.

5. (*a*) $d = \frac{1}{3}\sqrt[5]{\frac{Ln^2}{H}}$; $d = 3$.

6. $M = \frac{4\pi^2 I}{Ht^2}$; 497·1.

7. (*a*) $x = 2$. (*b*) $x = \frac{1}{11}$. (*c*) $l + 4x = 6$, $x = 8$ in., volume = 2560 cu.in.

8. (*a*) 192. (*b*) $y_1 = -53$, $y_2 = 58$, $y_1^2 - y_2^2 = -555$.

9. (*b*) $\frac{2(4x - x^2 - 1)}{(x-2)(x-3)}$.

10. (*a*) 2·236. (*b*) 2·234. Error 0·089 %.

CHAPTER 17

SIMULTANEOUS EQUATIONS

An equation of the type $x+y = 4$ is not soluble. It merely expresses a relationship between x and y. When some particular value is given to x then the value of y which corresponds is fixed, but unless a particular value is chosen for x or y no answer is possible. Let $x = 1$, then y must be equal to 3, while if $x = 7$, y must be equal to -3. There are an infinite number of values for x and y which will fit the statement, and, as we have seen in Chapter 27, the answers are the co-ordinates of all points which lie on a particular straight line. If, however, another relation between x and y is given simultaneously with the above equation, we have a pair of simultaneous equations which may be solved for x and y. The method of solution of simultaneous equations is to *eliminate* one of the unknowns.

Type 1. *Coefficients of one unknown identical, or differing merely in sign*

Example 1. Solve for x and y: $3x-2y = 4$, $4x-2y = 6$.

$$3x-2y = 4, \qquad (1)$$

$$4x-2y = 6. \qquad (2)$$

The unknown y may be eliminated by subtracting one equation from the other.

Subtracting (1) from (2), we have

$$x = 2.$$

To obtain y, substitute the value obtained for x in either of the two original equations.

Substituting 2 for x in (1), we have

$$6-2y = 4.$$

$$\therefore\ 2y = 2.$$

$$\therefore\ y = 1.$$

Hence $$x = 2, \quad y = 1.$$

As a check, the values obtained for x and y should be substituted in the equation which was not used for the previous substitution; in this case we must use equation (2).

Substituting 2 for x, and 1 for y in (2), we have

$$8-2 = 6,$$

which is correct.

Example 2. Solve for x and y: $5x+7y+17 = 0, y-5x-13 = 0$.

Rewriting the equations with unknowns on the left, and knowns on the right:

$$5x+7y = -17, \tag{1}$$

$$-5x+y = 13. \tag{2}$$

Adding (1) and (2), we have

$$8y = -4,$$

$$\therefore\ y = -\tfrac{1}{2}.$$

Substituting $-\frac{1}{2}$ for y in (1), we have

$$5x-3\tfrac{1}{2} = -17,$$

$$\therefore\ 5x = -\tfrac{27}{2}.$$

$$\therefore\ x = -2{\cdot}7.$$

Hence $\qquad x = -2{\cdot}7, \quad y = -0{\cdot}5.$

[Check: Substituting $-0{\cdot}5$ for y and $-2{\cdot}7$ for x in (2), we have $+13{\cdot}5-0{\cdot}5 = 13$, which is correct.]

Type 2. *Coefficients of one unknown made numerically equal by one multiplication*

Example 3. Solve for x and y: $5x-4y-40 = 0$, $3x-2y = 22$.

$$5x-4y = 40, \tag{1}$$

$$3x-2y = 22. \tag{2}$$

Multiplying (2) by 2 in order that the coefficients of y shall be the same, we have

$$5x-4y = 40, \tag{1}$$

$$6x-4y = 44. \tag{3}$$

Subtracting (1) from (3),

$$x = 4.$$

Substituting 4 for x in (1), we have

$$20-4y = 40,$$

$$\therefore\ 4y = -20.$$

$$\therefore\ y = -5.$$

Hence $$x = 4, \quad y = -5.$$

[Check: Substituting 4 for x, and -5 for y in (2), $12+10 = 22$, which is correct.]

It will be left as an exercise for the student to check the rest of the examples.

Type 3. *Coefficients of one unknown made equal by two multiplications*

Example 4. Solve for x and y: $6x-3y = 5$, $5x+2y = \frac{7}{6}$.

$$6x-3y = 5, \tag{1}$$

$$5x+2y = \tfrac{7}{6}. \tag{2}$$

It is possible to eliminate y by multiplying (1) by 2, and (2) by 3, when the coefficients of y will be -6 and $+6$. (Similarly x could be eliminated, if we chose, by multiplying (1) by 5, and (2) by 6.)

Multiplying (1) by 2, and (2) by 3, we have

$$12x-6y = 10, \tag{3}$$

$$15x+6y = 3\tfrac{1}{2}. \tag{4}$$

Adding (3) and (4),

$$27x = 13\tfrac{1}{2}.$$

$$\therefore\ x = \tfrac{1}{2}.$$

Substituting $\frac{1}{2}$ for x in (1), we have

$$3-3y = 5.$$

$$\therefore\ 3y = -2.$$

$$\therefore\ y = -\tfrac{2}{3}.$$

Hence $$x = \tfrac{1}{2}, \quad y = -\tfrac{2}{3}.$$

[This may be checked by substitution in (2).]

Solution by immediate substitution

All the above equations may be solved by using one equation to write one unknown in terms of the other unknown, and then substituting in the other equation. The latter is then solved as a simple equation.

Example 1 (repeated). Solve for x and y:

$$3x-2y=4, \quad 4x-2y=6.$$

$$3x-2y=4, \tag{1}$$

$$4x-2y=6. \tag{2}$$

From (1)
$$3x=4+2y.$$

$$\therefore\ x=\frac{4+2y}{3}. \tag{3}$$

Substituting $\frac{4+2y}{3}$ for x in (2), we have

$$4\frac{(4+2y)}{3}-2y=6.$$

$$\therefore\ \frac{16}{3}+\frac{8y}{3}-2y=6,$$

or $\quad 16+8y-6y=18 \quad$ (Multiplying throughout by 3).

$$\therefore\ 2y=2.$$

$$\therefore\ y=1.$$

Now, from (3), $\quad x=\frac{4+2y}{3}=\frac{4+2}{3}=2.$

Hence $\quad x=2, \quad y=1.$

Example 4 (repeated). Solve for x and y:

$$6x-3y=5, \quad 5x+2y=\tfrac{7}{6}.$$

$$6x-3y=5, \tag{1}$$

$$5x+2y=\tfrac{7}{6}. \tag{2}$$

From (1) $\quad 6x=5+3y, \quad \therefore\ x=\frac{5+3y}{6}. \qquad (3)$

Substituting $\frac{5+3y}{6}$ for x in (2), we have

$$5\frac{(5+3y)}{6}+2y=\tfrac{7}{6}.$$

$\therefore\ 5(5+3y)+12y = 7$ (Multiplying throughout by 6).

$\therefore\ 25+15y+12y = 7.$

$$27y = -18,$$

$$\therefore\ y = -\tfrac{2}{3}.$$

Now, from (3), $x = \dfrac{5+3y}{6} = \dfrac{5-2}{6} = \dfrac{1}{2}.$

Hence $x = \tfrac{1}{2},\quad y = -\tfrac{2}{3}.$

Examples involving reciprocals

Example. Solve for x and y: $\dfrac{1}{x}+\dfrac{3}{y} = \dfrac{3}{2}$, $\dfrac{5}{x}-\dfrac{4}{y} = \dfrac{7}{6}$.

It will be convenient to consider the unknowns as $\dfrac{1}{x}$ and $\dfrac{1}{y}$, to solve for these, and, finally, to invert the solutions:

$$\left(\frac{1}{x}\right)+3\left(\frac{1}{y}\right) = \frac{3}{2}, \tag{1}$$

$$5\left(\frac{1}{x}\right)-4\left(\frac{1}{y}\right) = \frac{7}{6}. \tag{2}$$

Multiplying (1) by 5, we have

$$5\left(\frac{1}{x}\right)+15\left(\frac{1}{y}\right) = \frac{15}{2}, \tag{3}$$

$$5\left(\frac{1}{x}\right)-4\left(\frac{1}{y}\right) = \frac{7}{6}. \tag{2}$$

Subtracting (2) from (3), we have

$$19\left(\frac{1}{y}\right) = \frac{38}{6}.$$

$$\therefore\ \frac{1}{y} = \frac{1}{3}.$$

Substituting $\tfrac{1}{3}$ for $\dfrac{1}{y}$ in (1), we have

$$\frac{1}{x}+1 = \frac{3}{2}.$$

$$\therefore\ \frac{1}{x} = \frac{1}{2}.$$

Hence $x = 2,\quad y = 3.$

Example. Solve for x and y: $\frac{2}{x+1} - 3y = -\frac{1}{5}$, $\frac{5}{x+1} + 4y = \frac{9}{5}$.

Consider the unknowns first, as $\frac{1}{x+1}$ and y.

$$2\left(\frac{1}{x+1}\right) - 3y = -\frac{1}{5}, \tag{1}$$

$$5\left(\frac{1}{x+1}\right) + 4y = \frac{9}{5}. \tag{2}$$

Multiplying (1) by 5, and (2) by 2, to eliminate $\frac{1}{x+1}$, we have

$$10\left(\frac{1}{x+1}\right) - 15y = -1, \tag{3}$$

$$10\left(\frac{1}{x+1}\right) + 8y = \frac{18}{5}. \tag{4}$$

Subtracting (3) from (4),

$$23y = \tfrac{23}{5}.$$

$$\therefore\ y = \tfrac{1}{5}.$$

Substituting $\frac{1}{5}$ for y in (1),

$$2\left(\frac{1}{x+1}\right) - \frac{3}{5} = -\frac{1}{4},$$

$$\therefore\ 2\left(\frac{1}{x+1}\right) = \frac{2}{5}.$$

$$\therefore\ \frac{1}{x+1} = \frac{1}{5},$$

or $$x + 1 = 5 \quad \text{(By inversion).}$$

$$\therefore\ x = 4.$$

Hence $$x = 4, \quad y = \tfrac{1}{5}.$$

Examples requiring simplification

Example. Solve for x and y:

$$\frac{4+x}{3} - \frac{2+y}{4} = \frac{13}{24}, \quad \frac{2-x}{5} + \frac{y-2}{3} = 0.$$

$$\frac{4+x}{3} - \frac{2+y}{4} = \frac{13}{24}, \tag{1}$$

$$\frac{2-x}{5} + \frac{y-2}{3} = 0. \tag{2}$$

Multiplying (1) by 24, and (2) by 15, we have

$$\left.\begin{aligned} 8(4+x)-6(2+y) &= 13 \\ \text{and}\quad 3(2-x)+5(y-2) &= 0 \end{aligned}\right\}$$

i.e.

$$\left.\begin{aligned} 32+8x-12-6y &= 13 \\ \text{and}\quad 6-3x+5y-10 &= 0 \end{aligned}\right\}$$

i.e.

$$8x-6y = -7, \tag{3}$$

and

$$-3x+5y = 4. \tag{4}$$

Multiplying (3) by 3, and (4) by 8, we have

$$24x-18y = -21, \tag{5}$$

$$-24x+40y = 32. \tag{6}$$

Adding (5) and (6), $22y = 11$.

$$\therefore\ y = \tfrac{1}{2}.$$

Substituting $\frac{1}{2}$ for y in (1), $\dfrac{4+x}{3}-\dfrac{2\frac{1}{2}}{4} = \dfrac{13}{24}$,

$$8(4+x)-6\times 2\tfrac{1}{2} = 13,$$

$$32+8x-15 = 13,$$

$$8x = -4.$$

$$\therefore\ x = -\tfrac{1}{2}.$$

Hence $\underline{x = -\frac{1}{2},\ y = \frac{1}{2}.}$

Examples for Chapter 17

1. Solve for x and y (Type 1):

(*a*) $6x-y = 17,\ 4x+y = 13.$

(*b*) $y+2x = 0,\ 4y = 2x = 5.$

(*c*) $29x-14y = 143,\ 105x-14y = 371.$

(*d*) $4x+5y = \frac{19}{3},\ 6y-4x = 1.$

(*e*) $10x+6y = 31{\cdot}4,\ 11x+6y = 33{\cdot}7.$

(*f*) $\dfrac{3x}{5}-\dfrac{y}{2} = 4,\ \dfrac{3x}{4}+\dfrac{y}{2} = 2\tfrac{3}{4}.$

(*g*) $0{\cdot}9x+3{\cdot}2y = 26{\cdot}9,\ 0{\cdot}9x-7{\cdot}3y+46{\cdot}6 = 0.$

(*h*) $0{\cdot}5x-1{\cdot}2y = 0{\cdot}679,\ 1{\cdot}2y-5x+2{\cdot}254 = 0.$ (*Chap.* 17)

2. Solve for the two unknowns (Type 2):
   - (*a*) $4x - 2y = 6,\ 3x + 6y + 33 = 0.$
   - (*b*) $12s - 11t = 0,\ s + 2t = 35.$
   - (*c*) $6l - 7m = 2\tfrac{7}{15},\ 7l + 14m = 10\tfrac{4}{15}.$
   - (*d*) $12x - 7y + 1{\cdot}7 = 0,\ 9x - 14y + 21{\cdot}4 = 0.$
   - (*e*) $15c + 13d = 2,\ 25c - 26d + \tfrac{1}{3} = 0.$
   - (*f*) $1\tfrac{1}{2}x - 5\tfrac{1}{2}y + 44 = 0,\ 2\tfrac{1}{2}x + 11y = 209.$
   - (*g*) $12x + 11y + 411 = 0,\ 4x + 2y + 102 = 0.$
   - (*h*) $15a + 14b = 57\tfrac{1}{2},\ 30a + 7b = 62\tfrac{1}{2}.$ (*Chap.* 17)

3. Solve for the two unknowns (Type 3):
   - (*a*) $2x - 3y + 4 = 0,\ 3x - 2y + 1 = 0.$
   - (*b*) $16p - 15q = 12,\ 4p = 3q.$
   - (*c*) $21r + 5s = 42,\ 16s - 4r = -8.$
   - (*d*) $0{\cdot}8x + 1{\cdot}2y + 0{\cdot}4 = 0,\ x + y = 0.$
   - (*e*) $2y + 3z = 48,\ 7y - 5z = 13.$
   - (*f*) $5x + 9y = 13\tfrac{1}{5},\ 2x + 13y = 12\tfrac{4}{5}.$
   - (*g*) $2l - 4m = 820,\ 3l + 5m + 200 = 0.$
   - (*h*) $2u = 9v + 47,\ 5v = 7u - 138.$ (*Chap.* 17)

4. (*a*) Solve for x and y: $\dfrac{5}{x} + \dfrac{9}{y} = \dfrac{19}{4},\ \dfrac{1}{x} - \dfrac{2}{y} = 0.$

   (*b*) Solve for p and q: $\dfrac{1}{2+p} + \dfrac{3}{q} = 3{\cdot}25,\ \dfrac{5}{q} - \dfrac{3}{2+p} = 4{\cdot}25.$

   (*c*) Solve for a and b: $\dfrac{1}{2a} - \dfrac{2}{3b} = \dfrac{5}{3},\ \dfrac{5}{2a} + \dfrac{1}{3b} = 1.$ (*Chap.* 17)

5. (*a*) Solve for x and y:

$$\frac{2x+y}{3} + \frac{y-3x}{2} = \frac{5}{6},\quad \frac{x-y}{4} - \frac{2x+y}{5} = -\frac{21}{20}.$$

   (*b*) Solve for s and t: $\dfrac{s-2t}{3t} + \dfrac{1}{3} = -\dfrac{1}{6},\ s + 3t + 7 = 0.$

   (*c*) Solve for p and q: $\dfrac{q-p}{3} + \dfrac{2q+3p}{2} = 11{\cdot}5,\ \dfrac{1}{p} = \dfrac{2}{q}.$

(*Chap.* 17)

6. The velocity, in cm. per second, of sound waves through a gas is given by the formula $v = \sqrt{\frac{\gamma P}{d}}$, where $\gamma = 1{\cdot}4$, P is the pressure in dynes per sq.cm., and d is the density in g. per c.c. Find the velocity of sound through a gas whose density is $1{\cdot}29 \times 10^{-3}$ g. per c.c., and which is under a pressure of $13{\cdot}6 \times 76 \times 981$ dynes per sq.cm. (*Chap.* 10)

7. (*a*) From the formula $R = r\sqrt{\left(\frac{f+p}{f-p}\right)}$, express

(i) f in terms of R, r and p,

(ii) p in terms of R, r and f.

(*b*) Evaluate p when $f = 3500$, $R = 5$, $r = 4$. (*Chap.* 16)

8. (*a*) Solve for x: $\frac{3x-2}{5x} = \frac{13}{25}$. (*Chap.* 14)

(*b*) When a solid body floats in a liquid, the fraction of the volume of the body which is below the surface of the liquid is equal to the ratio of the density of the solid to that of the liquid.

A block of wood floats with $\frac{2}{3}$ of its volume immersed in water of density $62\frac{1}{2}$ lb. per cu.ft. Find the fraction of the wood which would be beneath the surface when the block was floated in sea water of density 64 lb. per cu.ft. (*Chap.* 15)

9. (*a*) Determine the value of π to four significant figures from the following approximations:

(i) $\pi = \frac{22}{7}$, (ii) $\pi^2 = 10$. (*Chaps.* 2 *and* 4)

(*b*) Assuming that the value of π correct to four significant figures is $3{\cdot}142$, find the percentage error in each of the above approximations, giving the answer correct to three significant figures. (*Chap.* 3)

10. Factorise the following expressions:

(*a*) $c^2 - a^2 - 2ab - b^2$.

(*b*) $10x^2 - 29xy + 10y^2$.

(*c*) $1 - x^3 + x - x^4$.

(*d*) $2x^3 + x^2 - 12x + 9$. (*Chaps.* 12 *and* 13)

Answers to Examples for Chapter 17

1. (*a*) $x = 3,\ y = 1.$ (*b*) $x = -\frac{1}{2},\ y = 1.$ (*c*) $x = 3, y = -4.$
(*d*) $x = \frac{3}{4},\ y = \frac{2}{3}.$ (*e*) $x = 2{\cdot}3,\ y = 1{\cdot}4.$
(*f*) $x = 5,\ y = -2.$ (*g*) $x = 5,\ y = 7.$
(*h*) $x = 0{\cdot}35,\ y = -0{\cdot}42.$

2. (*a*) $x = -1,\ y = -5.$ (*b*) $s = 11,\ t = 12.$
(*c*) $l = \frac{4}{5},\ m = \frac{1}{3}.$ (*d*) $x = 1{\cdot}2,\ y = 2{\cdot}3.$
(*e*) $c = \frac{1}{15},\ d = \frac{1}{13}.$ (*f*) $x = 22,\ y = 14.$
(*g*) $x = -15,\ y = -21.$ (*h*) $a = 1\frac{1}{2},\ b = 2\frac{1}{2}.$

3. (*a*) $x = 1,\ y = 2.$ (*b*) $p = -3,\ q = -4.$
(*c*) $r = 2,\ s = 0.$ (*d*) $x = 1,\ y = -1.$
(*e*) $y = 9,\ z = 10.$ (*f*) $x = 1\frac{1}{5},\ y = \frac{4}{5}.$
(*g*) $l = 150,\ m = -130.$ (*h*) $u = 19,\ v = -1.$

4. (*a*) $x = 2,\ y = 4.$ (*b*) $p = 2,\ q = 1.$ (*c*) $a = \frac{3}{2}, b = -\frac{1}{2}.$

5. (*a*) $x = 1,\ y = 2.$ (*b*) $s = -1,\ t = -2.$
(*c*) $p = 3,\ q = 6.$

6. 33,170 cm. per sec.

7. (*a*) (i) $f = p\left(\frac{R^2 + r^2}{R^2 - r^2}\right)$; (ii) $p = f\left(\frac{R^2 - r^2}{R^2 + r^2}\right).$ (*b*) 768·3.

8. (*a*) $x = 5.$ (*b*) $\frac{125}{192}.$

9. (*a*) 3·143; 3·162. (*b*) 0·0318 %; 0·637 %.

10. (*a*) $(c - a - b)(c + a + b).$ (*b*) $(5x - 2y)(2x - 5y).$
(*c*) $(1 + x)(1 - x)(1 + x + x^2).$ (*d*) $(x - 1)(x + 3)(2x - 3).$

CHAPTER 18

PROBLEMS ON SIMULTANEOUS EQUATIONS

Literal equations

The meaning of literal equations has been explained in Chapter 15. We shall apply the rules of simultaneous equations to literal types.

Example. Solve for x and y:

$$3x-4y = 3a-8b, \quad 5x-3y = 5a-6b.$$

$$3x-4y = 3a-8b, \tag{1}$$

$$5x-3y = 5a-6b. \tag{2}$$

Multiplying (1) by 5, and (2) by 3, we have

$$15x-20y = 15a-40b, \tag{3}$$

$$15x-9y = 15a-18b. \tag{4}$$

Subtracting (3) from (4),

$$11y = 22b,$$

$$\therefore\ y = 2b.$$

Substituting $2b$ for y in (1),

$$3x-8b = 3a-8b,$$

$$\therefore\ 3x = 3a.$$

$$\therefore\ x = a.$$

Hence $$x = a, \quad y = 2b.$$

Example. Find the values of x and y in terms of l and m from

$$\frac{3x}{2}-\frac{4y}{5} = \frac{7l+23m}{10} \quad \text{and} \quad \frac{x}{4}-\frac{3y}{2} = \frac{7m-5l}{4}.$$

$$\frac{3x}{2}-\frac{4y}{5} = \frac{7l+23m}{10}, \tag{1}$$

$$\frac{x}{4}-\frac{3y}{2} = \frac{7m-5l}{4}. \tag{2}$$

Multiplying (1) throughout by 10: $15x - 8y = 7l + 23m.$ (3)

Multiplying (2) throughout by 4: $x - 6y = 7m - 5l.$ (4)

Multiplying (4) by 15: $15x - 90y = 105m - 75l.$ (5)

Subtracting (5) from (3): $82y = 82l - 82m.$

$$\therefore\ y = l - m.$$

Substituting $l - m$ for y in (1): $\dfrac{3x}{2} - \dfrac{4(l-m)}{5} = \dfrac{7l + 23m}{10},$

or

$$15x - 8l + 8m = 7l + 23m.$$

$$\therefore\ 15x = 15l + 15m.$$

$$\therefore\ x = l + m.$$

Hence $x = l + m, \quad y = l - m.$

Types of problems

Example. A contractor sent 5 crates and 6 boxes by rail for a short journey, the total charge being 54*s*. The next week 8 crates and 5 boxes were carried for 62*s*. 3*d*. Find the charge for a box and for a crate.

Let x be the charge for one crate, and y be the charge for one box, both in shillings.

Then $$5x + 6y = 54, \tag{1}$$

and $$8x + 5y = 62\tfrac{1}{4}. \tag{2}$$

Multiplying (1) by 5, and (2) by 6:

$$25x + 30y = 270, \tag{3}$$

$$48x + 30y = 373\tfrac{1}{2}. \tag{4}$$

Subtracting (3) from (4): $23x = 103\frac{1}{2},$

$$\therefore\ x = 4\tfrac{1}{2}.$$

Substituting $4\frac{1}{2}$ for x in (1): $22\frac{1}{2} + 6y = 54.$

$$\therefore\ 6y = 54 - 22\tfrac{1}{2}$$

$$= 31\tfrac{1}{2}.$$

$$\therefore\ y = 5\tfrac{1}{4}.$$

Hence, charge for one crate = 4*s*. 6*d*., charge for one box = 5*s*. 3*d*.

Equation of a straight line, given co-ordinates of two points on the line. It has been shown in Chapter 27 that the equation of any straight line may be written in the form $y = mx + c$, where m and c are constants. It happens frequently in science that the relation between two variables is linear, that is, if values of one variable are plotted vertically, and values of the other variable horizontally, all the points will fall on a straight line. When only two points on the line are known, the equation of the line may be found by a pair of simultaneous equations.

Example. Find the equation of the straight line which passes through the points $(-1, 3)$ and $(2, -4)$.

Let the equation of the line be $y = mx + c$, where m and c are to be determined.

Then substituting -1 and 3 for x and y we obtain one equation in m and c, and substituting 2 and -4 for x and y we obtain a second equation in m and c:

$$3 = -1m + c, \tag{1}$$

$$-4 = 2m + c. \tag{2}$$

Subtracting (1) from (2): $-7 = 3m$.

$$\therefore\ m = -\tfrac{7}{3}.$$

Substituting $-\frac{7}{3}$ for m in (1): $3 = \frac{7}{3} + c$.

$$\therefore\ c = 3 - \tfrac{7}{3}$$

$$= \tfrac{2}{3}.$$

Hence, the equation of the line is $y = -\frac{7}{3}x + \frac{2}{3}$,

or $$\underline{3y + 7x - 2 = 0.}$$

[The problem may also be solved by the method of Chapter 27.] (Illustrated in fig. 31.)

The linear law in mechanics. Experiments on many simple machines reveal that the effort required to raise a load and the load itself are related by a linear law of the form

$$\text{Effort} = m \times \text{Load} + c,$$

where m and c are constants. Consider the simple pulley system of fig. 32. If the weight of the lower pulley could be neglected, the principles of mechanics show that Effort = $\frac{1}{2}$ Load. In practice the law is Effort = $\frac{1}{2}$ Load + constant, this constant being due to the weight of the lower pulley block.

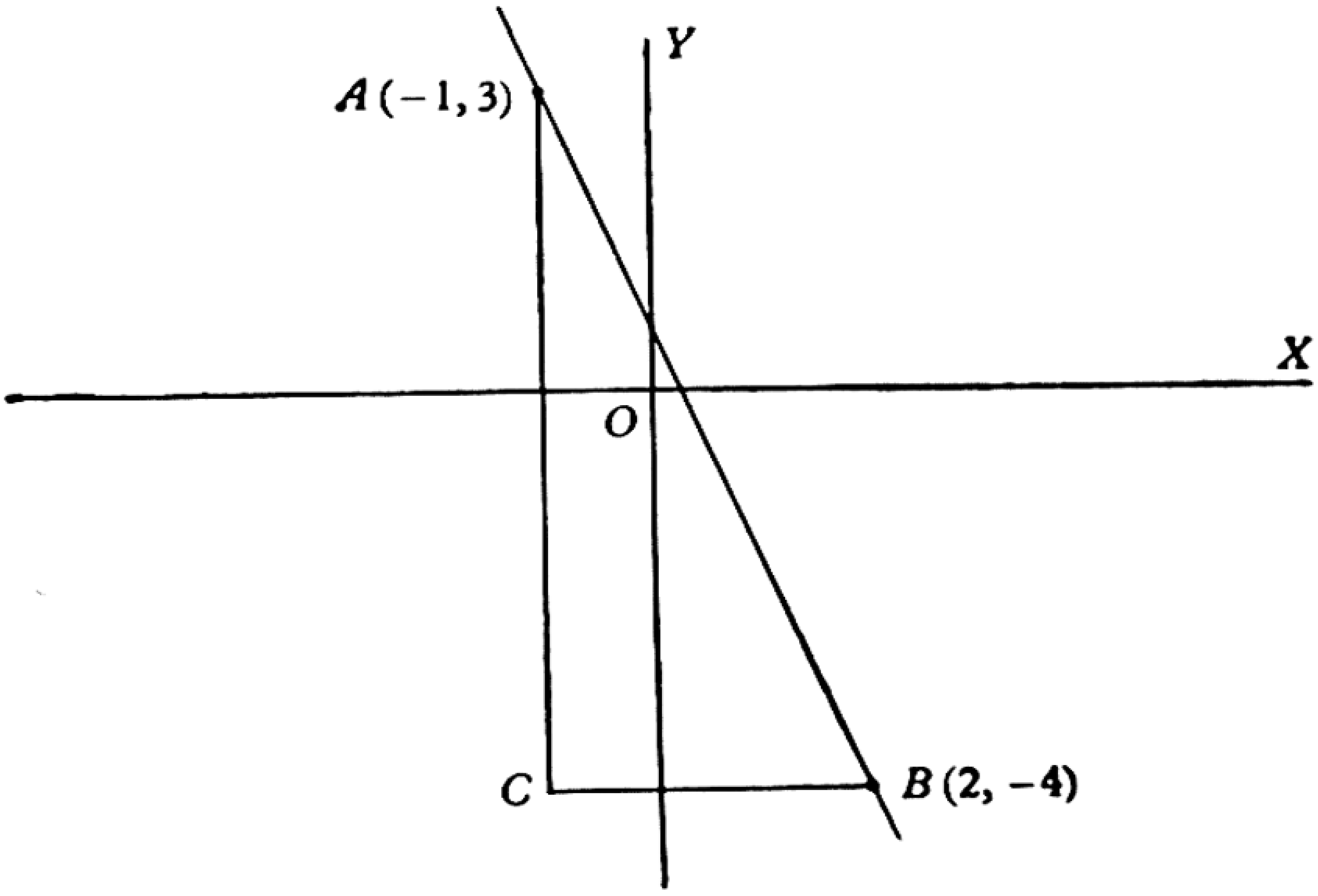


Fig. 31

Example. The law of a machine is of the form $P = aW + b$, where P represents the effort in lb. wt., W the load in lb wt., and a and b are constants. It is found that an effort of 4·3 lb. is required to raise a load of 20 lb. wt., and an effort of 6·3 lb. wt. is required to raise a load of 36 lb. wt. Find the law of the machine, and from this law determine the effort required to raise a load of 50 lb. wt.

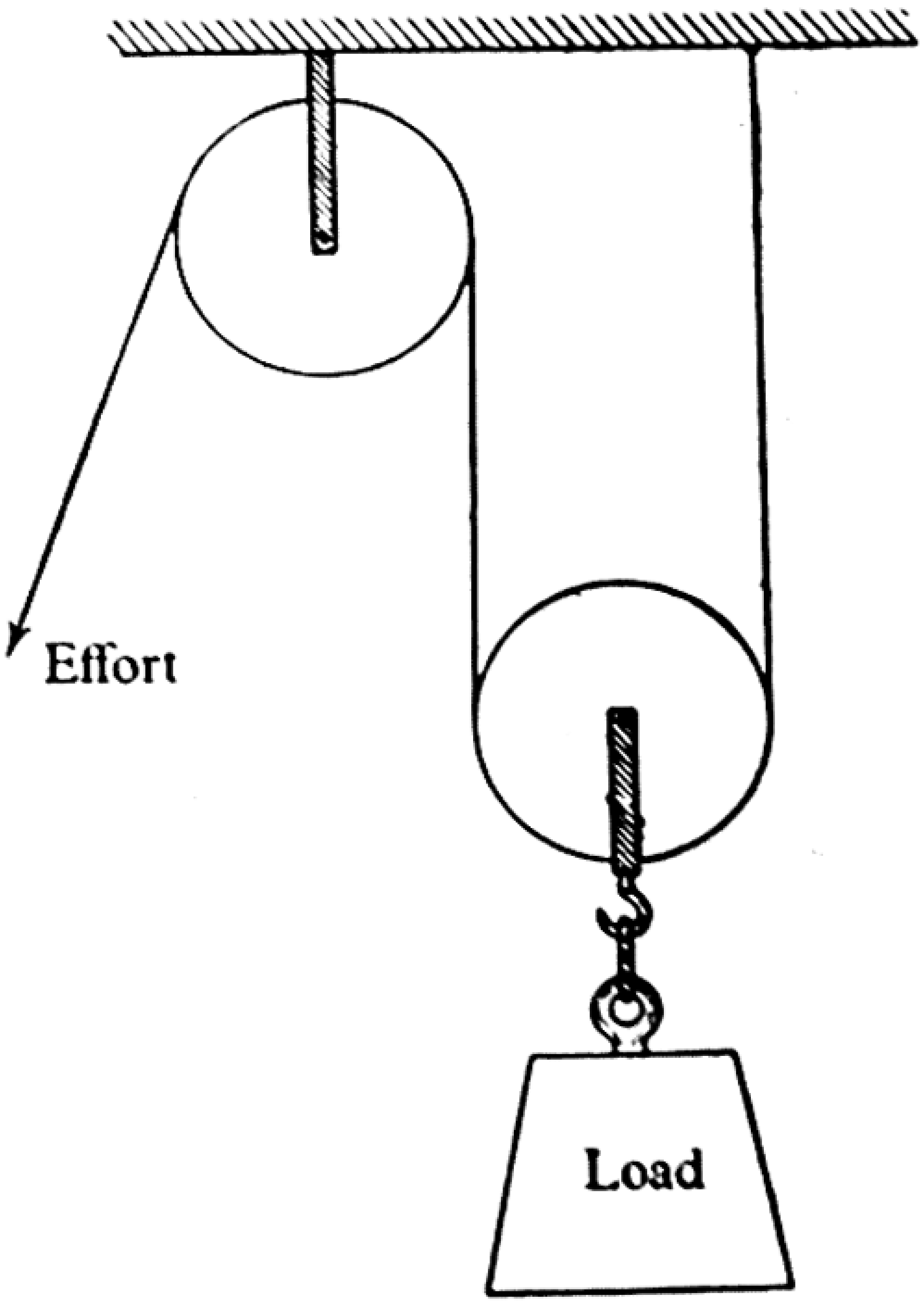


Fig. 32

$$P = aW + b,$$

when $P = 4{\cdot}3$, $W = 20$, and when $P = 6{\cdot}3$, $W = 36$.

Hence $4{\cdot}3 = 20a + b$, (1)

and $6{\cdot}3 = 36a + b$. (2)

Subtracting (1) from (2):

$$2 = 16a.$$

$$\therefore\ a = \tfrac{1}{8} \text{ or } 0{\cdot}125.$$

Substituting 0·125 for a in (1): $4{\cdot}3 = 2{\cdot}5 + b$.

$$\therefore\ b = 1{\cdot}8.$$

Hence the law of the machine is $P = 0{\cdot}125W + 1{\cdot}8$.

Writing 50 for W, we have
$$\begin{aligned} P &= 0{\cdot}125 \times 50 + 1{\cdot}8 \\ &= 6{\cdot}25 + 1{\cdot}8 \\ &= 8{\cdot}05. \end{aligned}$$

Thus, the effort required to raise a load of 50 lb. wt. is 8·05 lb.

[Students will realise that this is fundamentally the same type of problem as the previous one, that is, to find the equation of a line in the form $P = aW + b$ given two points (20, 4·3) and (36, 6·3) on the line.]

Electrical example of linear law. When wires made of such substances as copper, manganin or platinum are heated, their electrical resistance increases. For small rises of temperature, the following relationship holds good: $R_t = R_0(1 + \alpha t)$, where R_t is the resistance of the wire at a temperature t° C., R_0 is the resistance of the same wire at 0° C., t is the temperature of the wire corresponding to R_t, and α is the *temperature coefficient* of resistance of the metal of the wire, given as a fraction per degree Centigrade. In order to determine the value of α for a metal, the resistance of a coil of wire of that metal is measured at two known temperatures. It is not usually convenient to measure its resistance at 0° C., so the constant R_0 may be eliminated (and determined later if necessary) by two equations.

Example. A coil of copper wire was found to have a resistance of 8·57 ohms at 100° C., and a resistance of 7·28 ohms at 50° C. Assuming the law $R_t = R_0(1 + \alpha t)$, find the temperature coefficient of resistance of copper:

$$R_t = R_0(1 + \alpha t).$$

Substituting values given for R_t and t, we have two equations,

$$8{\cdot}57 = R_0(1 + 100\alpha), \tag{1}$$

$$7{\cdot}28 = R_0(1 + 50\alpha). \tag{2}$$

To eliminate R_0, it is convenient to divide (1) by (2).

$$\therefore\ \frac{8{\cdot}57}{7{\cdot}28} = \frac{1 + 100\alpha}{1 + 50\alpha}.$$

Cross-multiplication gives $8{\cdot}57(1+50\alpha) = 7{\cdot}28(1+100\alpha)$,

$$8{\cdot}57+428{\cdot}5\alpha = 7{\cdot}28+728\alpha.$$

$$\therefore\ 1{\cdot}29 = 299{\cdot}5\alpha.$$

$$\therefore\ \alpha = \frac{1{\cdot}29}{299{\cdot}5} = 0{\cdot}0043.$$

Hence, temperature coefficient of resistance of copper

$$= 0{\cdot}0043 \text{ per } ^\circ\text{C}.$$

Example on dynamics

An aircraft flying with the wind makes a journey of 260 miles in 1 hr. 44 min. On the return journey the time taken is 2 hr. Find the average speed of the aircraft in still air, and find the speed of the wind.

Let v (m.p.h.) be the velocity of the aircraft in still air, and let u be the velocity of the wind.

Since distance = velocity × time, we have two equations:

$$(v+u)\,1\tfrac{11}{15} = 260, \tag{1}$$

$$(v-u)\,2 = 260, \tag{2}$$

i.e. $$\frac{26v}{15}+\frac{26u}{15} = 260 \quad\text{or}\quad 26v+26u = 3900, \tag{3}$$

and $$2v-2u = 260. \tag{4}$$

Multiplying (4) by 13 and adding, we have

$$52v = 3900+260\times 13$$

$$= 7280.$$

$$\therefore\ v = \frac{7280}{52} = 140.$$

Substituting 140 for v in (2), we have

$$(140-u)\,2 = 260,$$

$$280-2u = 260.$$

$$\therefore\ u = 10.$$

Hence, speed of aircraft in still air = 140 m.p.h., and speed of wind = 10 m.p.h.

Examples for Chapter 18

1. (*a*) Solve for x and y: $2x - y = b - 4a$, $5x + y = 6b - 10a$.

(*b*) The cost of 3 tons of coal and 4 tons of slack was £23. 15*s*., while the cost of 5 tons of coal and $3\frac{1}{2}$ tons of slack was £28. 10*s*. Find the cost of 1 ton of coal, and of 1 ton of slack. (*Chap.* 18)

2. (*a*) Solve for x and y:

$$\frac{3x}{2} - y = \frac{8l - k}{2}, \quad 5x + \frac{y}{3} = \frac{17k + 29l}{3}.$$

(*b*) Find the equation of the straight line which passes through the points (2, 4) and (– 1, – 1).

(*c*) A straight line cuts the axis of y at the point where $y = 2$, and also cuts the axis of x at 1 unit to the left of the origin. Find its equation. (*Chap.* 18)

3. (*a*) Solve for x and y $\left(\text{solve first for } \frac{1}{x} \text{ and } \frac{1}{y}\right)$:

$$\frac{1}{x} + \frac{2}{y} = \frac{3a + b}{(a^2 - b^2)}, \quad \frac{3}{x} - \frac{2}{y} = \frac{a - 5b}{(a^2 - b^2)}.$$

(*b*) The law of a machine is of the form $P = aW + b$, where a and b are constants.
When $W = 28$, $P = 7$ and when $W = 52$, $P = 11{\cdot}8$ (all values in lb. wt.). Find the law of the machine, and hence determine the value of P when $W = 60$. (*Chap.* 18)

4. (*a*) The electrical resistance of a coil of wire is found to be 5·6 ohms at a temperature of 20° C., while its resistance is 6·78 ohms at 80° C. Find the temperature coefficient of resistance of the metal, and the resistance of the coil at 0° C. (See example in text.)

(*b*) Find the equations of the straight lines which pass through (i) (1, 5) and (4, 14), (ii) (– 4, 3) and (8, 0), and show that they intersect on the axis of y. (*Chap.* 18)

5. (*a*) The weight of fuel consumed per hour (W lb.), and the brake horse-power (H) of an engine are related by the law

$W = a + bH$, where a and b are constants. When $H = 4$, $W = 3{\cdot}2$, and when $H = 5{\cdot}8$, $W = 4{\cdot}46$. Find the law connecting weight of fuel consumed per hour, and horse-power.

(*b*) An aircraft flying with the wind completes a journey of 180 miles in 1 hr. 12 min. When flying against the wind the return journey takes 2 hr. Find the average speed of the aircraft in still air and the speed of the wind. (*Chap.* 18)

6. (*a*) When a body is travelling with constant acceleration (f), its velocity (v) at a time t sec. after passing a given point is $v = u + ft$, where u is the velocity with which it passed the fixed point. The velocity of a particular body 4 sec. after passing a certain point was 30 ft. per sec., and 6 sec. later its velocity was 42 ft. per sec. Find its acceleration, in feet per second per second, and its velocity at the beginning of the timed period.

(*b*) For the same type of motion $s = ut + \frac{1}{2}ft^2$, where s is the distance travelled from a fixed point in t sec., and u has the same meaning as above.

In a certain case $s = 200$ when $t = 4$, and $s = 318$ when $t = 6$. Find u and f. (*Chap.* 18)

7. (*a*) A formula for the transverse vibrations of strings is $n = \frac{1}{2l}\sqrt{\frac{T}{M}}$. Transform the formula to give M in terms of n, l and T.

(*b*) Evaluate M when $T = 9 \times 10^5$, $n = 50$, $l = 80$.

(*Chap.* 16)

8. (*a*) Solve for a from the formula $s = \frac{n}{2}\{2a + (n-1)d\}$, when $s = 84$, $n = 7$, $d = 3$.

(*b*) When the price of an article was raised by 10*s*., it was found that the new turnover for a sale of 100 articles was £20 less than the turnover on the sale of 120 articles at the original price. Find the original price of each article. (*Chap.* 14)

9. (*a*) A casting is in the shape of a prism whose base is a square of side a in., with the area of a quarter circle, radius x in.,

cut from each corner of the square. Another casting has the same length, and a square base, and has the same volume as the other prism. Write an expression for the length of one side of the square

(*b*) Find the length of the side of the square given $a = 20$ in., $x = 7$ in. and $\pi = \frac{22}{7}$. (*Chap.* 5)

10. (*a*) If $3 = 10^{0\cdot4771}$ and $5 = 10^{0\cdot6990}$, write 75 as a power of 10, and find, without tables, log 0·6, log 1·667 and log $0\cdot\dot{5}$.

(*b*) A cone of height h, radius r, is melted down and recast into a sphere of radius R. Find an expression for R in terms of r and h, and evaluate R when $r = 3$, $h = 1\frac{1}{2}$. (*Chaps.* 10 *and* 16)

Answers to Examples for Chapter 18

1. (*a*) $x = b-2a$, $y = b$. (*b*) Coal £3. 5*s*., slack £3. 10*s*.

2. (*a*) $x = k+2l$, $y = 2k-l$. (*b*) $3y = 5x+2$.
(*c*) $y-2x-2 = 0$.

3. (*a*) $x = (a+b)$, $y = (a-b)$.
(*b*) $P = 0\cdot2W+1\cdot4$, $P = 13\cdot4$ lb. wt.

4. (*a*) $\alpha = 0\cdot0038$ per ° C., $R_0 = 5\cdot2$ ohms.
(*b*) (i) $y = 3x+2$, (ii) $y = -\frac{x}{2}+2$. Intersect at (0, 2).

5. (*a*) $W = 0\cdot4+0\cdot7H$.
(*b*) Speed of aircraft = 120 m.p.h., speed of wind = 30 m.p.h.

6. (*a*) $f = 2$ ft. per sec. per sec., $u = 22$ ft. per sec.
(*b*) $f = 3$ ft. per sec. per sec., $u = 44$ ft. per sec.

7. (*a*) $M = \frac{T}{4n^2l^2}$. (*b*) $M = 0\cdot014$.

8. (*a*) $a = 3$. (*b*) £3. 10*s*.

9. (*a*) Length of side $= \sqrt{(a^2-\pi x^2)}$ in. (*b*) 15·7 in.

10. (*a*) $10^{1\cdot8751}$, $\bar{1}\cdot7781$, 0·2219, $\bar{1}\cdot7448$.
(*b*) $R = \sqrt[3]{\frac{r^2h}{4}}$; $R = \frac{3}{2}$.

CHAPTER 19

VARIATION

Direct variation

Proportionality has been discussed in Chapter 3. Now we have seen that science frequently exhibits cases of two variable quantities such that doubling the numerical value of one of these variables automatically results in the doubling of the other. For instance, it is proved in mechanics that if a spring is loaded with weights, then increase of the load results in a proportionate extension of the spring. Similarly, the volume of a gas is directly proportional to the absolute temperature under constant conditions of pressure. When a change in the value of one variable produces a proportional change in the value of a related variable, the variables are said to vary directly.

A simple electrical experiment is illustrated in fig. 33. The two instruments labelled V and A are a voltmeter and an ammeter respectively, to measure the voltage across the fixed resistance, and to measure the current flowing through it. Both of these quantities may be changed by means of the variable resistance. Readings of current in amperes, and volts are shown in the table:

Potential difference across coil in volts (V)	4	3	2·5	2	1·5	1
Current through coil in amperes (I)	0·8	0·6	0·5	0·4	0·3	0·2

It is easily seen that when the voltage is doubled, either from 1 to 2 volts, or from 2 to 4 volts, the current is doubled. In fact the ratio voltage/current is 5 in every case. Thus, the experiment shows that under the given conditions, current is proportional to voltage, or current varies directly as voltage. This is written mathematically as Current $\propto$ Voltage (where $\propto$ means 'varies directly as' or 'is directly proportional to'), or $I \propto V$. Similarly $V \propto I$. Now we have seen that V/I gives a constant ratio, so the statement may also be written as $V/I =$ constant, or $V = KI$

(where K is a constant). In this particular experiment the constant is called the *resistance* of the coil of wire, and the relationship $V = IR$ is called Ohm's Law. (The resistance, however, may not remain constant when the coil becomes heated.)

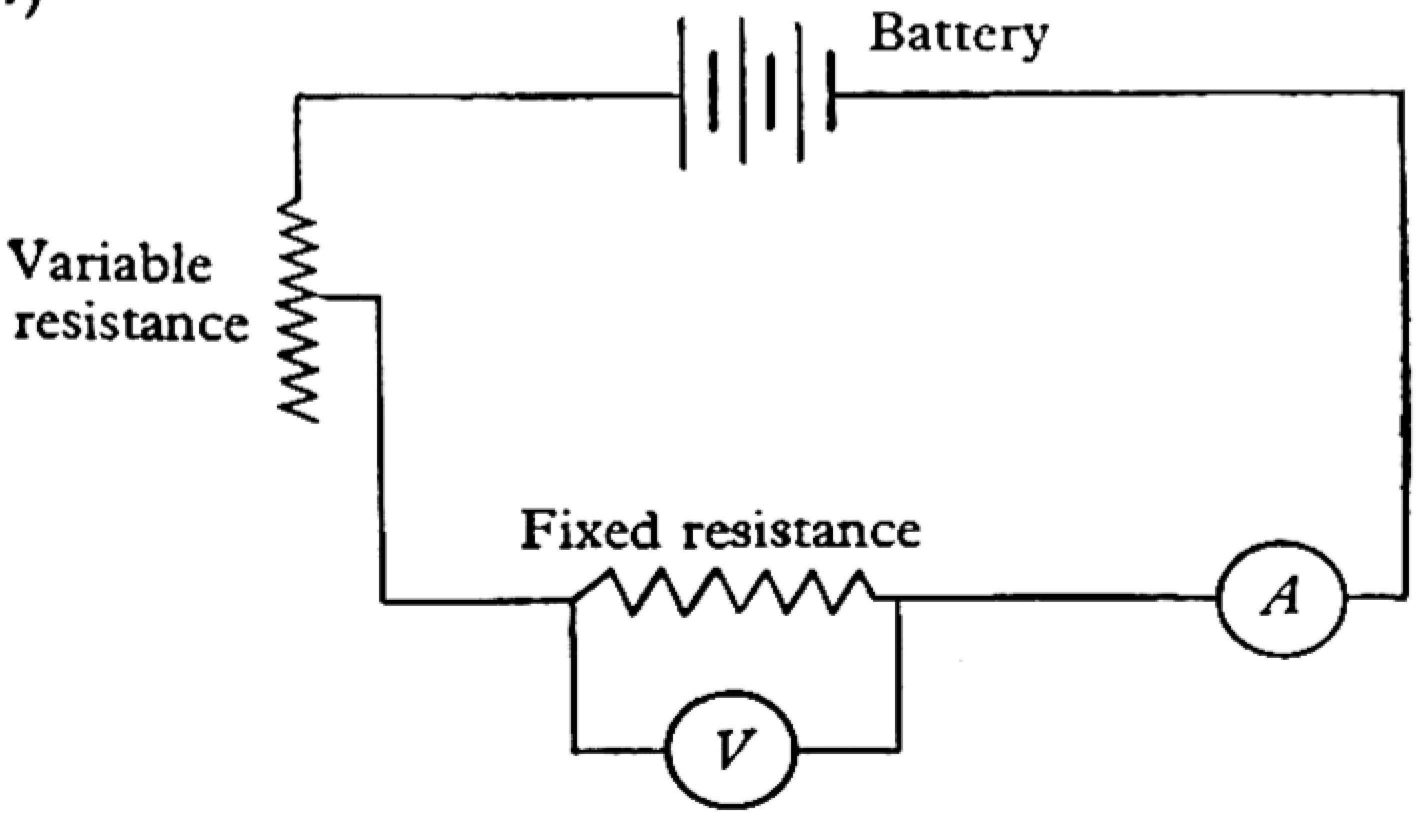


Fig. 33

Assuming the truth of Ohm's Law, consider the following example: The voltage (potential difference) across the ends of a coil of wire is 5·5 volts when the current is 0·5 amp.; find the current when the potential difference is 4·2 volts.

Given $$V \propto I.$$

$$\therefore\ V = KI \quad \text{(where } K \text{ is constant).}$$

In the given case $$5{\cdot}5 = K \times 0{\cdot}5.$$

$$\therefore\ K = \frac{5{\cdot}5}{0{\cdot}5} = 11.$$

Thus $V = 11I$ is the law for this particular coil.

Thus $4{\cdot}2 = 11I$ in the second case, or $I = \frac{4{\cdot}2}{11} = 0{\cdot}38$ amp.

Current through the coil for a potential difference of 4·2 volts = 0·38 amp.

Variation problems may all be worked in this manner. First write a mathematical statement for the proportionality, then replace the variation sign by a sign of equality and a constant. When the value of the constant has been determined from the given conditions, the problem is easily solved.

Example. The extension of a spring is directly proportional to the load producing it. It is found that a weight of 3 lb.

produces an extension of $4\frac{1}{2}$ in. Find the extension produced by a load of 5 lb. and find also what load would be required to produce an extension of 5 in.

Extension $\propto$ load (Statement given in the problem)

or $$e \propto W.$$

$$\therefore\ e = KW \quad \text{(where } K \text{ is a constant).} \qquad (1)$$

Hence $$4{\cdot}5 = K\,.\,3 \quad \text{(Using given conditions).}$$

$$\therefore\ K = \frac{4{\cdot}5}{3} = 1{\cdot}5.$$

To find extension produced by a load of 5 lb., substitute 5 for W in statement (1), and use the value of K determined, i.e.

$$e = 1{\cdot}5 \times 5$$
$$= 7{\cdot}5.$$

Hence, the extension produced by a load of 5 lb. is 7·5 in.

To find the load required to produce an extension of 5 in., substitute 5 for e in (1) and use the same value of K, i.e.

$$5 = 1{\cdot}5 \times W.$$

$$\therefore\ W = \frac{5}{1{\cdot}5} = 3\tfrac{1}{3}.$$

Hence, the load required to produce an extension of 5 in. would be $3\frac{1}{3}$ lb.

The method is the same when one quantity varies as the square, or cube, or any power of the related quantity.

Example. The central force on a rotating body is directly proportional to the square of its velocity. When the velocity is 10 ft. per sec., the central force is 125 poundals. Find the central force when the velocity is 12 ft. per sec., and find the velocity which would produce a central force of 200 poundals:

$$\text{Central force} \propto (\text{velocity})^2,$$

$$F \propto V^2.$$

$$\therefore\ F = KV^2, \text{ where } K \text{ is constant.} \qquad (1)$$

For the instance given, $125 = K \, . \, 100$.

$$\therefore \; K = \tfrac{125}{100} = \tfrac{5}{4}.$$

Substituting 12 for V in (1) and using $\frac{5}{4}$ for K:

$$\begin{aligned} F &= \tfrac{5}{4} \, . \, 144 \\ &= 180. \end{aligned}$$

Thus, central force from velocity of 12 ft. per sec. = 180 poundals.

To find the velocity which would give a central force of 200 poundals, substitute 200 for F in (1), using the value of K already determined.

Then $$200 = \tfrac{5}{4}v^2.$$

$$\begin{aligned} \therefore \; v^2 &= \frac{200 \times 4}{5} \\ &= 160. \end{aligned}$$

Hence $$v = \sqrt{160} = 12{\cdot}65 \text{ (approx.)}.$$

Thus, the velocity required to produce a central force of 200 poundals is 12·65 ft. per sec.

Inverse variation

Boyle's Law (Chapter 3) provides an example of a different type of variation or proportionality. Increase of pressure on a cylinder of gas produces a decrease in volume. When the pressure is doubled, the volume is halved. Consider the table below:

Pressure on gas P (lb. per sq.in.)	15	20	24	30	36	40	A
Volume of gas V (cu.ft.)	12	9	7·5	6	5	4·5	B

It is evident from the first and fourth columns that the doubling of the pressure produces a halving of the volume. It is also evident that the product of the values of P and V in each column is equal to 180 in every case, so a convenient method of expressing Boyle's Law is PV = constant.

Let us write a line of reciprocals of the values of V:

$$1/V \quad \tfrac{1}{12} \quad \tfrac{1}{9} \quad \tfrac{2}{15} \quad \tfrac{1}{6} \quad \tfrac{1}{5} \quad \tfrac{2}{9} \quad \text{C}$$

Considering lines A and C we find that P and $\frac{1}{V}$ are directly proportional, thus

$$15 \div \tfrac{1}{12} = 20 \div \tfrac{1}{9} = 24 \div \tfrac{2}{15} = 180.$$

It follows, then, that our inverse proportionality may be written as $P \propto \frac{1}{V}$ or $P = \frac{K}{V}$. This is also obvious from the previous statement $PV =$ constant, for if $PV = K$, then $P = \frac{K}{V}$, or

$$V = \frac{K}{P}.$$

We have now a means of writing a mathematical statement for inverse variation. If a quantity x is known to vary inversely as a quantity y, then $x \propto \frac{1}{y}$, i.e. $x = \frac{K}{y}$, or $y = \frac{K'}{x}$.

Example. A mass of gas occupies 25 cu.ft. at atmospheric pressure (15 lb. per sq.in.). Find its volume at a pressure of 60 lb. per sq.in., and find the pressure necessary to compress it to a volume of 15 cu.ft., assuming no change of temperature occurs in either case.

$$P \propto \frac{1}{V}.$$

$$\therefore\ P = \frac{K}{V}. \qquad \textbf{(1)}$$

From the given conditions $15 = \frac{K}{25}$.

$$\therefore\ K = 15 \times 25.$$

Substituting 60 for P, and 15×25 for K in **(1)**:

$$60 = \frac{15 \times 25}{V}.$$

$$\therefore\ V = \frac{15 \times 25}{60}$$

$$= 6{\cdot}25.$$

Thus the volume is 6·25 cu.ft. when the pressure is 60 lb. per sq.in.

Substituting 15 for V, and 15×25 for K in (1),

$$P = \frac{15 \times 25}{15} = 25.$$

Thus, the pressure necessary to compress the gas to a volume of 15 cu.ft. is 25 lb. per sq.in.

Example. The force of repulsion between two magnetic poles varies inversely as the square of the distance between them. When their distance apart is 10 cm., the force is 6·25. Find their distance apart when the force is 9, and find the force when their distance apart is 12 cm.

$$\text{Force} \propto \frac{1}{(\text{Distance})^2}.$$

$$\therefore \; F = \frac{K}{d^2}. \qquad (1)$$

From the given facts: $6{\cdot}25 = \frac{K}{100}$. $\therefore \; K = 625$.

Substituting 9 for F, and 625 for K in (1):

$$9 = \frac{625}{d^2}.$$

$$\therefore \; d^2 = \tfrac{625}{9}. \quad \therefore \; d = \tfrac{25}{3} = 8\tfrac{1}{3} \text{ cm.}$$

Hence, the distance apart for a force of 9 units is $8\frac{1}{3}$ cm.

Also, substituting 12 for d in (1), with K as before,

$$F = \tfrac{625}{144}.$$

$$\therefore \; F = 4{\cdot}34.$$

Hence the force, when the distance is 12 cm., is 4·34 units.

Examples for Chapter 19

1. (*a*) If x varies directly as y, and $y = 6$ when $x = 9$, find (i) y when $x = 6$, (ii) x when $y = 11$.

(*b*) The pressure in a liquid is directly proportional to the depth. When the depth is 10 ft., the pressure is 625 lb. per sq.ft. Find the pressure at a depth of 18 ft., and find the depth when the pressure is 600 lb. per sq.ft. (*Chap.* 19)

2. (*a*) When the potential difference across the ends of a coil is 8·8 volts, the current in the coil is 0·8 amp. Find the current when the potential difference is reduced to 4·9 volts. (See example in text.)

(*b*) The extension of a spring is directly proportional to the load producing it. For a load of 50 g. the extension is 3·2 cm. Find the extension for a load of 83·7 g. (*Chap.* 19)

3. (*a*) The volume of a gas is proportional to its absolute temperature (where absolute temperature is equal to ° C. + 273). When the temperature is 15° C., the volume is 2000 c.c. Find the volume when the temperature is raised to 85° C.

(*b*) The heat produced by an electric current in a wire in equal intervals of time is proportional to the square of the current flowing in the wire. When the current was 1·5 amp. the heat produced was 980 calories. Find the heat produced when the current is 2·5 amp. (*Chap.* 19)

4. (*a*) The pressure on 45 c.c. of air is 76 cm. of mercury. The air is compressed in accordance with Boyle's Law until its volume is 32 c.c. Find the pressure of the gas. (See example in text.)

(*b*) The force exerted between two electric charges is inversely proportional to the square of their distance apart. A force of 590 dynes is exerted between two charges 12 cm. apart. Find the force when the distance is 9 cm. (*Chap.* 19)

5. (*a*) The quantity I varies as d^3, and $I = 240$ when $d = 2{\cdot}9$. Find I when $d = 3{\cdot}4$.

(*b*) The deflecting force on a tangent magnetometer varies as the tangent of the angle of deflection. When the force is 25 units, the angle is 45°. Find the force when the angle is 60°. (*Chap.* 19)

6. (*a*) If s is inversely proportional to the square root of t, and if $s = 28$ when $t = 64$, find (i) s when $t = 81$, (ii) t when $s = 60$.

(*b*) If l varies inversely as the square of θ, and $l = 520$ when $\theta = 28$, find θ when $l = 450$. (*Chap.* 19)

7. (a) Solve for x and y:

$$3(x+y)-2(x-y)+2=0, \quad 5(x-2y)+4(2x-y)=53.$$

(b) Find the law of a machine in the form $E = aW + b$ from the figures $E = 11 \cdot 7$ when $W = 112$, and $E = 3 \cdot 3$ when $W = 28$. (*Chaps.* 17 *and* 18)

8. From the compound interest formula $A = P\left(1+\frac{r}{100}\right)^n$, show that $r = 100\left(\sqrt[n]{\frac{A}{P}}-1\right)$. (*Chap.* 16)

Evaluate r when $n = 3$, $A = 265 \cdot 3$, $P = 250$. (*Chap.* 9)

9. (a) Factorise: (i) $g^2-h^2-2hk-k^2$.

(ii) $6x^2-19xy+10y^2$.

(iii) a^2b-b+a^2c-c. (*Chap.* 12)

(b) Find the factor which is common to the three expressions, x^3-27, $2x^2-3x-9$, x^3-2x^2-4x+3. (*Chap.* 13)

10. A train weighs W tons. The locomotive is exerting a force equal to x lb. wt. per ton of train when travelling at v m.p.h. The work done per second is the product of the force exerted in lb. wt. and the distance in feet travelled per second. The horse-power developed by the locomotive is expressed as $\frac{\text{work per second}}{550}$.

Find an expression for the horse-power in the above case, and evaluate it when $x = 20$, $W = 500$, $v = 30$. (*Chap.* 5)

Answers to Examples for Chapter 19

1. (a) $y = 4$, $x = 16\frac{1}{2}$. (b) 1125 lb. per sq.ft., 9·6 ft.

2. (a) 0·45 amp. (b) 5·36 cm.

3. (a) 2486 c.c. (b) 2722 calories.

4. (a) 107 cm. of mercury (approx.).

(b) 1049 dynes (approx.).

5. (*a*) 386·8. (*b*) 43·3 units.

6. (*a*) $s = 24{\cdot}9$, $t = 13{\cdot}94$. (*b*) $\theta = 30{\cdot}1$.

7. (*a*) $x = 3$, $y = -1$. (*b*) $E = 0{\cdot}1W + 0{\cdot}5$.

8. $r = 2$.

9. (*a*) (i) $(g-h-k)(g+h+k)$.
   (ii) $(3a-2y)(2x-5y)$.
   (iii) $(a-1)(a+1)(b+c)$.
   (*b*) $x-3$.

10. $\text{H.P.} = \dfrac{Wxv}{375}$, 800.

CHAPTER 20

QUADRATIC EQUATIONS

We have not yet attempted the solution of an equation which contains terms in x^2. Consider the equation $x^2 = 1$. It is satisfied by $x = 1$, and by $x = -1$, since the square of either of these quantities is $+1$. In general we must expect to find two solutions, or 'roots', to an equation of the second degree in x. Such equations of the second degree are called quadratic equations. In this chapter we shall examine two methods for their solution.

Solution by factors

The equation above may be written $x^2 - 1 = 0$. In this form, the left-hand side will factorise into

$$(x-1)(x+1) = 0.$$

Now if we can find a value of x which will make one of these brackets zero, then the equation is satisfied, and that chosen value of x is a root, or solution, of the equation, for no matter what is the numerical value of the other bracket, the fact of one bracket being zero balances the equation. Writing $+1$ for x, the first bracket vanishes, $[(1-1) = 0]$, so $+1$ is a root of the equation. Also, writing -1 for x, the second bracket vanishes, so -1 is a root of the equation. Hence $x = 1$ and $x = -1$ are the answers.

Quadratic equations should be written first with right-hand side zero, as above. The left-hand side is examined, and, if it will factorise, the roots are obtained as in the example just completed.

Consider $\qquad x^2 + 2x + 1 = 0.$

Factorising $\qquad (x+1)(x+1) = 0.$

The value -1 for x will make both brackets equal to zero. Thus -1 is the answer. Because there are two factors, an equation of this type is said to have two equal roots, -1 and -1.

Example. Solve for x: $x^2 - 5x + 6 = 0$.

$$x^2 - 5x + 6 = 0.$$

Factorising $(x-3)(x-2) = 0.$

$$\therefore \quad x = 3 \quad \text{and} \quad x = 2.$$

Check: Writing 3 for x in the original equation, we have

$$9-15+6 = 0, \quad \text{which is correct.}$$

Writing 2 for x in the original equation, we have

$$4-10+6 = 0, \quad \text{which is correct.}$$

Example. Solve for x: $x^2-5x = 6$.

$$x^2-5x-6 = 0.$$

Factorising $(x-6)(x+1) = 0.$

$$\therefore \quad x = 6 \quad \text{and} \quad x = -1.$$

Fractional roots

Example. Solve for x: $2x^2-7x+6 = 0$.

Factorising $(2x-3)(x-2) = 0.$

If we write $+3$ in place of $2x$ in the first bracket, the equation is satisfied, i.e. $x = \frac{3}{2}$. The second bracket disappears when $x = 2$.

Hence the roots are $x = \frac{3}{2}$ and $x = 2$.

Example. Solve for x: $6x^2-5x = 4$.

$$6x^2-5x-4 = 0.$$

Factorising $(3x-4)(2x+1) = 0.$

Write 4 for $3x$ and the first bracket vanishes, so $x = \frac{4}{3}$.

Write -1 for $2x$, and the second bracket vanishes, so $x = -\frac{1}{2}$.

Hence the roots are $x = \frac{4}{3}$ and $x = -\frac{1}{2}$.

Example. Solve for x: $4-4x = 15x^2$.

Rewriting with all terms given on the left,

$$4-4x-15x^2 = 0.$$

It is easier to factorise an expression when the term in x^2 is positive, so *multiplying both sides* of the equation by -1, we have

$$15x^2+4x-4 = 0 \quad \text{(There is no necessity to write } -0\text{).}$$

Factorising $(5x-2)(3x+2) = 0,$

whence $x = \frac{2}{5}$ and $x = -\frac{2}{3}$.

When the expression on the left will not resolve into simple factors, another method of solution must be employed.

Solution by completing the square

The term 'completing the square' requires a little explanation before applying it to a problem. We have seen in Chapter 11 that

$$(a+b)^2 = a^2+2ab+b^2$$

and

$$(a-b)^2 = a^2-2ab+b^2.$$

Notice that a binomial gives three terms on squaring, as follows:

(1) the square of the first term of the binomial,
(2) the square of the second term of the binomial,
(3) twice the product of the two terms of the binomial.

Now let us write certain perfect squares from facts (1), (2) and (3):

$$(x+1)^2 = \underset{(1)}{x^2}+\underset{(2)}{1}+\underset{(3)}{2x},$$

$$(x+\tfrac{1}{2})^2 = \underset{(1)}{x^2}+\underset{(2)}{\tfrac{1}{4}}+\underset{(3)}{x},$$

$$(x+5)^2 = \underset{(1)}{x^2}+\underset{(2)}{25}+\underset{(3)}{10x},$$

$$(x+\tfrac{3}{2})^2 = \underset{(1)}{x^2}+\underset{(2)}{\tfrac{9}{4}}+\underset{(3)}{3x}.$$

To complete a square given terms (1) and (3):

Let x^2+16x be two terms of a perfect square, then $16x$ represents twice the product of the terms of the binomial which is squared. One term of the binomial is obviously x, since this gives x^2, so the other term must be *half of* 16, i.e. 8. Thus

$$x^2+16x+64 = (x+8)^2.$$

Consider $x^2+\dfrac{5x}{2}$. If $\dfrac{5x}{2}$ represents twice the product of the two terms of the binomial, then the other term must be $(\tfrac{5}{4})^2$.

Hence $x^2+\dfrac{5x}{2}+\tfrac{25}{16} = (x+\tfrac{5}{4})^2$.

To complete a square of a binomial, given the term in x^2, and the term in x, *halve* the coefficient of x, and square the half coefficient. This gives the third term of the perfect square.

Hence $x^2 + \frac{1}{3}x$ becomes a perfect square when $(\frac{1}{2} \text{ of } \frac{1}{3})^2$ is added, i.e.

$$x^2 + \tfrac{1}{3}x + \tfrac{1}{36} = (x + \tfrac{1}{6})^2.$$

The same rule holds good when the term in x is negative.

Thus $x^2 - \frac{5}{2}x$ is a perfect square when $(\frac{1}{2} \text{ of } -\frac{5}{2})^2$ is added, so $x^2 - \frac{5}{2}x + \frac{25}{16} = (x - \frac{5}{4})^2$.

This method is applied to quadratic equations when the expression on the left will not factorise, in order to have a perfect square on one side of the equation.

Example. Solve for x: $x^2 + 5x - 2 = 0$.

$$x^2 + 5x - 2 = 0.$$

$$\therefore\ x^2 + 5x = 2 \quad \text{(Writing terms in } x \text{ on the left).}$$

$$\therefore\ x^2 + 5x + (\tfrac{5}{2})^2 = 2 + (\tfrac{5}{2})^2$$

(Complete the square on the left by the method shown above, then add the extra term to *both* sides of the equation).

$$\therefore\ (x + \tfrac{5}{2})^2 = 2 + \tfrac{25}{4}$$

(Write the left-hand side as the square of a binomial)

$$= \tfrac{33}{4}.$$

$$\therefore\ x + \tfrac{5}{2} = \pm\sqrt{\tfrac{33}{4}}$$

(Extract the square root of both sides noting that two signs are possible)

$$= \pm\frac{\sqrt{33}}{2}.$$

$$\therefore\ x = -\tfrac{5}{2} \pm \frac{\sqrt{33}}{2}$$

(Remove the numerical term from the left)

$$= -2{\cdot}5 \pm \frac{5{\cdot}745}{2} \quad \text{(Evaluate the required root)}$$

$$= -2{\cdot}5 \pm 2{\cdot}873$$

$$= -5{\cdot}373 \text{ and } 0{\cdot}373.$$

Thus, the roots of $x^2 + 5x - 2 = 0$ are $-5{\cdot}37$ and $0{\cdot}37$ correct to two decimal places.

When the coefficient of x^2 is not $+1$, the equation must be divided throughout by the coefficient of x^2.

Example. Solve for x: $3x^2 - 4x - 1 = 0$.

$$3x^2 - 4x - 1 = 0.$$

Dividing throughout by 3, we have

$$x^2-\tfrac{4}{3}x-\tfrac{1}{3}=0.$$

$$\therefore\ x^2-\frac{4x}{3}=\frac{1}{3}\quad\text{(Removing constant term to right).}$$

$$\therefore\ x^2-\frac{4x}{3}+\left(\frac{2}{3}\right)^2=\frac{1}{3}+\left(\frac{2}{3}\right)^2\quad\text{(Completing square on left and adding extra term to both sides).}$$

$$(x-\tfrac{2}{3})^2=\tfrac{1}{3}+\tfrac{4}{9}$$

$$=\tfrac{7}{9}.$$

$$\therefore\ x=+\frac{2}{3}\pm\frac{\sqrt{7}}{3}$$

$$=0{\cdot}6667\pm\frac{2{\cdot}646}{3}$$

$$=0{\cdot}6667\pm0{\cdot}882$$

$$=1{\cdot}5487\ \text{and}\ -0{\cdot}2153.$$

Hence the roots of $3x^2-4x-1=0$ are $1{\cdot}55$ and $-0{\cdot}22$ correct to two decimal places.

Example from mechanics

The height (h ft.) of an object thrown vertically upwards, at a time t sec. after it has been projected, is given by the formula

$$h=ut-\tfrac{1}{2}gt^2,$$

where u is the velocity of projection and g is the acceleration due to gravity. Taking $u=80$ ft. per sec. and $g=32$, find the times when the object is 24 ft. above the point of projection.

$$h=ut-\tfrac{1}{2}gt^2.$$

$$\therefore\ 24=80t-16t^2.$$

$$\therefore\ 16t^2-80t+24=0.$$

$$\therefore\ t^2-5t+\tfrac{3}{2}=0\quad\text{(Dividing throughout by 16).}$$

$$\therefore\ t^2-5t=-\tfrac{3}{2}.$$

$$\therefore\ t^2-5t+(\tfrac{5}{2})^2=-\tfrac{3}{2}+(\tfrac{5}{2})^2$$

$$=-\tfrac{3}{2}+\tfrac{25}{4}$$

$$=\tfrac{19}{4}.$$

$$\therefore\ (t-\tfrac{5}{2}) = \pm\frac{\sqrt{(19)}}{2}.$$

$$\begin{aligned}\therefore\ t &= \frac{5}{2}\pm\frac{\sqrt{(19)}}{2}\\ &= 2{\cdot}5\pm\frac{4{\cdot}359}{2}\\ &= 2{\cdot}5\pm 2{\cdot}18\\ &= 4{\cdot}68 \text{ and } 0{\cdot}32.\end{aligned}$$

The roots of $16t^2-80t+24=0$ are 4·68 and 0·32.

These values of t correspond to the times at which the body passed the 24 ft. mark on the upward and on the downward journeys. Thus the times when the body is 24 ft. above the point of projection are 0·32 sec. and 4·68 sec. after projection.

Examples for Chapter 20

1. Solve the following equations:

(*a*) $x^2-3x=0$.
(*b*) $x^2-3x+2=0$.
(*c*) $x^2+3x+2=0$.
(*d*) $x^2-x-20=0$.
(*e*) $2x^2-x-3=0$.
(*f*) $6x^2-11x-10=0$.
(*g*) $10x^2-3x-1=0$.
(*h*) $2x^2-x-15=0$.
(*i*) $4x^2-4x+1=0$.
(*j*) $x^2-a^2=0$.
(*k*) $x^2-\frac{5}{6}x+\frac{1}{6}=0$.
(*l*) $6x^2-13x-110=0$.

(*Chap.* 20)

2. Solve the following equations, giving the roots correct to two decimal places:

(*a*) $x^2-2x-1=0$.
(*b*) $x^2-3x+1=0$.
(*c*) $x^2+5x-3=0$.
(*d*) $x^2-4x+2=0$.

(*Chap.* 20)

3. Solve the following equations, giving the roots correct to two decimal places:

(*a*) $2x^2-5x-1=0$.
(*b*) $3x^2+x-1=0$.
(*c*) $5x^2+8x+2=0$.
(*d*) $10x^2+4x-3=0$.

(*Chap.* 20)

4. The height in feet, after t sec., of a body projected vertically, is given by

$$h = 80t - 16t^2.$$

Find the times at which it is 32 ft. above its point of projection, and find how many seconds it is in flight. (*Chap.* 20)

5. The sum (s) of a number of terms of a series is given by

$$s = \frac{n}{2}\{2a + (n-1)d\},$$

where n is the number of terms to be added, a is the first term, and d is the common difference between the terms. Find how many terms are added to make a sum of 40, in a series whose first term is 2, and whose common difference is 3. (*Chap.* 20)

6. The bending moment for a certain uniform beam is given by the formula

$$M = 25x - \frac{wx^2}{2},$$

where M is the bending moment at a distance x ft. from one end, and w is the weight per foot of the beam. Find how far from one end the bending moment is 60, when the weight per foot of the beam is 5 lb. (*Chap.* 20)

7. (*a*) If x varies inversely as y, and $y = 20$ when $x = 5{\cdot}5$, find y when $x = 16$.

(*b*) The distance (s ft.) covered by a falling body in time t sec. is directly proportional to the square of the time. When $s = 64$ ft., $t = 2$ sec. Find s when $t = 2{\cdot}5$ sec., and find the time taken to fall 625 ft. (*Chap.* 19)

8. (*a*) Solve for x: $\dfrac{3}{x+2} - \dfrac{2}{x+3} = \dfrac{7}{x^2+5x+6}$. (*Chap.* 14)

(*b*) Find the equation of the straight line which passes through the points (1, 2) and (-3, -14), and find the ordinate of a point on the line whose abscissa is 3. (*Chap.* 18)

9. The moment of inertia of a rectangular metal plate of sides a and b ft. about an axis through the centre of the plate is given by

$$I = \frac{M}{12}(a^2 + b^2),$$

where M is the total mass of the plate.

Write a formula for a in terms of I, M and b, and evaluate a when $b = 3$ ft., $I = 50$ lb.ft.^{2}, and $M = 24$ lb. (*Chap.* 16)

10. The magnetic moment (M) of a bar magnet is given by the formula

$$M = (d^2 + l^2)^{\frac{3}{2}} H \tan\theta.$$

Evaluate M when $d = 22{\cdot}5$, $l = 4{\cdot}8$, $H = 0{\cdot}18$ and $\theta = 35°$. (*Chap.* 10)

Answers to Examples for Chapter 20

1. (*a*) $x = 0$ and $x = 3$. (*b*) $x = 1$, $x = 2$.
(*c*) $x = -1$, $x = -2$. (*d*) $x = 5$, $x = -4$.
(*e*) $x = \frac{3}{2}$, $x = -1$. (*f*) $x = -\frac{2}{3}$, $x = \frac{5}{2}$.
(*g*) $x = -\frac{1}{5}$, $x = \frac{1}{2}$. (*h*) $x = -\frac{5}{2}$, $x = 3$.
(*i*) $x = 1$ (2 equal roots). (*j*) $x = \pm a$.
(*k*) $x = \frac{1}{2}$, $x = \frac{1}{3}$. (*l*) $x = 5\frac{1}{2}$, $x = -3\frac{1}{3}$.

2. (*a*) $x = -0{\cdot}41$ and $x = 2{\cdot}41$. (*b*) $x = 2{\cdot}62$, $x = 0{\cdot}38$.
(*c*) $x = -5{\cdot}54$, $x = 0{\cdot}54$. (*d*) $x = 3{\cdot}41$, $x = 0{\cdot}59$.

3. (*a*) $x = 2{\cdot}69$, $x = -0{\cdot}19$. (*b*) $x = -0{\cdot}77$, $x = 0{\cdot}44$.
(*c*) $x = -1{\cdot}29$, $x = -0{\cdot}31$. (*d*) $x = -0{\cdot}78$, $x = 0{\cdot}38$.

4. 4·56 sec., 0·44 sec.; 5 sec. (write 0 for h and solve).

5. Answers obtained are $n = 5$, and $n = -5\frac{1}{3}$. The latter is neglected because it has no meaning. Number of terms is 5.

6. 4 ft. and 6 ft.

7. (*a*) $y = 6{\cdot}875$. (*b*) $s = 100$ ft.; $t = 6{\cdot}25$ sec.

8. (*a*) $x = 2$. (*b*) $y = 4x - 2$; $y = 10$.

9. $a = \sqrt{\left(\frac{12I}{M} - b^2\right)}$; $a = 4$ ft.

10. $M = 1535$.

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CHAPTER 21

USE OF GEOMETRICAL INSTRUMENTS

The unit of angular measure is the degree (°), which is $\frac{1}{360}$ of a complete circle. This may be subdivided into 60 minutes (′) and each minute into 60 seconds (″).

The protractor, shown in fig. 34, is used to measure angles. For this purpose, the point of the angle is placed at the point of the protractor marked X in the diagram, and one of the arms of the angle is made to lie along XY or XZ. Thus the angle PXQ in the diagram is 40°, while the angle RXQ is 140°. The former angle is called an *acute* angle (less than 90°), and the latter is called an *obtuse* angle (greater than 90° and less than 180°). The student will see the reason for the double figuring on the protractor.

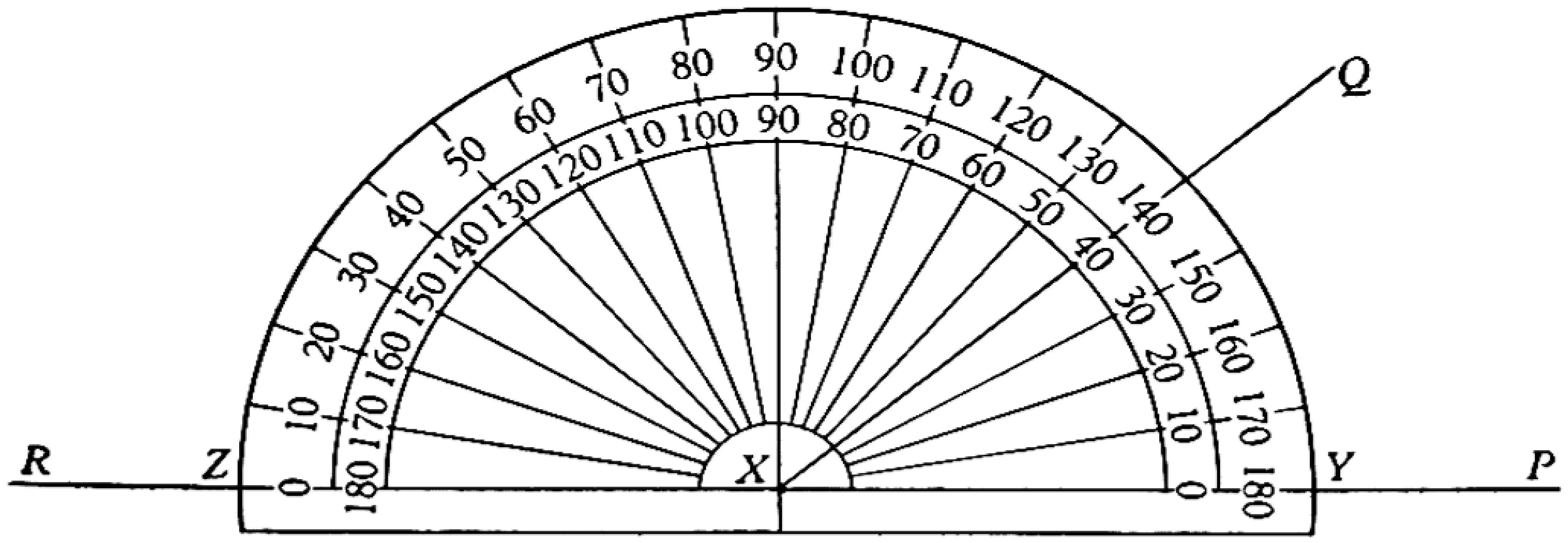


Fig. 34

An angle of 90° is called a right angle, and the two arms (i.e. lines forming the angle) of a right angle are said to be perpendicular to each other. Two acute angles which total 90° are said to be complementary angles, thus 40° is the complement of 50°, and 15° is the complement of 75°. Two angles whose total is 180° are said to be supplementary, thus 130° is the supplement of 50°.

Points of the compass. Bearings

The same units of angular measure are used to denote directions on the surface of the earth. The bearing of a point B from a point A is the angle the line AB makes with one of the standard directions known as points of the compass. The bearing of A from B is the angle the line BA makes with one of the standard directions. Fig. 35 shows the standard directions N., S., E. and W., and four lines OA, OB, OC and OD making different angles with these standards. There are several methods of writing bearings, and we shall use the directions of these four lines to illustrate three common methods.

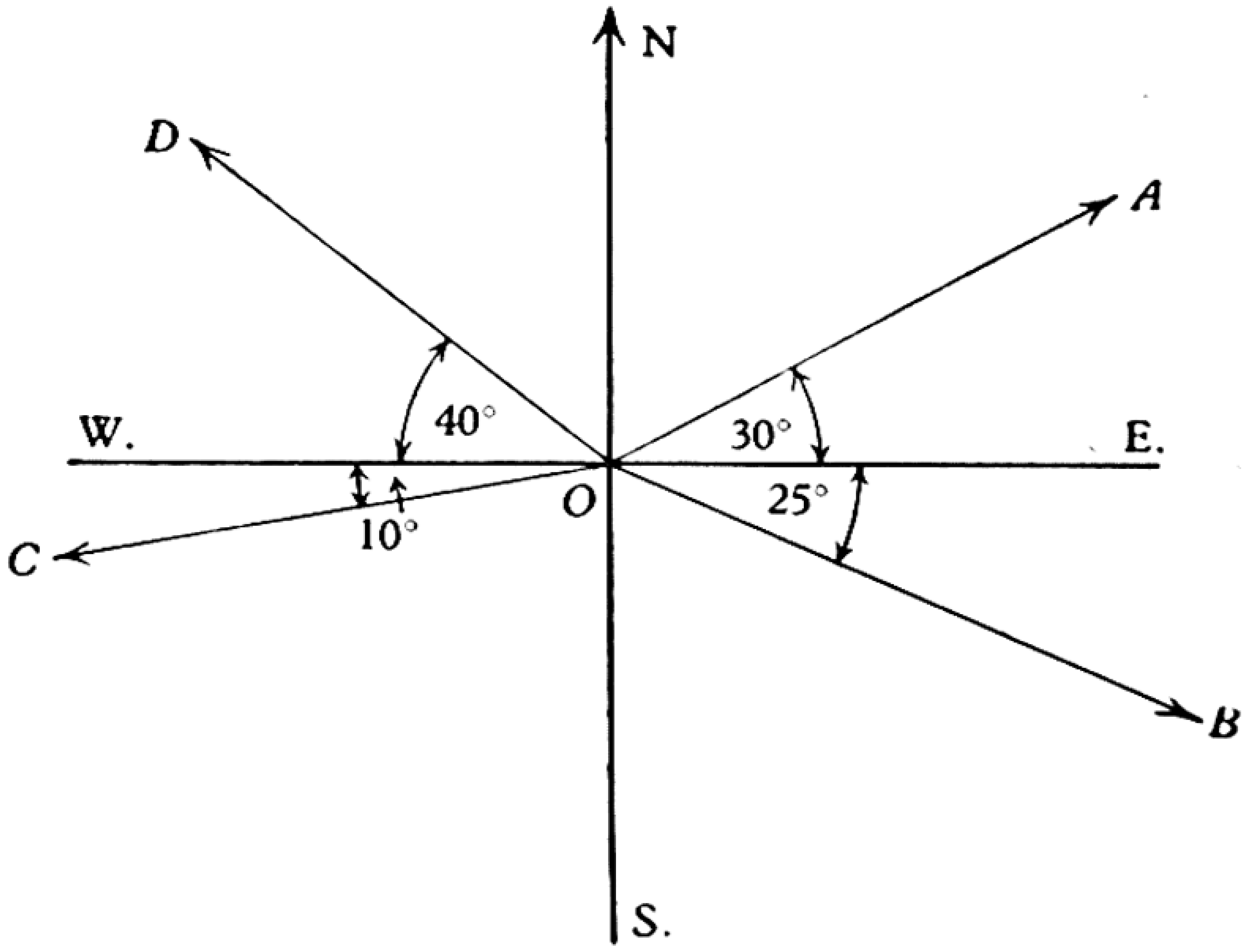


Fig. 35

Method 1. Write the angle the required line makes with respect to either of the standard directions between which it lies. Thus the direction OA is either 30° N. of E., or 60° E. of N.; the direction OB is either 25° S. of E., or 65° E. of S.; the direction OC is either 10° S. of W., or 80° W. of S.; the direction OD is either 40° N. of W. or 50° W. of N.

Method 2. Write N or S followed by the angle the required direction makes with ON or OS. If the angle is measured to the east of the NS line, the letter E finishes the statement, while angles to the west of the NS line finish with W.

Thus the direction OA is N. 60° E.; the direction OB is S. 65° E.; the direction OC is S. 80° W.; the direction OD is N. 50° W.

Method 3 (*R.A.F. method*). The directions N., S., E., W. are not used. The direction ON is called 000, and the direction of any line is given by the angle it makes with the direction 000, measured in a clockwise direction from 000 (i.e. measured round by what we have previously called the east direction). Three figures are used in order to avoid mistakes. If the angle is 6°, the bearing is given as 006, while an angle of 25° will be 025.

Thus the direction OA is 060; the direction OB is 115; the direction OC is 260; the direction OD is 310.

The list of methods is not complete, but these are the methods the student is most likely to meet.

Example. Two ships A and B leave port together at an average speed of 10 m.p.h. A steers a course N. 45° W., and B steers a course N. 10° E. Find by a scale drawing their distance apart after 2 hr., and the bearing of B from A.

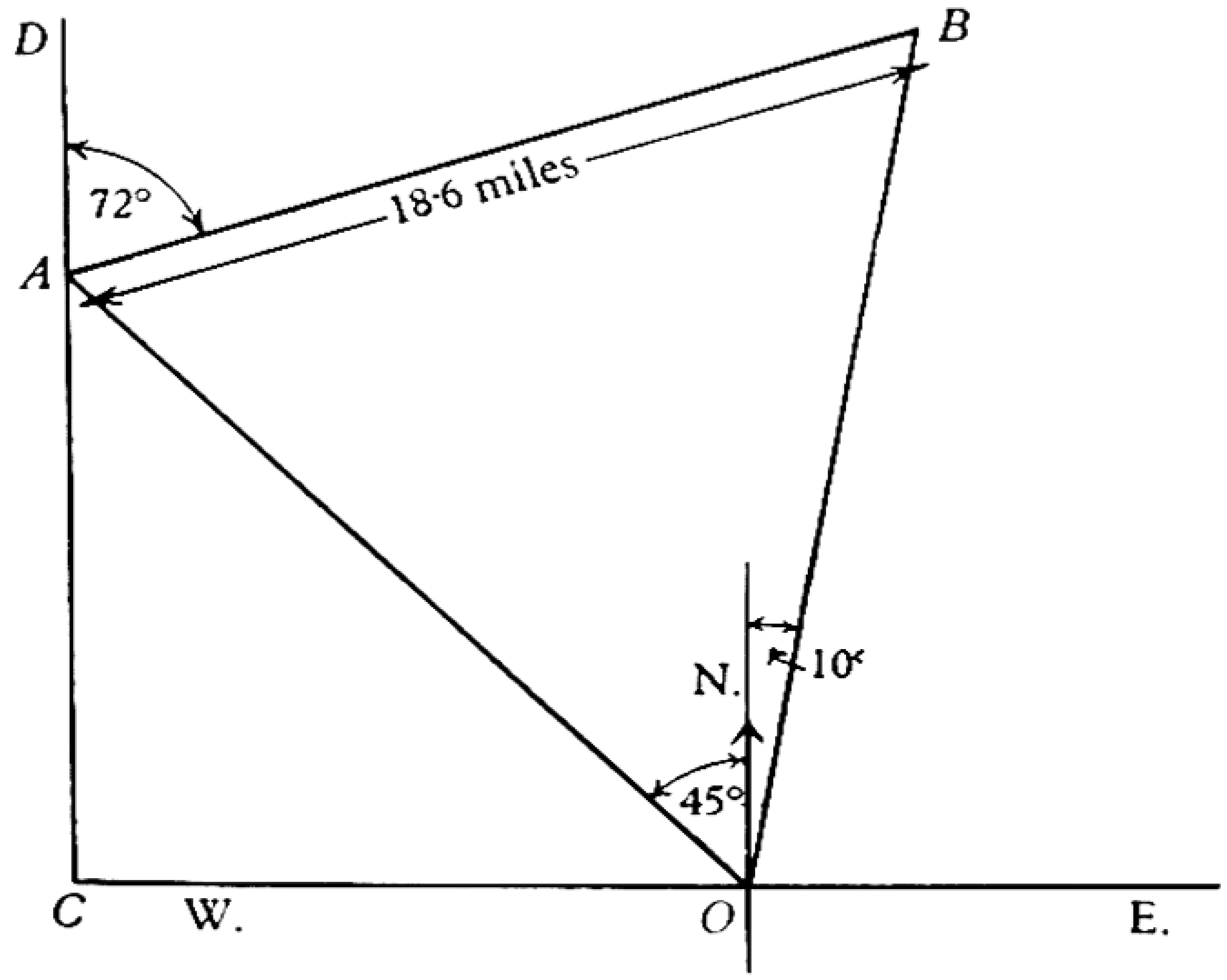


Fig. 36

In fig. 36, O is the port, and the directions of the ships make angles of 45° and 10° respectively with the direction N. OA and OB are drawn to the same scale and represent a distance of 20 miles (2 hr. at 10 m.p.h.). Then AB represents their distance apart after 2 hr. This measures $18\frac{1}{2}$ miles.

To obtain the bearing of B from A, we must measure the angle between AB and either a horizontal or a vertical line. AC is drawn perpendicular to WE, and the angle BAD is measured as $72\frac{1}{2}°$. This gives a bearing of N. $72\frac{1}{2}°$ E. for B.

Thus, after 2 hr., the distance apart of the ships is $18\frac{1}{2}$ miles, and the bearing of B from A is N. $72\frac{1}{2}°$ E.

Example. The jib of a crane, 24 ft. long, is attached to, and makes an angle of 40° with the vertical post. The tie, which joins the jib and the post, and is attached to the top end of the jib, makes an angle of 35° with the latter. Find, by a scale drawing, (*a*) the length of the tie to the nearest half foot, (*b*) the vertical height of the top above the bottom of the jib.

Fig. 37 shows the sketch. AC is vertical, $AB = 24$ units, and angle $CAB = 40°$. Angle CBA is measured as 35°. Then, by measurement, $BC = 16$ units. BD is drawn at an angle of 90° to AD, which is drawn perpendicular to AC. Then, by measurement, $BD = 18\frac{1}{2}$ units. Hence, length of tie = 16 ft., height of top above bottom of jib $= 18\frac{1}{2}$ ft.

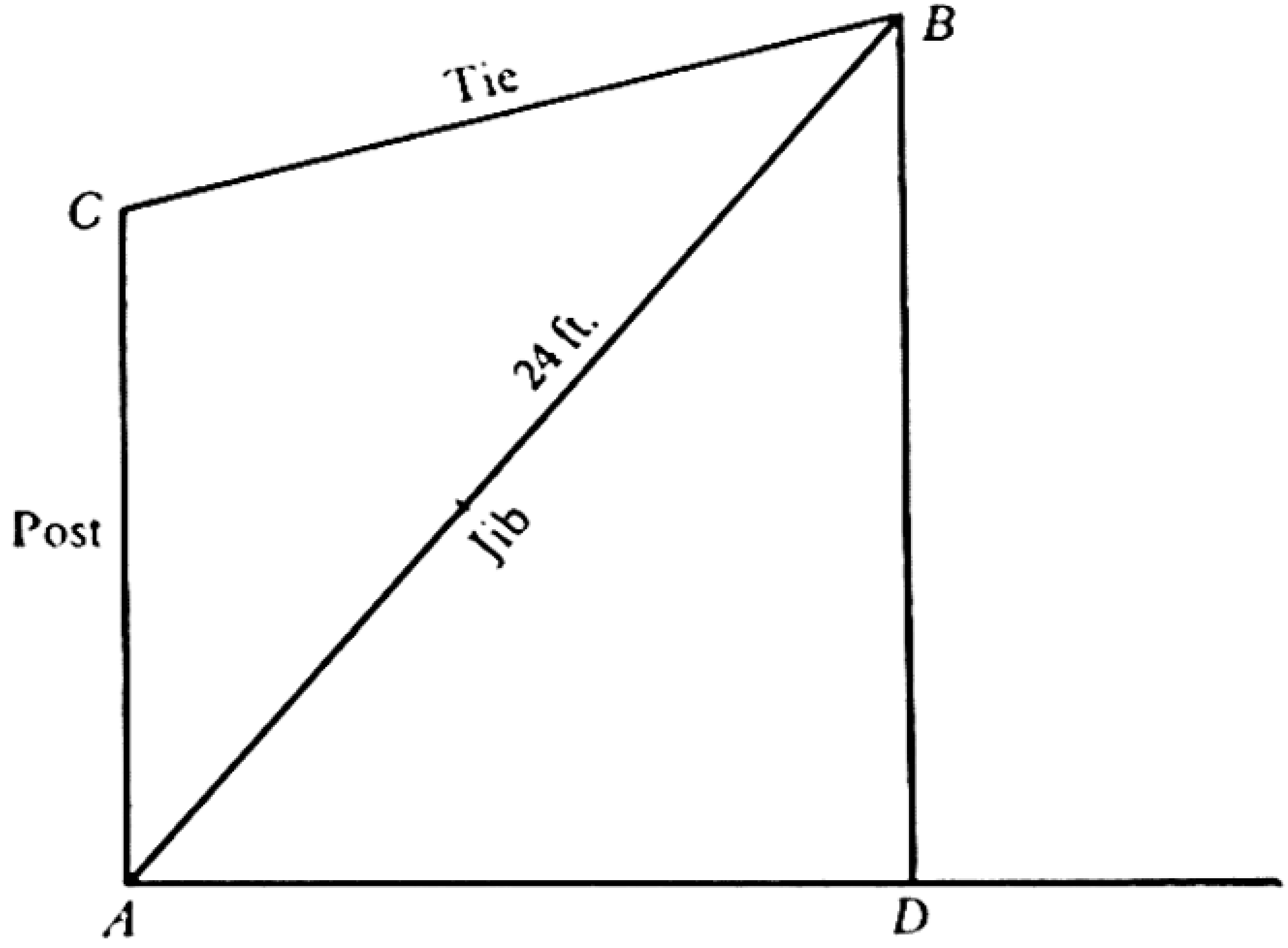


Fig. 37

Parallel lines

Two lines which have the same direction are said to be parallel. In two-dimensional space (e.g. on paper or a blackboard) parallel lines are lines which will not intersect.

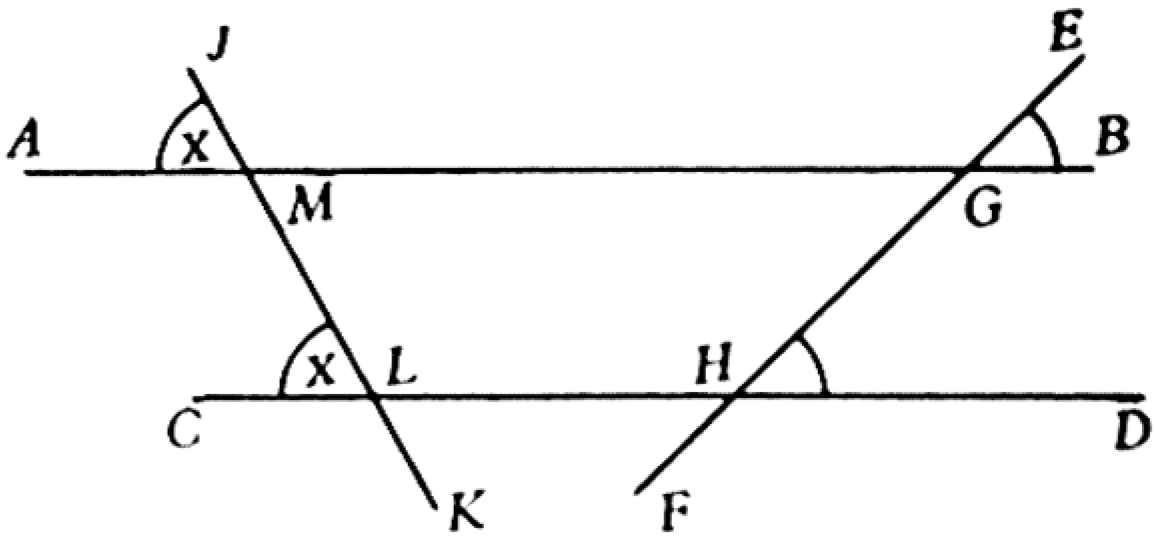


Fig. 38

In fig. 38, AB and CD are two parallel lines, and the lines EF and JK are called transversals. It will be obvious that the angles EGB and GHD are equal, and that angles JMA and MLC are equal. This fact is made use of in a simple method of drawing one line parallel to another. It is required to draw a line through a given point C (fig. 39) parallel to a given line AB. One edge of a set-square is placed along AB and a ruler brought up to touch another edge of the set-square. Keeping the ruler fixed, the set-square is slid along the ruler until C is on its upper edge. A line may then be drawn parallel to AB.

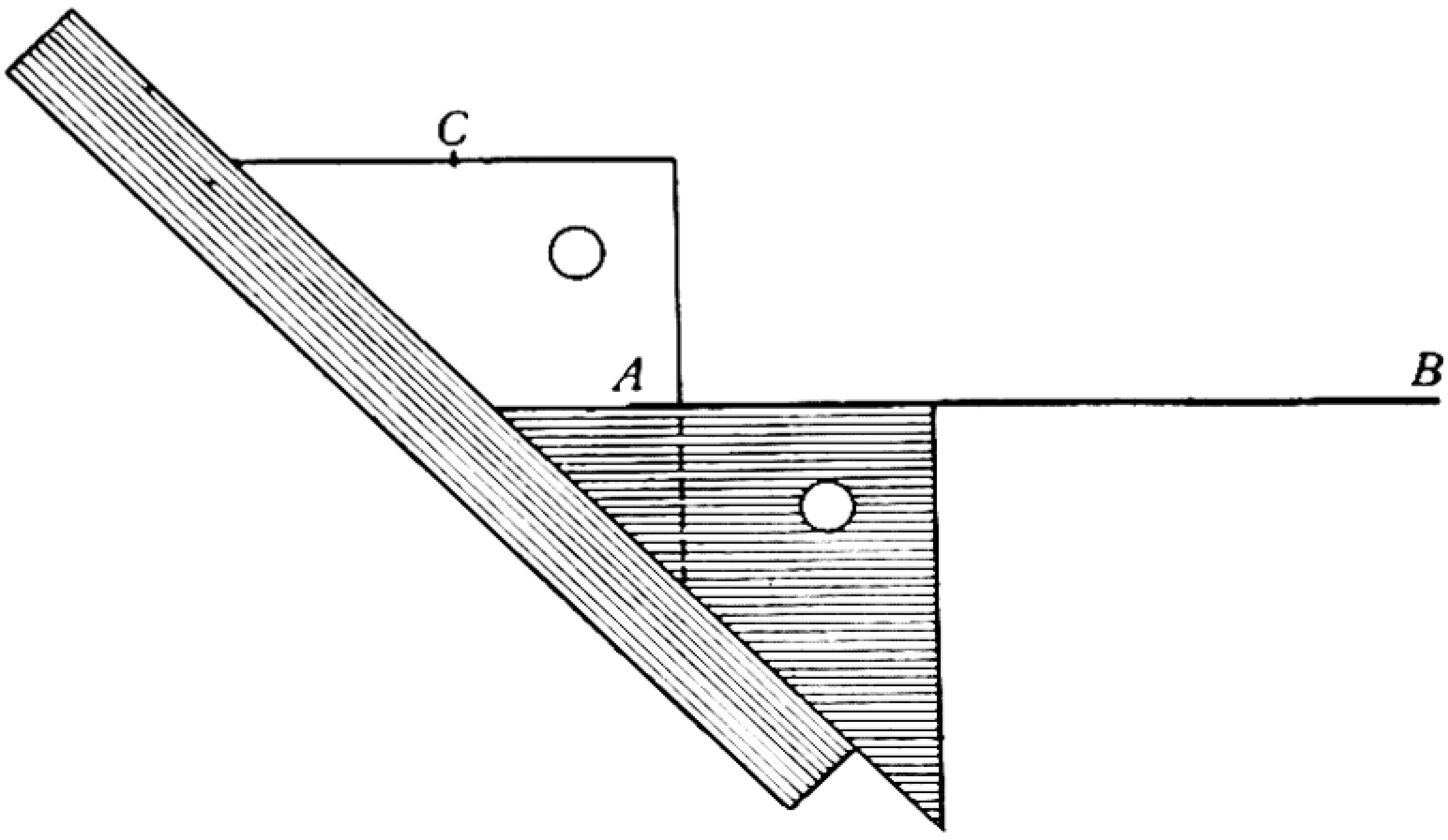


Fig. 39

Use of compasses

A pair of compasses is used in geometry, not merely to draw circles, but frequently to mark off a given distance. The part of the circle which is described by the pencil when the point is fixed is called an *arc* of the circle, an arc being part of the circumference of the circle. Every point on the circumference is the same distance from the centre. The geometrical facts of circles and angles in circles are left to Book II, but the main definitions must be known at this stage.

Fig. 40 shows a circle with lines drawn in and through it. OA is a *radius*, a line from the centre to any point on the circumference. BC is a *chord*, a line terminating at two points on the circumference.

If AO is produced to meet the circumference in L, then AL is a *diameter* of the circle, and the two parts into which the circle is thus divided are called semicircles. BC also divides the circle into two parts, which are unequal, and are called *segments*, a major and a minor segment. DG is a *secant*, a line intersecting the circumference in two points. HK is a *tangent*, a line touching the circumference. The curved lines BC, $EBCF$, $EAJF$, are called *arcs* of the circumference, the last one is a major arc, while the first is a minor arc. Concentric circles are circles which have the same centre.

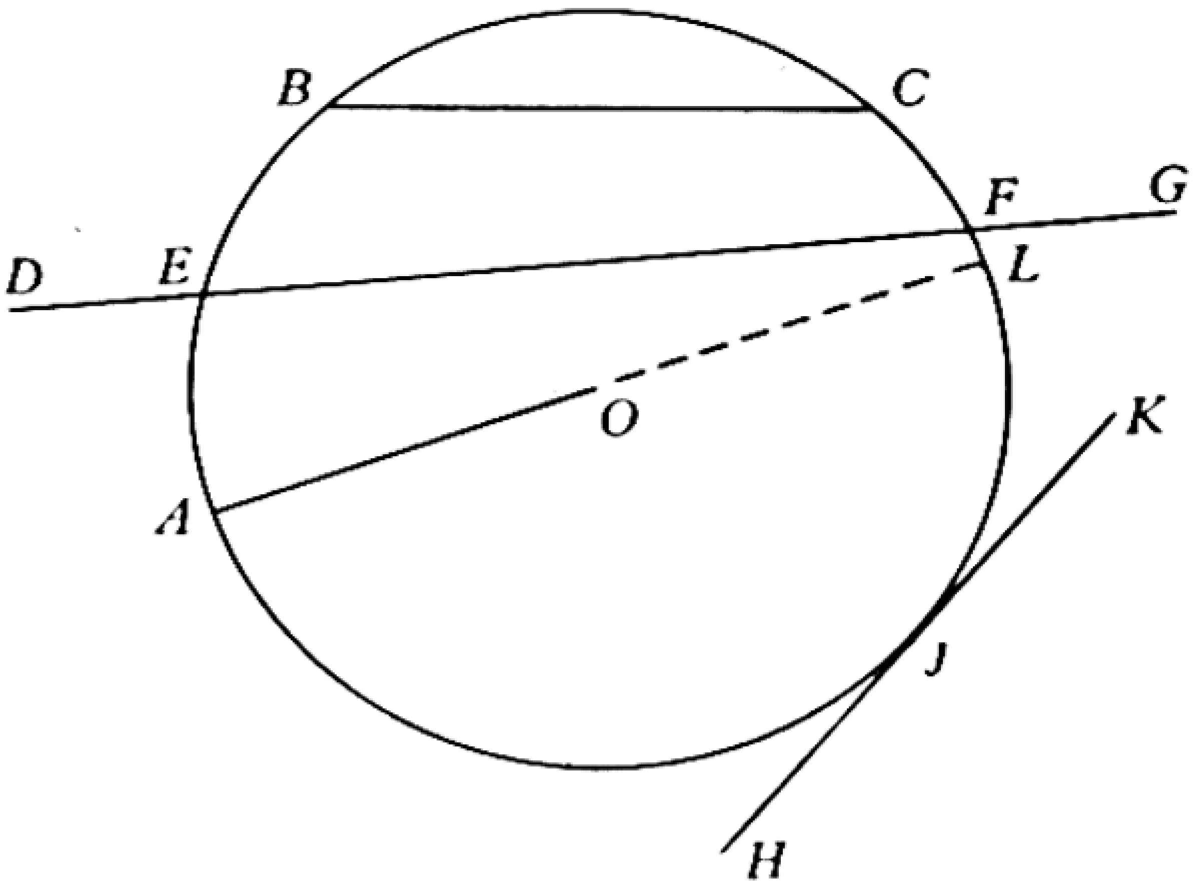


Fig. 40

Example. The radius of the 'crank circle' of an engine is 9 in., and the connecting rod AP is 2 ft. 6 in. long. If A is the cross-head, O the centre of the crank circle, find the change in length of OA as the angle AOP changes from 20° to 90°.

Fig. 41 shows the scale drawing. The crank circle has radius 9 units. Angle POA is made 20°, and then PA is measured as 30 units. Angle $P'OA = 90°$, and $P'A'$ (the new position of the connecting rod) is also 30 units. Then AA' represents the change in length of OA for the movement given.

By measurement $AA' = 9\frac{1}{2}$ units (approx.).

Thus, change in length of $OA = 9\frac{1}{2}$ in. (approx.).

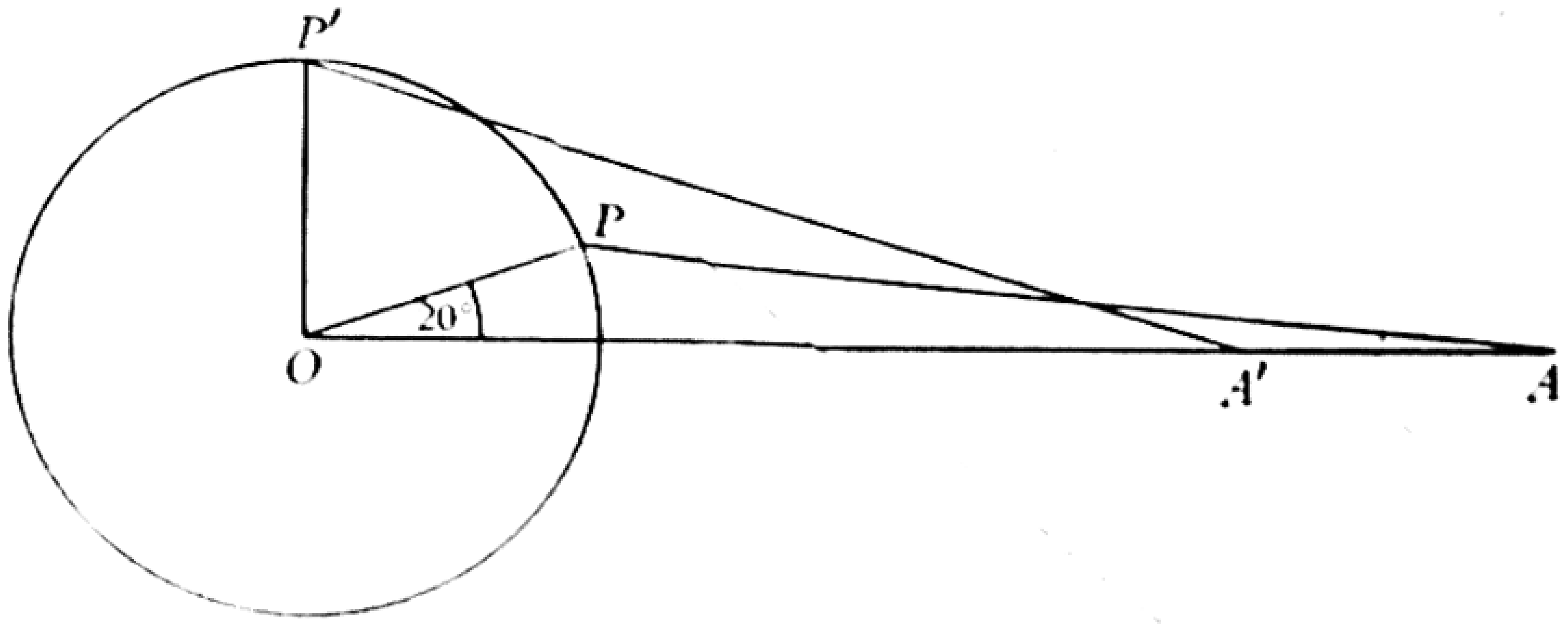


Fig. 41

Geometrical facts to be remembered

1. When one straight line stands upon another straight line, the two angles so formed are supplementary. In fig. 42 it is

obvious that the angles marked O and X are supplementary. In order to avoid confusion, an angle is usually denoted by three letters, the middle one of which represents the actual position of the angle. The angles O and X may thus be called angle ABD and angle CBD respectively. Other methods of writing the names of angles are $\angle ABD$ and $\angle CBD$, or $A\hat{B}D$ and $C\hat{B}D$.

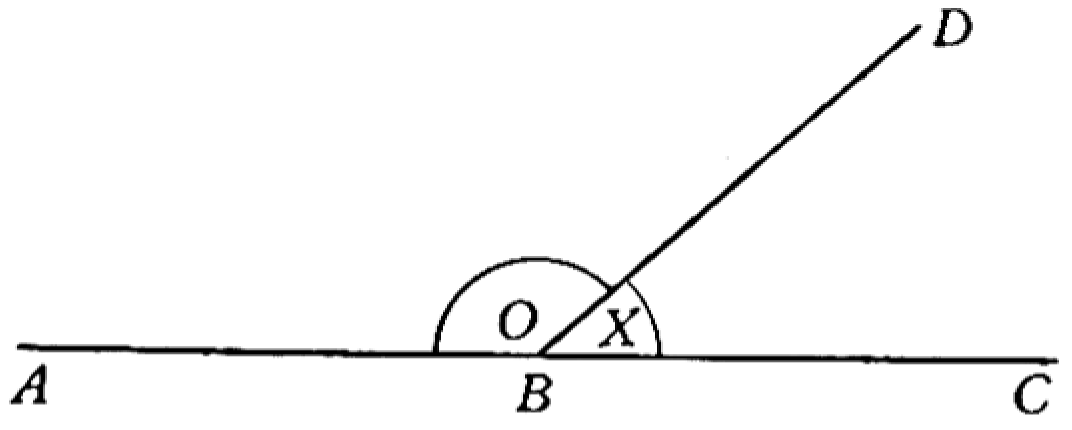


Fig. 42

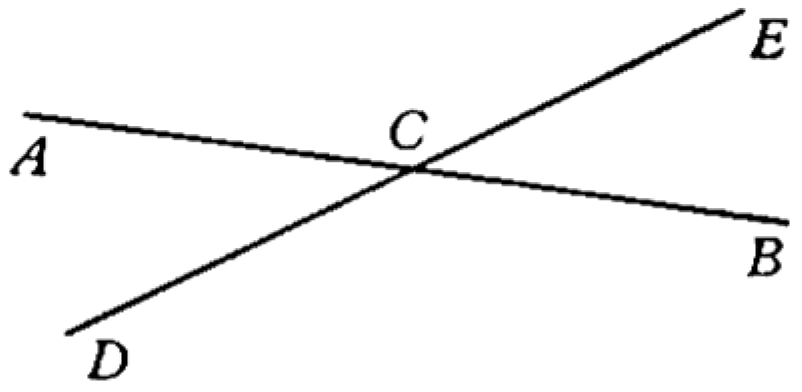


Fig. 43

2. When two straight lines intersect, the vertically opposite angles are equal. Fig. 43 shows AB and DE intersecting at C. It is obvious that the vertically opposite angles ECB and ACD are equal, as also are the vertically opposite angles ACE and DCB.

3. If a straight line cuts two parallel lines, the alternate angles so formed are equal. Fig. 44 shows two parallel lines AB, CD cut by a transversal EH. Eight angles are formed, lettered s, t, u, v, w, x, y, z. For convenience we shall call s, t, y and z exterior angles, and u, v, w, x interior angles. Also u and x are called alternate angles, as well as v and w.

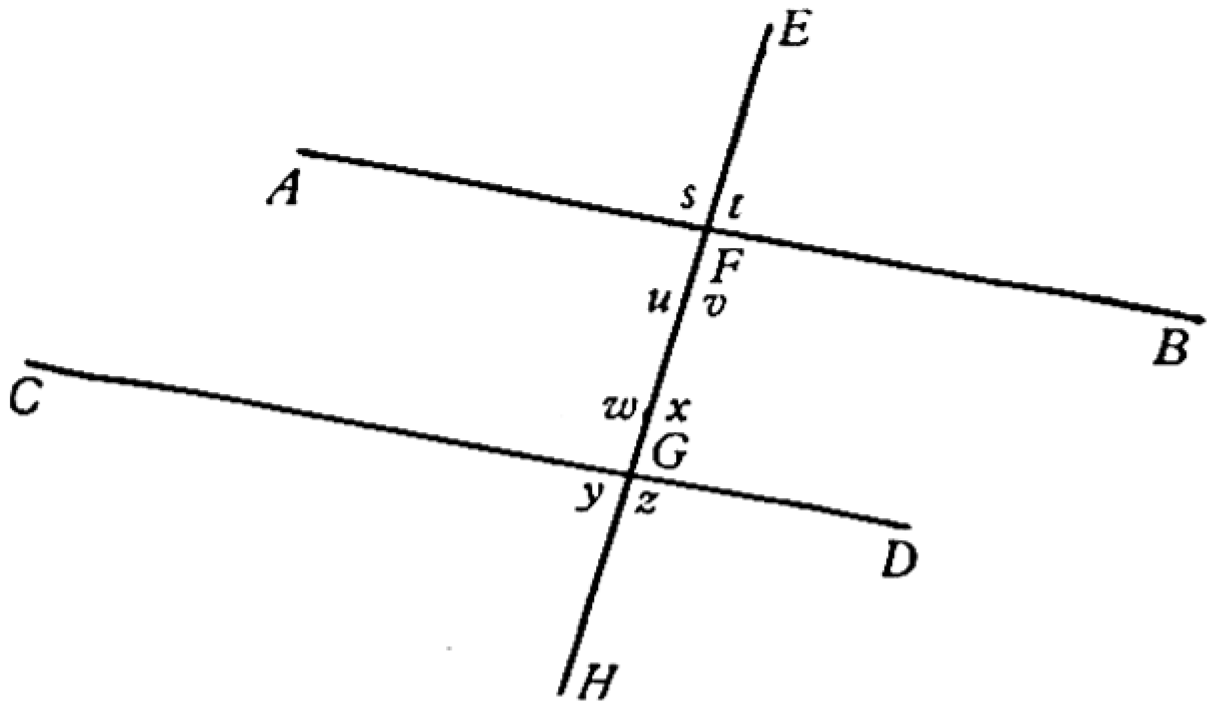


Fig. 44

Now from our method of constructing parallel lines, we know that angle t = angle x, angle s = angle w, with two other similar equalities. We also know from fact no. 2 above that angle t = angle u, so it follows that angle u = angle x, and angle v = angle w. Therefore the 'alternate' angles are equal.

4. When a transversal cuts two parallel lines, it makes the two interior angles on the same side of itself together equal to two right angles. Referring again to fig. 44, we have just shown that angle u = angle x. Now, from the fact that angle u + angle v = 180°, it follows that angle x + angle v = 180°. Similarly, angle u + angle w = 180°.

Examples for Chapter 21

1. (*a*) A direction N. 75° E. may also be written 075. Express similarly the following directions: (i) N. 35° E., (ii) 20° W. of S., (iii) N. 35° W., (iv) S. 24° E.

(*b*) A point A is due E. of, and 10 miles distant from, a point B. A point C is 10 miles N. of B. What is the bearing of A from C, and the bearing of C from A? (*Chap.* 21)

2. A ship sails due S. from a port for 30 min. at an average speed of 15 m.p.h. It then changes course and sails in a direction N. 60° E. for 1 hr. Find, by means of a scale drawing, its distance from the port at this time, and its bearing from the port. (*Chap.* 21)

3. An aircraft flies due E. in still air at a speed of 120 m.p.h. After 45 min. it changes course to a direction 045. Find its distance from the starting-point after 45 min. on this course, and find its bearing from the starting-point. (*Chap.* 21)

4. The jib of a crane (AB) is 20 ft. long, and makes an angle of 35° with the vertical post at A. The tie joins the jib at B, making with it an angle of 30°, and joins the post at C. Find, by a scale drawing, the approximate lengths of the tie and the post. (*Chap.* 21)

5. Describe a circle of radius 3 in. In the circle draw a chord AB, 4 in. long. If C is the mid-point of the chord AB, join C to the centre, O, of the circle. Draw any other chord, 5 in. long, in the same circle. If D and E are the extremities of this chord, and

F its mid-point, join OF. Measure the angles ACO, BCO, DFO and EFO, and hence complete the statement: 'A line joining the centre of a circle to the mid-point of a chord is . . . to the chord.' (*Chap.* 21)

6. Describe any circle and in it draw a chord AB. Choose any three points on the circumference on the same side of AB. Join each of the points C, D and E to both A and B. Measure the angles ACB, ADB and AEB, and hence complete the statement, 'Angles in the same segment of a circle are . . .'. (*Chap.* 21)

7. (*a*) AB and CD are two parallel lines. Another line EF intersects AB in J and CD in K, while a fourth line GH intersects AB in L, and CD in M. If angle LJK + angle $LMC = 180°$, what conclusion may be drawn?

(*b*) AB and CD are two straight lines which intersect at O. A line EOF is drawn so that angle AOE = angle COE. Show that angle AOE + angle BOF + angle $COB = 180°$. (*Chap.* 21)

8. In an arrangement similar to that of fig. 41, the radius of the crank circle is 7 in., and the length of the connecting rod is 25 in. What is the farthest distance of A from O? Find the movement of A from this position, to its position when angle $AOP = 90°$. (*Chap.* 21)

9. A framework $ABDC$ consists of four members AB, BD, DC, CA, bolted at their ends, and a cross-strut BC which is 4 ft. long. Angle ABC = angle $ACB = 60°$, and angle DBC = angle $DCB = 30°$. Find the lengths of the four members. (*Chap.* 21)

10. A and B are two small ports on a north coastline, and B is 15 miles E. of A. A boat sailing from A at 12 m.p.h. and a boat from B sailing at 9 m.p.h. both reach the same island in 1 hr. What is the bearing of the island from A and from B? (*Chap.* 21)

Answers to Examples for Chapter 21

1. (*a*) (i) 035; (ii) 200; (iii) 325; (iv) 156.
   (*b*) S. 45° E., N. 45° W.

2. 13 miles (approx.); due E.

3. $166\frac{1}{2}$ miles (approx.); $022\frac{1}{2}$.

4. $12\frac{1}{2}$ ft. (approx.); 11 ft. (approx.).

5. 'Perpendicular.'

6. 'Equal.'

7. Lines EF and GH are parallel.

8. 32 in.; 8 in.

9. $AB = AC = 4$ ft.; $BD = CD = 2\frac{3}{10}$ ft. (approx.).

10. N. 53° E.; N. 37° W. (both approx.).

CHAPTER 22

TRIANGLES AND POLYGONS

A triangle is a figure enclosed by three straight lines. Triangles may be classified as follows:

Equilateral triangles have all three sides the same length. (It follows that equilateral triangles have also three equal angles. See fig. 45.)

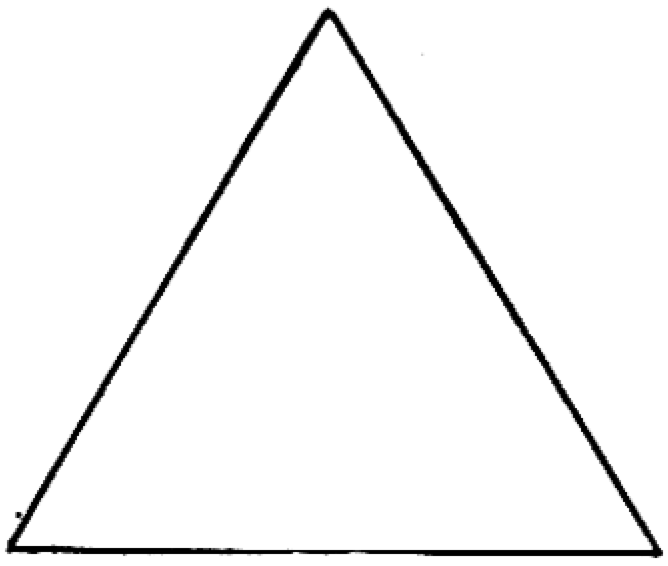


Fig. 45

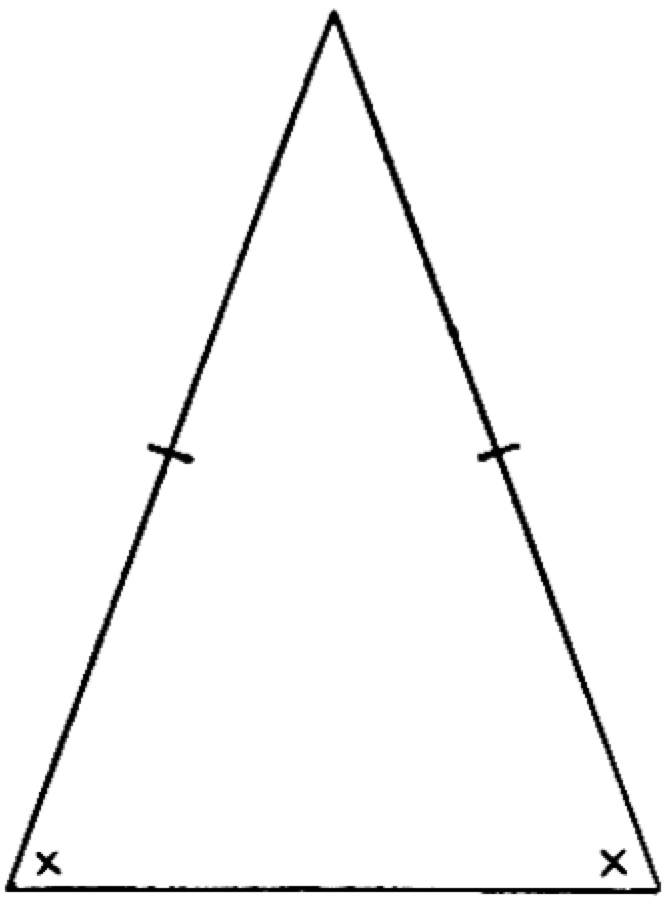


Fig. 46

Isosceles triangles have two sides equal in length. (It follows that the angles opposite these sides are equal. In fig. 46 the equal sides and equal angles are marked.)

Scalene triangles have sides of three different lengths (fig. 47).

Right-angled triangles have one of their angles 90°. Fig. 48 shows two such triangles, one of which is also isosceles.

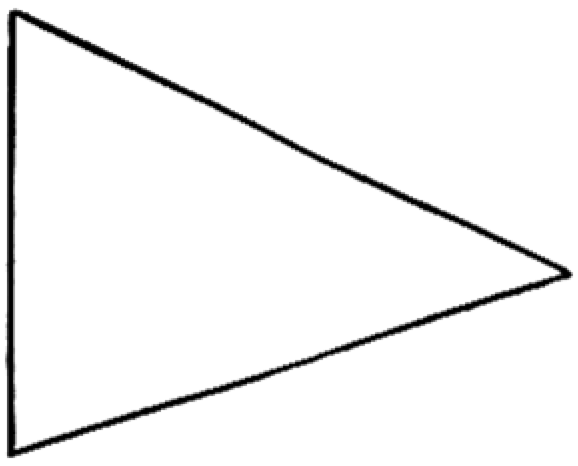
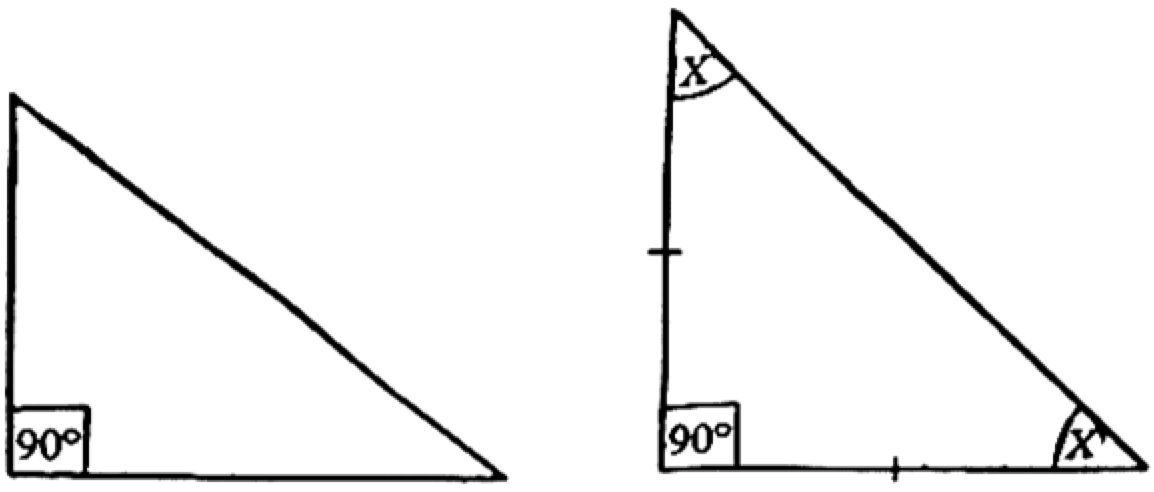


Fig. 47



(1)

(2)

Fig. 48

Polygons

A polygon is a figure bounded by straight lines. A polygon of four sides is called a quadrilateral (which category includes squares and rectangles). Polygons of five sides, six sides and eight sides are called pentagons, hexagons and octagons respectively. A regular polygon is one whose sides are all the same length. Fig. 49 shows two regular polygons.

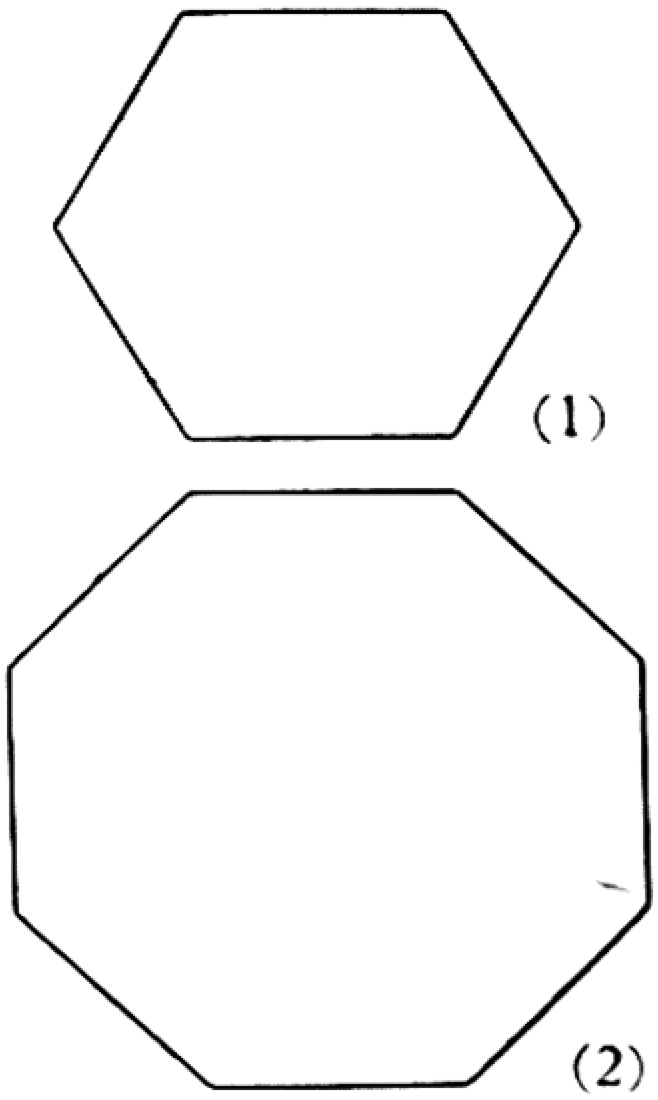

(1)
(2)

Fig. 49

A convex polygon is one whose interior angles are all less than 180°. Fig. 50 shows a convex polygon and one which contains a re-entrant angle.

Sum of angles of a polygon. The interior angles of a polygon are the angles between adjacent sides, inside the polygon, while the exterior angles are the angles made with one side produced, and the adjacent side. Fig. 51 shows a convex polygon $ABCDE$. The sides AB, BC, CD, DE, EA are produced to F, G, H, J, K respectively, and the angles FBC, GCD, HDE, JEA, KAB are exterior angles. Let us imagine a pencil as shown lying along the side AB. Slide the pencil to B and turn it through the angle FBC so that it lies along BC. Slide again and turn through the angle GCD. Repeat until the pencil is again in the position it occupied at first. It has obviously turned through a complete circle (i.e. 360°, or four right angles), and to do so has turned through each of the exterior angles. It follows that the sum of the exterior angles of a convex polygon is four right angles.

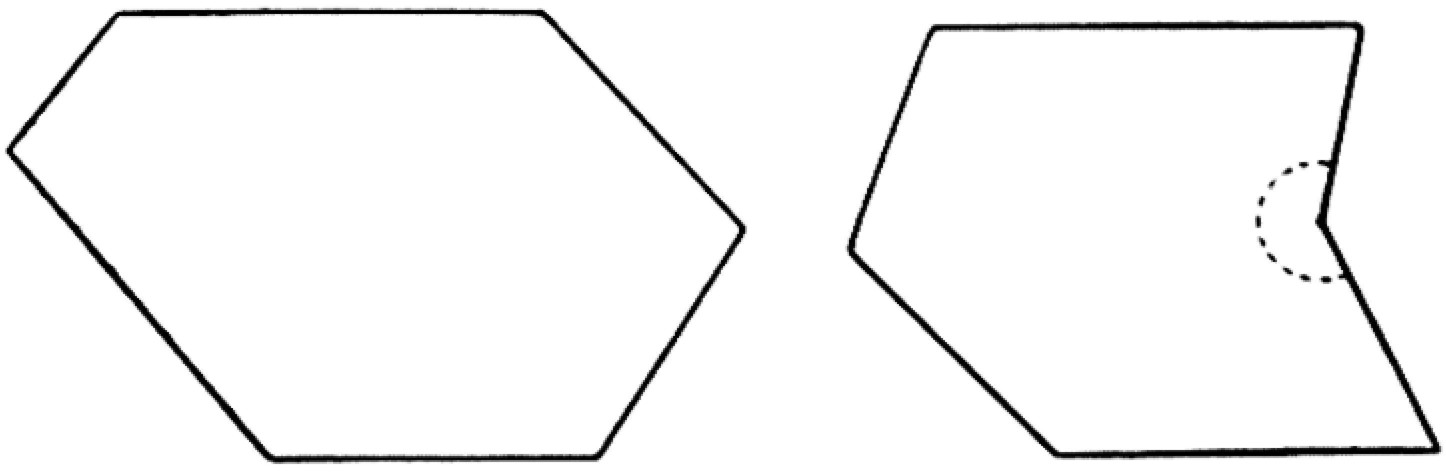

(1)
(2)

Fig. 50

Further, the sum of the exterior and interior angles at any one point must be equal to 180°, because at each of these points one line stands upon another, so it follows that the total of the interior and exterior angles in the diagram is 5 × 180°, i.e. 10

right angles. Hence the sum of the interior angles is $10-4$ right angles $=6$ right angles. In general, if a convex polygon has n sides, then the sum of the interior angles is $2n-4$ right angles.

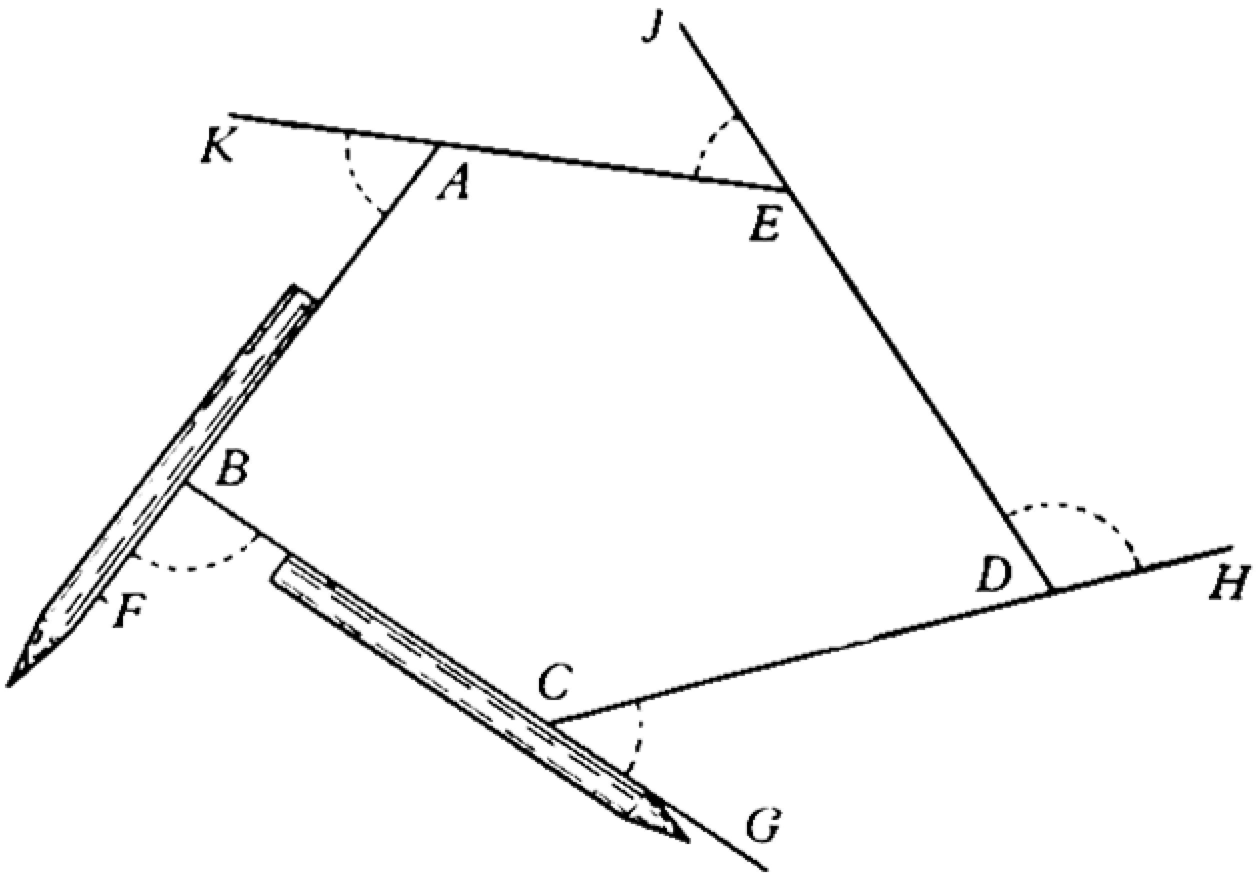


Fig. 51

Example. To find the angles of a regular hexagon.

Number of sides $=6$. Thus, sum of interior angles

$$= 2\times 6-4 \text{ right angles}$$
$$= 8 \text{ right angles.}$$

Since the hexagon is regular, all these angles are equal; thus each angle is $\dfrac{8\times 90^\circ}{6}=120^\circ$.

Example. To find the angles of a regular octagon.

Number of sides $=8$.

$$\therefore \text{ Sum of interior angles} = 2\times 8-4 \text{ right angles}$$
$$= 12 \text{ right angles.}$$
$$\therefore \text{ Each interior angle is } \frac{12\times 90^\circ}{8}=135^\circ.$$

[The symbol $\therefore$ is used to mean 'therefore', and the symbol $\because$ may be used to mean 'because'.]

Applying the same formula to a triangle, we see that the sum of the interior angles of a triangle is $2\times 3-4$ right angles, i.e. 2 right angles. This is an important fact to be memorised, viz. the sum of the interior angles of any triangle is 180°. It follows that each angle of an equilateral triangle is 60°.

Similarly, if we know the value of the odd angle of an isosceles triangle, we may find the two equal angles by subtracting the odd angle from 180°, and dividing the answer by 2.

Example. The angle opposite the smallest side of an isosceles triangle is 25°. Find the other angles:

$$180^\circ - 25^\circ = 155^\circ = \text{sum of the two equal angles.}$$

$$\therefore \text{ The equal angles are each } 77\tfrac{1}{2}^\circ.$$

To prove that the exterior angle of a triangle is equal to the sum of the two interior opposite angles (see fig. 52).

Proof. Angle ABC + angle BAC + angle $ACB = 180^\circ$ ($\because$ interior angles of a triangle).

But angle ACB + angle $ACD = 180^\circ$ ($\because$ AC stands on BD).

$\therefore$ angle ABC + angle BAC + angle ACB = angle ACB + angle ACD.

$\therefore$ angle ABC + angle BAC = angle ACD. Q.E.D.

[The letters Q.E.D., after a proof, stand for 'Quod erat demonstrandum', meaning 'which was to be proved'.]

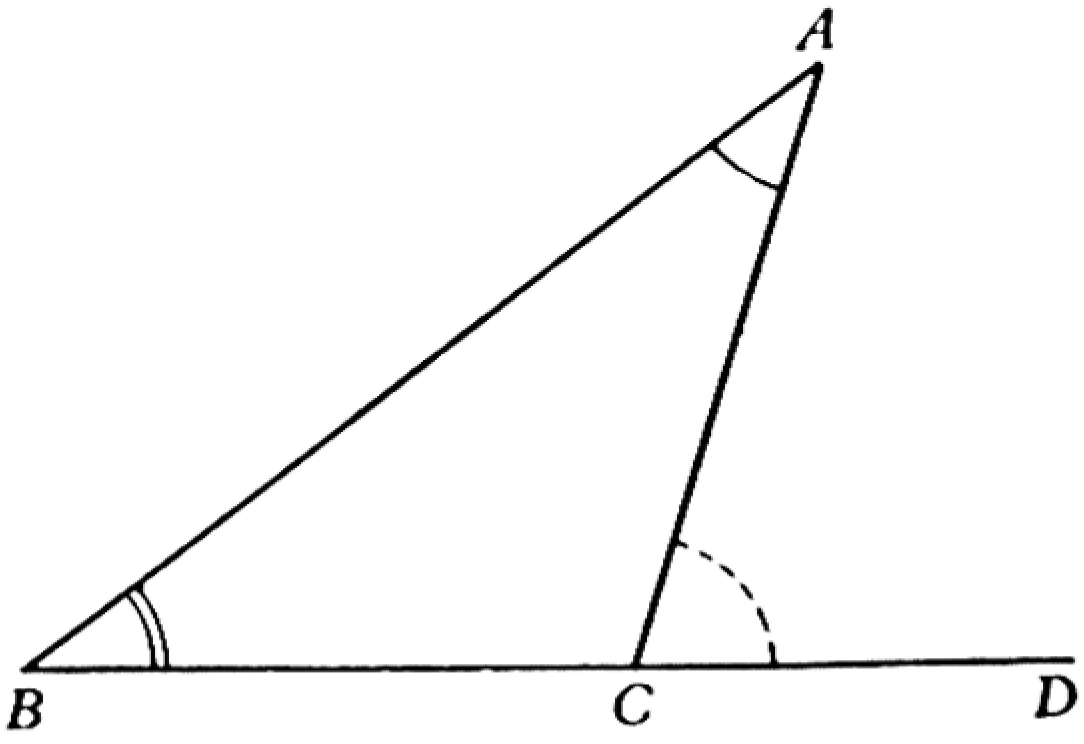


Fig. 52

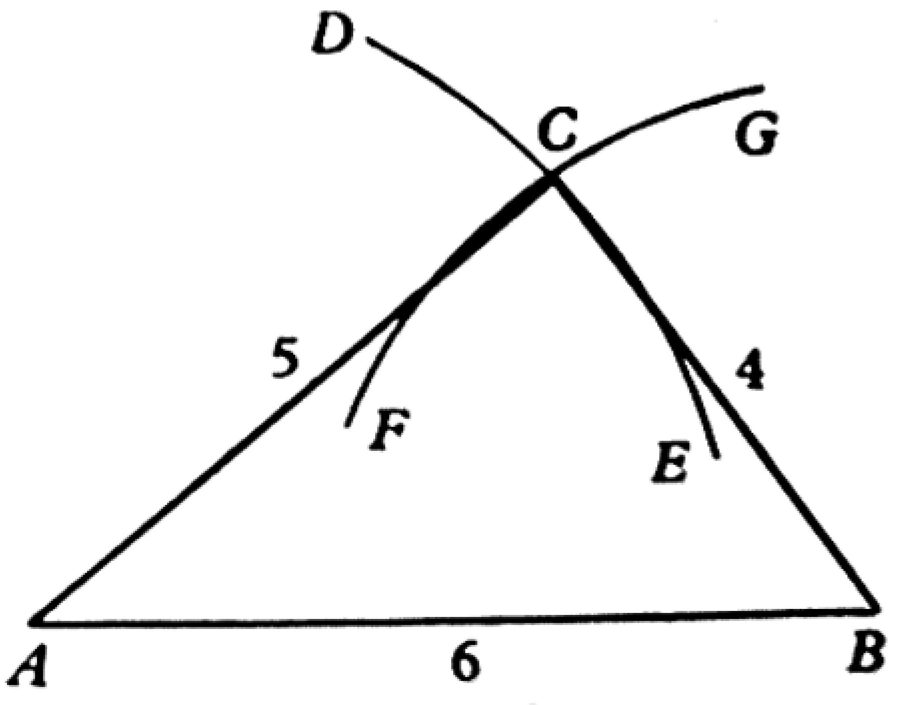


Fig. 53

Construction of triangles

(1) *Given the lengths of three sides.*

Example. Construct a triangle whose sides are 6, 5 and 4 units. One side is chosen as the base, $AB = 6$ units, in fig. 53. The compass is set to a radius of 5 units, centred at A, and the arc DE is drawn. The compass is then centred at B, radius 4 units, and the arc FG is drawn. Since every point on DE is 5 units from A, and every point on FG is 4 units from B, the point C, where the arcs intersect, must be the point C of the triangle. Join AC and BC; then ABC is the required triangle.

It will be noticed that if the radii of the two arcs drawn did not at least total 6 units they would not intersect, so it follows that any two sides of a triangle are together greater than the third side.

(2) *Given the length of one side, and the values of two angles.*

Example. Construct a triangle ABC, in which $AC = 6$ units, angle $A = 40°$, and angle $C = 55°$.

In fig. 54, AC, 6 units long, is chosen as the base, and angles of 40° and 55° are constructed at A and C respectively. When the arms are produced they intersect at B.

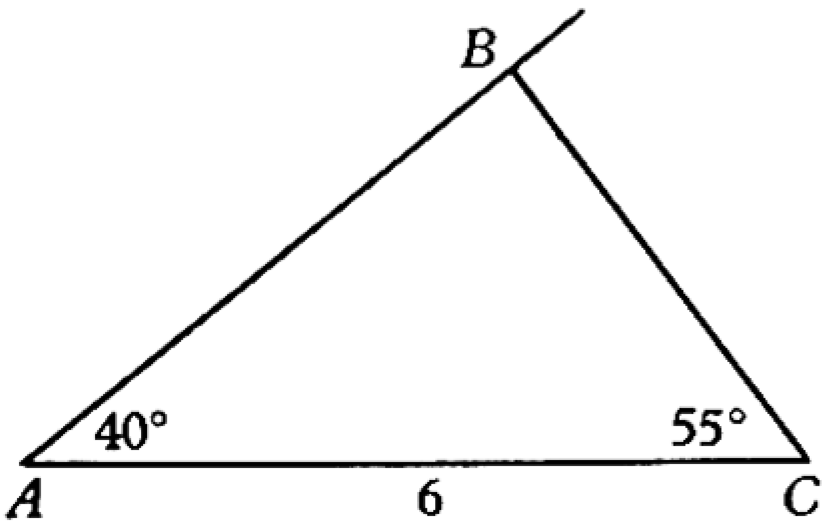


Fig. 54

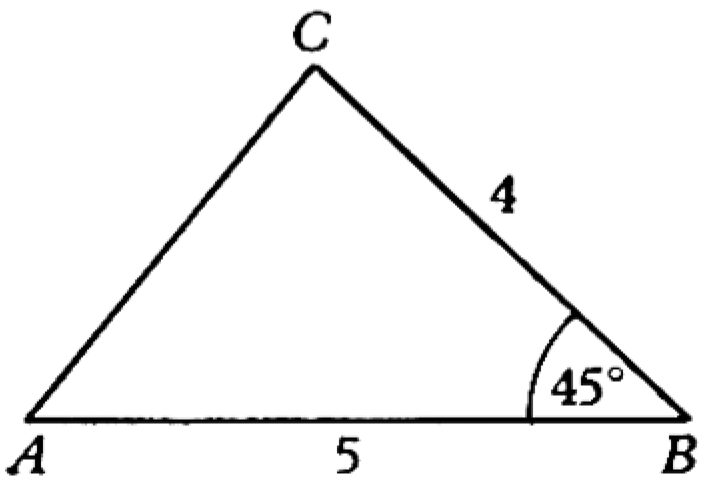


Fig. 55

(3) *Given the lengths of two sides, and the angle included by them.*

Example. Construct a triangle in which angle $B = 45°$, $AB = 5$ units, $BC = 4$ units.

In fig. 55, AB, 5 units long, is drawn, and at B an angle of 45° is made. The arm is continued 4 units to C, then A and C are joined.

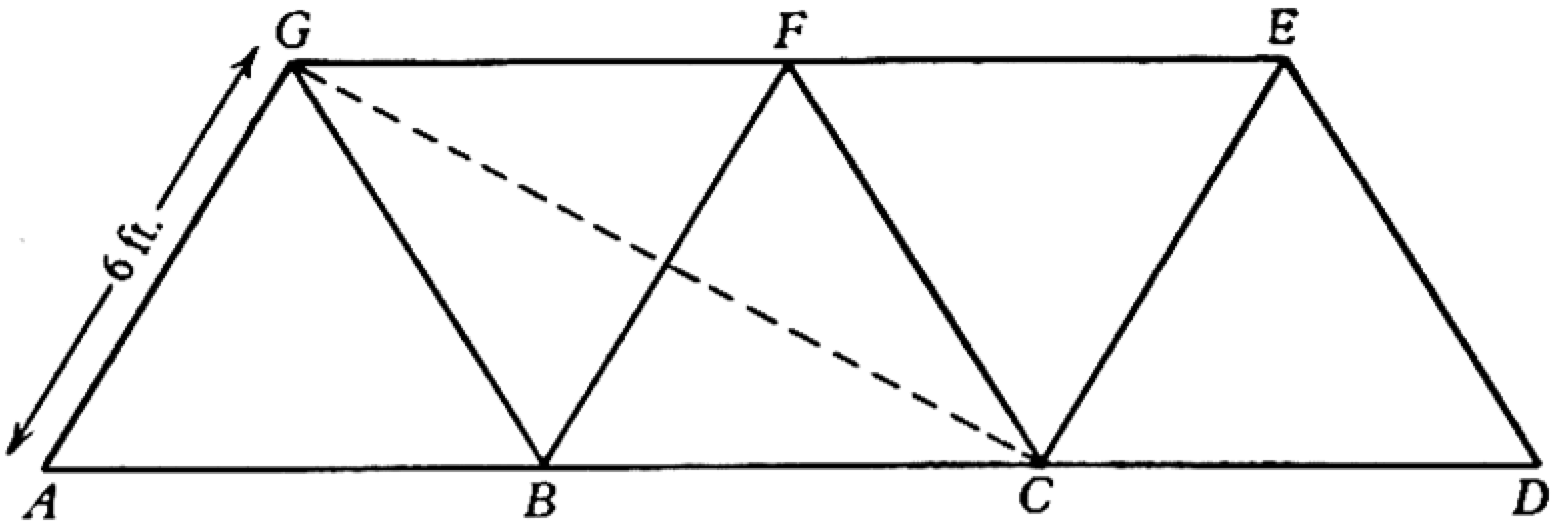


Fig. 56

Example. Fig. 56 shows a Warren girder forming one side of a bridge. The member AG is 6 ft. long, and all triangles are equilateral. Prove, without measurement,

(*a*) that the width of the span is 18 ft.,

(*b*) that AG is parallel to BF and to CE,

(*c*) that a line joining GC is perpendicular to AG.

(*a*) If all triangles are equilateral, all members are 6 ft. long, thus the width of the span is 6 + 6 + 6 ft. = 18 ft.

(*b*) Since all triangles are equilateral, angle AGB = angle GBF, therefore AG is parallel to BF (alternate angles equal, Chapter 21).

(*c*) Join GC, then triangle GBC is isosceles ($\because GB = BC$); thus angle BGC = angle BCG. Also, as angle $GBC = 120°$, then angle $BGC = 30°$. Hence angle AGB + angle $BGC = 90°$. (Q.E.D.)

Example. A mast 50 ft. long is held by a stay 25 ft. long attached to a point in the mast 18 ft. from the bottom. The mast and stay are attached to the deck of the ship at points 15 ft. apart. Find, by means of a scale drawing, the height of the top of the mast above the level of the deck.

(See fig. 57.) Let A and B be the feet of the stay and mast respectively, and let AB be 15 units.

Centre A radius 25 units, and centre B radius 18 units. Let C be the point of intersection of the two arcs, then C is the point on the mast to which the stay is fixed. Produce BC to D, where $BD = 50$ units. With a set-square, draw DE perpendicular to AB produced. Measure DE. By measurement, DE = 49 units (approx.).

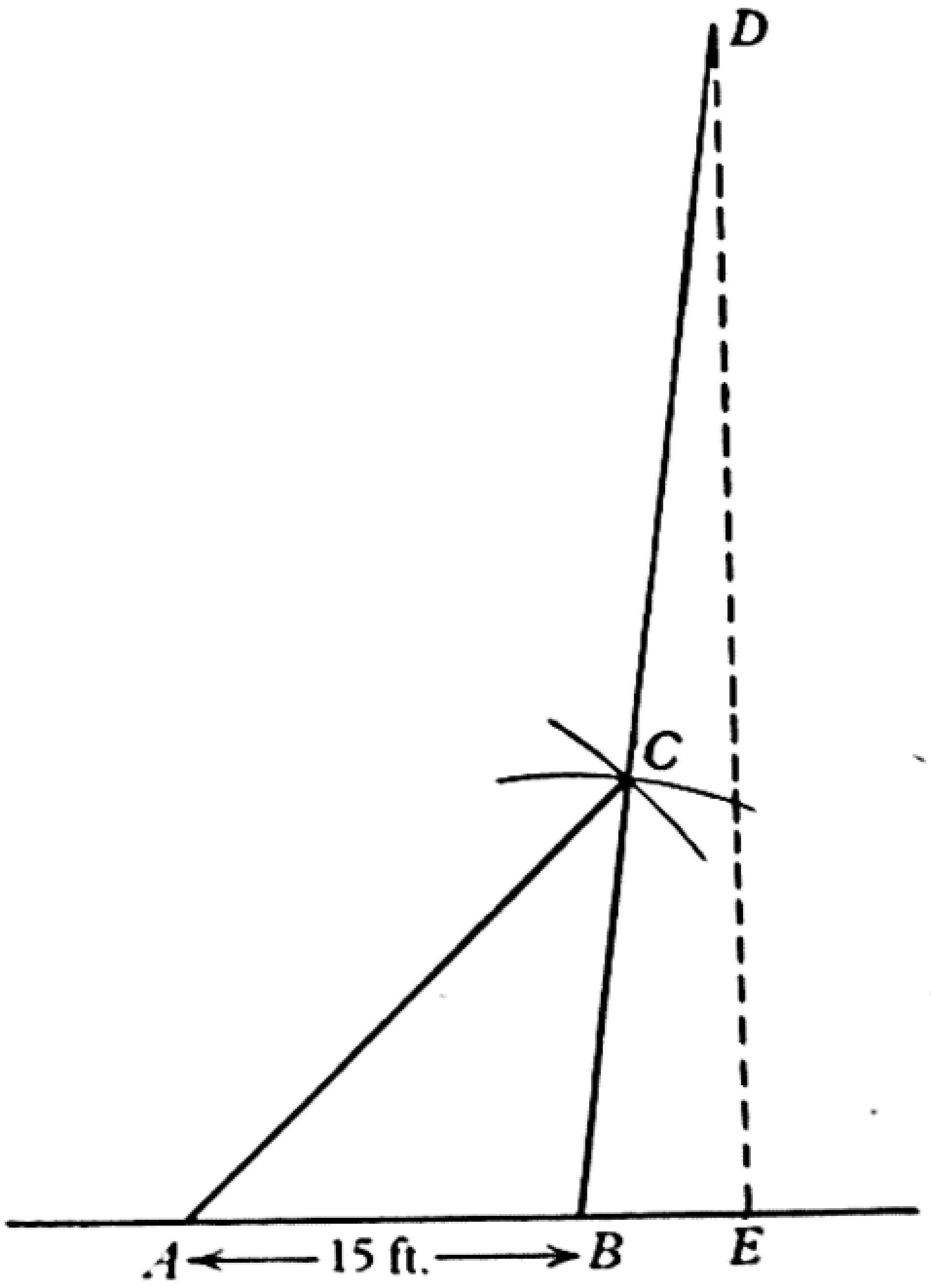


Fig. 57

Thus, height of top of mast above deck = 49 ft. (approx.).

Examples for Chapter 22

1. Fig. 58 represents a Warren girder. All three triangles are isosceles and right-angled:

(*a*) Prove that AC is parallel to DE.

(*b*) If the width of the span is 40 ft., what is the height?

(*c*) Find the sum of the interior angles of the quadrilateral $ACED$.

(*d*) Draw the girder to scale, and measure the lengths of the four equal members. (*Chap.* 22)

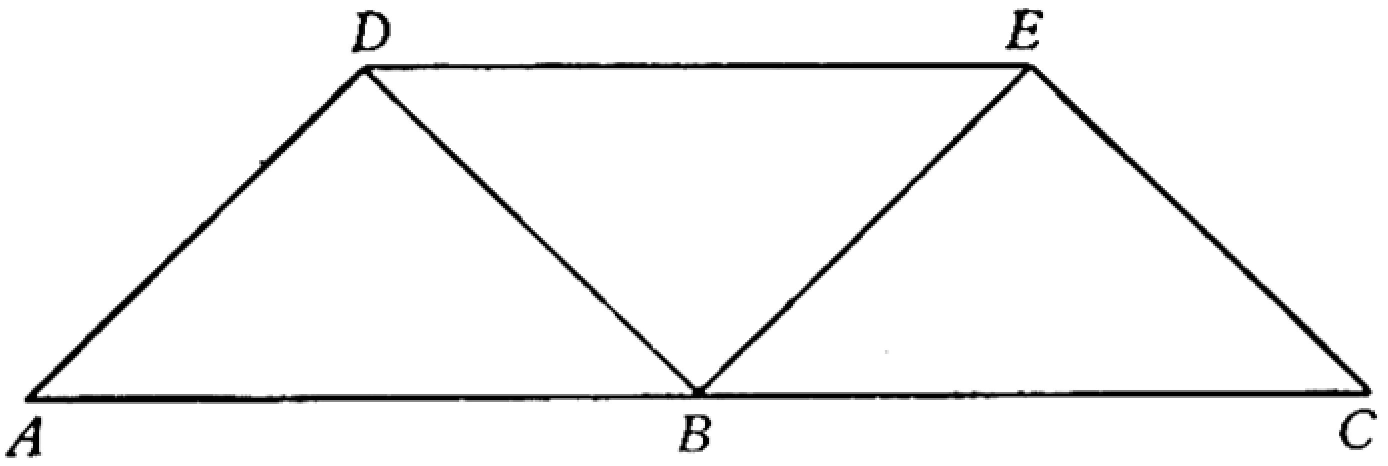


Fig. 58

2. Fig. 59 represents a jib crane (not drawn to scale). If angle $ABC = 130°$ and angle $BDC = 110°$, calculate the angle between the jib and the tie. Find also the angle between the jib and the chain.

Draw the figure to scale with the jib 20 ft. long and the distance AB 2 ft. Measure the distance of C from the vertical through BD, and measure the height of C above the level of A. (*Chap.* 22)

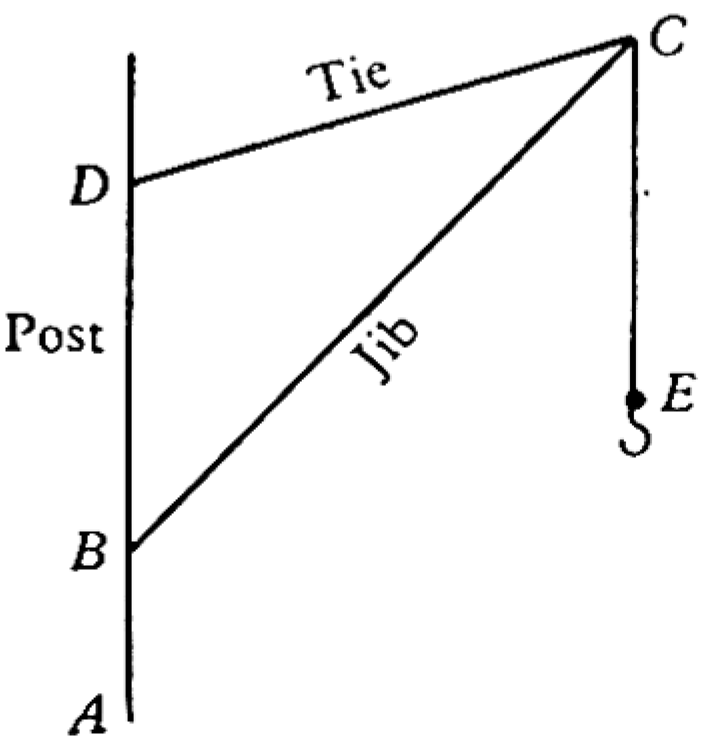


Fig. 59

3. Construct a regular octagon by describing a circle radius 2 in., and in it eight isosceles triangles whose equal sides are radii. Measure the length of one side of the octagon as accurately as possible. Measure also the distance from the centre to the mid-point of one of the sides. If the area of a rectangle is equal to the length multiplied by the breadth, find an approximate answer for the area of the octagon. (*Chap.* 22)

4. Construct a regular hexagon by a similar method to that in Question 3, and calculate its area. The area of a hexagon is given approximately by the formula

$$\text{Area} = 2{\cdot}6 \times \text{length of one side} \times \text{length of one side}.$$

Compare answers by the two methods. (*Chap.* 22)

5. Fig. 60 represents a chain ABC carrying a load at C. The chain is attached to two walls at A and B, and B is 6 ft. lower than A. If the chain is 20 ft. long, the walls are 16 ft. apart, and it is

known that the parts BC and AC are equally inclined to the horizontal; draw the figure to scale and measure the lengths AC and BC. (*Chap.* 22)

6. Fig. 61 represents a girder structure. ABC is a straight line, AD is vertical, and the triangle ABD is isosceles

$$(AB = BD = 10\,\text{ft.}).$$

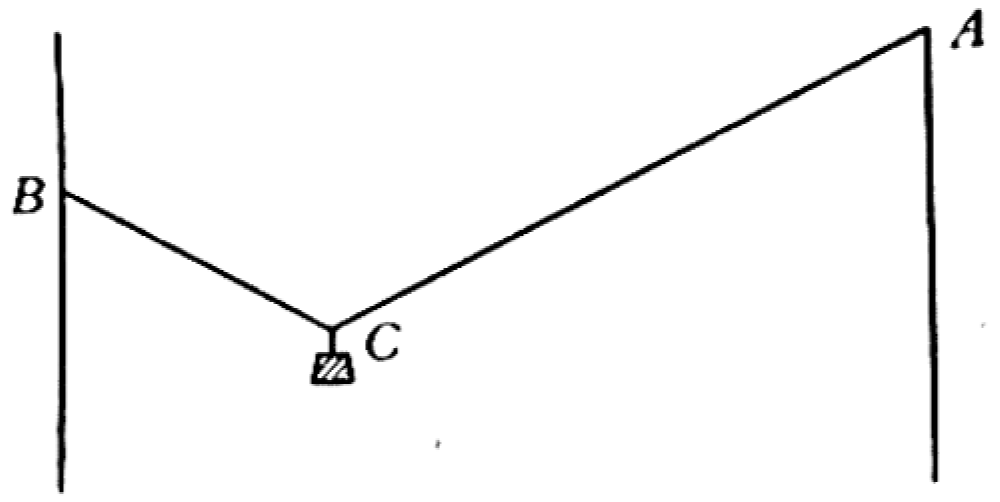


Fig. 60

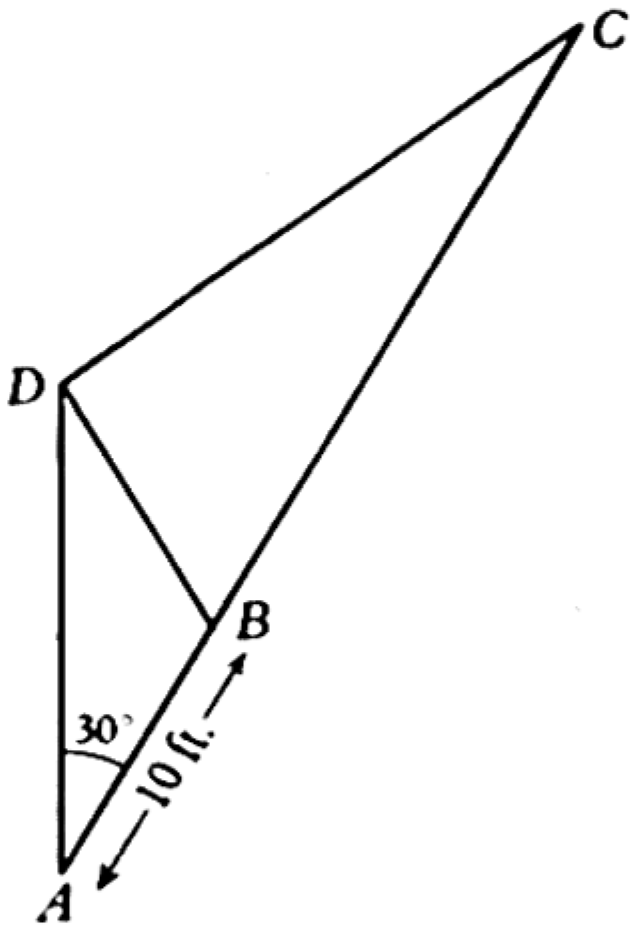


Fig. 61

If $AC = 30$ ft., (a) calculate the angle DBC, (b) prove that a line from D to the mid-point of BC will be horizontal, (c) calculate the angle CDB. Draw the figure to scale and measure the length of the member DC. (*Chap.* 22)

7. Draw AB 5 in. long. With centre A draw a circle of radius 4 in., and with centre B draw a circle of radius 3 in. Call the two points in which these circles intersect C and D. Join AC, AD, BC and BD, and measure the angles ACB and ADB. Hence write the area of the figure $ACBD$. (*Chap.* 21)

8. Draw any circle, and in it draw any two chords AB and CD, intersecting at E. Measure the lengths of AE, EB, CE and ED as accurately as possible. Compare the products $AE \times EB$, and $CE \times ED$. (*Chap.* 21)

9. A ship sailing on a course S. 30° W. at 16 m.p.h. is 8 miles due E. of a lighthouse. What will be its bearing and distance from the lighthouse after half an hour? How much later will it be before the ship is due S. of the lighthouse? (*Chap.* 21)

10. A straight road runs E.–W. A power station C lies to the north of the road, and is 5 miles distant from each of two points B and D on the road, 6 miles apart. From C two straight cables, each 8 miles long, run to the road, joining it at points A and E. By means of a scale drawing find the distance between the points A and E. (*Chap.* 21)

Answers to Examples for Chapter 22

1. (*b*) 10 ft. (*c*) 360°. (*d*) 14·1 ft. (approx.).

2. Angle $DCB = 20°$, angle $BCE = 50°$; 15·3 ft., 14·9 ft.

3. 1·53 in.; 1·85 in.; 11·32 sq.in.

5. Centre A, radius 20, and mark off point D on wall below B. Measure angle BDA, and construct equal angle DBC. Join BC. Ans. 15·4 ft. and 4·6 ft.

6. Angle $DBC = 60°$; angle $CDB = 90°$; $DC = 17{\cdot}3$ ft. (approx.).

7. Angle ACB = angle $ADB = 90°$. Area of $ACBD = 12$ sq.in.

8. $AE \,.\, EB = CE \,.\, ED$.

9. S. 30° E.; 8 miles; 30 min.

10. 13·9 miles (approx.).

CHAPTER 23

CONGRUENT AND SIMILAR TRIANGLES

Six facts determine a triangle completely; these are, the lengths of three sides, and the values of three angles. For practical purposes, many triangles may be determined from a knowledge of three of these facts only. They are: (*a*) the lengths of the three sides, (*b*) the lengths of two sides and the value of the angle included by these sides, (*c*) the values of two angles and the length of one side.

Two triangles are *congruent* if they are equal in all respects. The conditions of congruency are important because they are used throughout geometry as parts of further proofs.

(1) *Two triangles are congruent if the three sides of one are equal to the three sides of the other.*

Fig. 62 shows two triangles ABC and DEF, in which $AB = DE$, $AC = DF$, $BC = EF$. In order to construct either of these triangles by the method of Chapter 22, we should draw a line of length AB for the base, then find the point C by the intersection of arcs of circles radii AC and BC respectively. Since the arcs could only intersect in one point, the point C is fixed. Also, since the same lengths are chosen for the radii of the arcs in the second case, the point F must lie in the position corresponding to the point C. Therefore the two triangles ABC and DEF are equal in all respects.

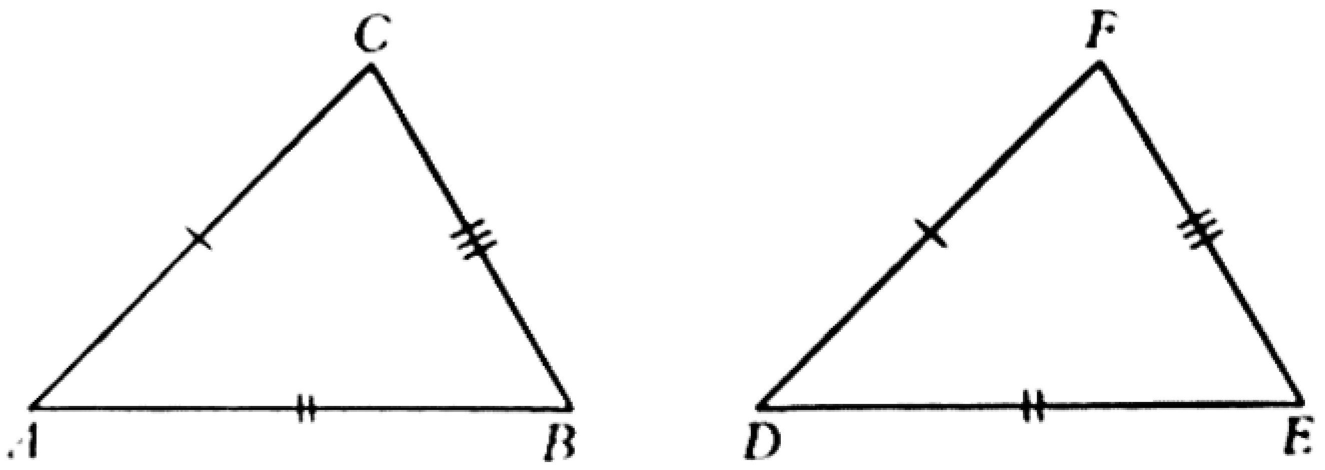


Fig. 62

(2) *Two triangles are congruent if two angles of the one are equal to two angles of the other, and any side of the first equal to the corresponding side of the other.*

Fig. 63 shows triangles ABC and DEF in which $AB = DE$, angle A = angle D, angle B = angle E. (It makes no difference which pair of angles has been given equal, because the sum of the interior angles of a triangle is 180°, thus the third angles are also equal.) As in the previous case, the method of constructing one of the triangles, using ruler and protractor, makes it obvious that only one triangle can be drawn from the given facts. Therefore the triangles ABC and DEF are equal in all respects.

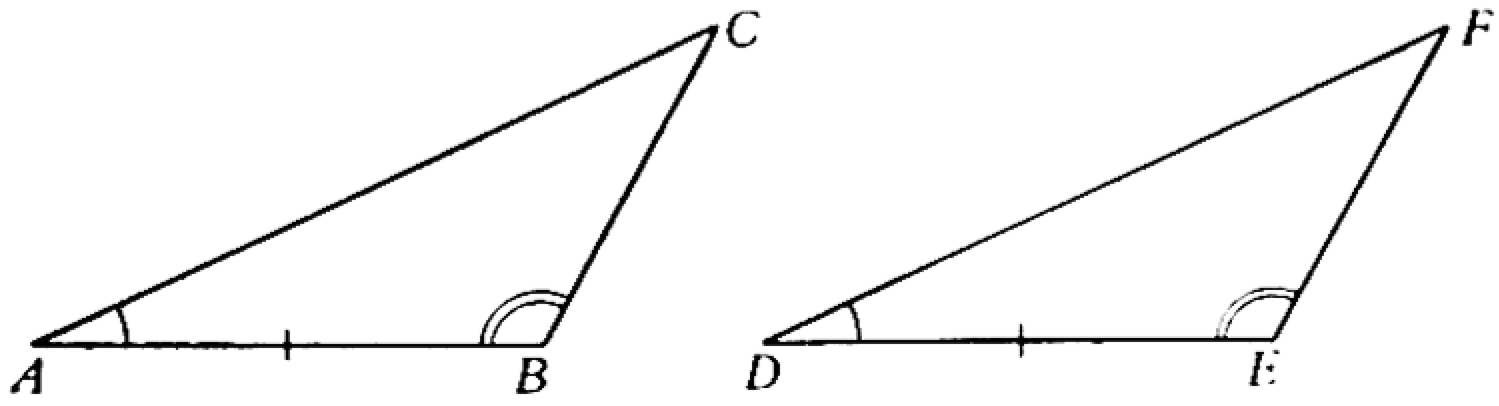


Fig. 63

(3) *Two triangles are congruent if they have two sides of the one equal to two sides of the other, and the angles included by those sides equal.*

Fig. 64 shows triangles ABC and DEF in which $AC = DF$, $BC = EF$, angle C = angle F. To construct ABC, choose AC as base, then measure the angle C and continue the arm CB to the required length. Join AB. Only one triangle can be drawn from these data. Hence the triangles ABC and DEF are equal in all respects.

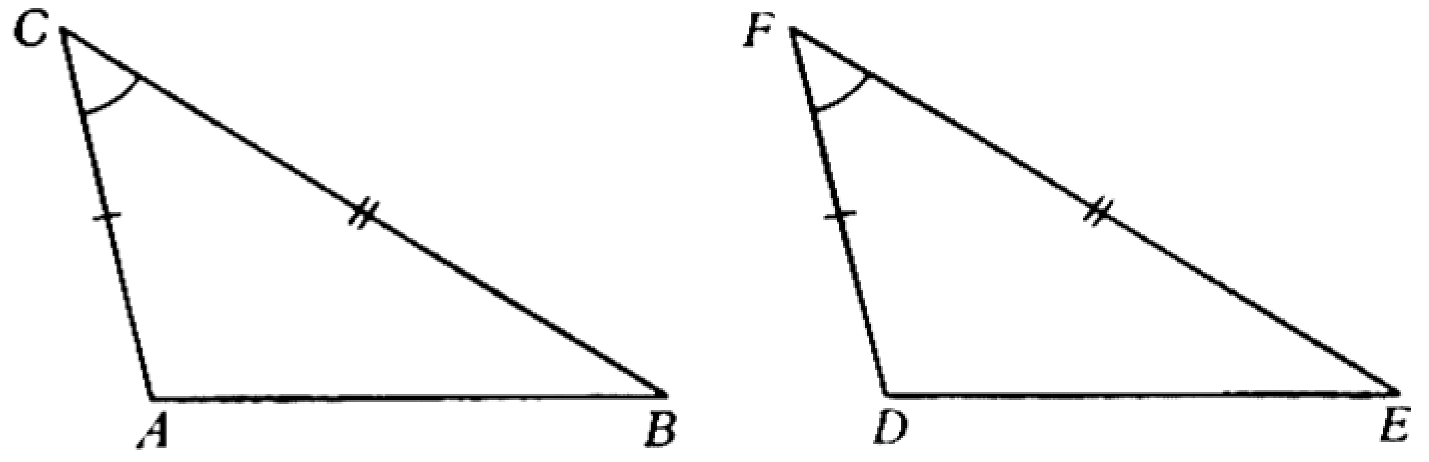


Fig. 64

Note that the angles given equal in the data are specified as those whose arms are the given sides. Triangles are not necessarily congruent if two pairs of sides, and a pair of angles not included by those sides, are given equal.

(4) *Two right-angled triangles are congruent if their hypotenuses are equal, and one other side of one of them equal to a side of the other.*

Fig. 65 shows two right-angled triangles ABC and DEF in which $BC = EF$, and $AB = DE$. (BC and EF are the *hypotenuses* of the triangles, i.e. the longest sides of the right-angled triangles.) To construct either of the triangles a vertical is drawn to correspond to the direction AC or DF (length not known). Then from B (or E) an arc of the required length for the hypotenuse is drawn to intersect the vertical line at C (or F). Obviously, only one triangle is possible, therefore the triangles ABC and DEF are equal in all respects.

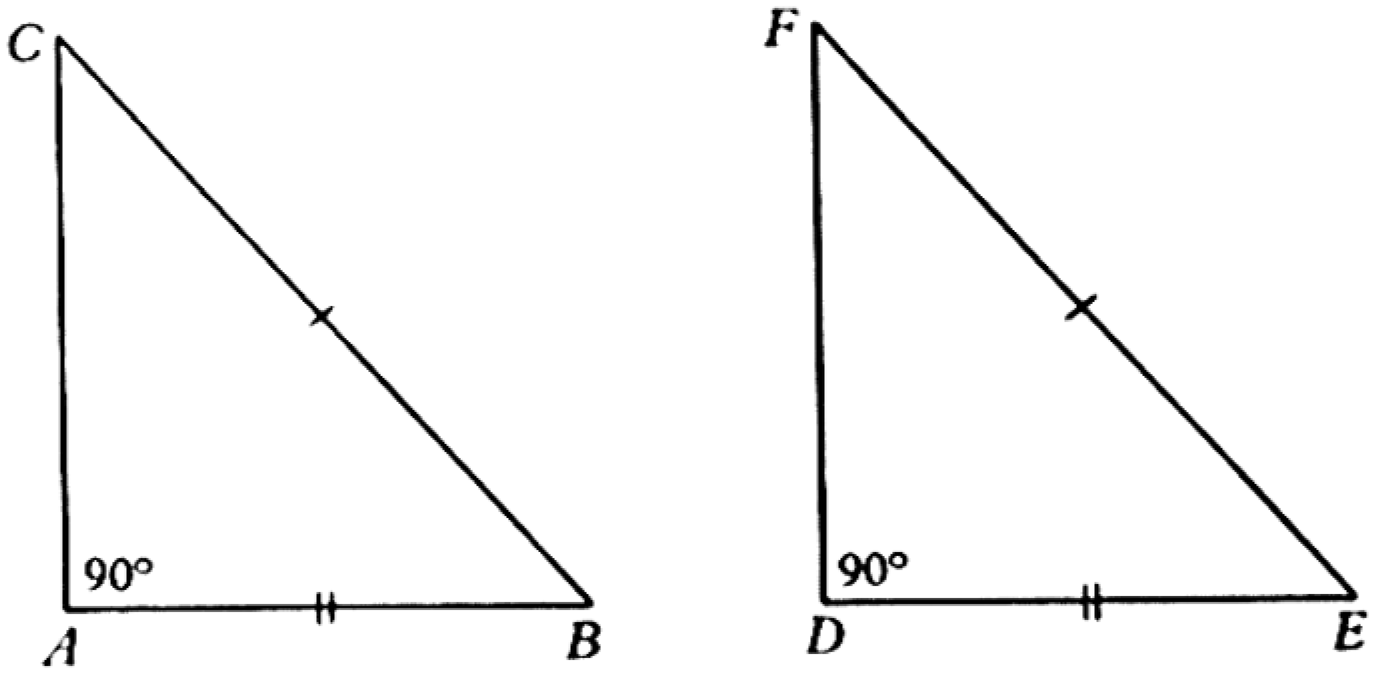


Fig. 65

The ambiguous case

A warning was given after the third case of congruency. Fig. 66 shows two triangles ABC and DEF in which angle A = angle D, $AB = DE$ and $BC = EF$ (i.e. the given angle is not included between the given sides). Obviously triangles ABC and DEF are not congruent. To construct a triangle from the

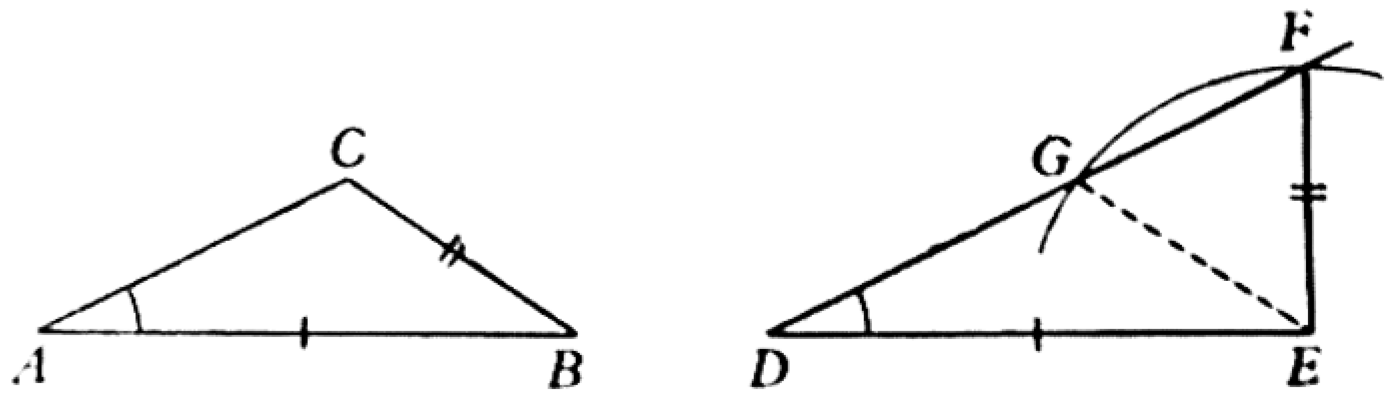


Fig. 66

given data, the base DE is drawn to scale, and a line in the direction DF is drawn from D. To fix F, an arc radius BC is drawn from E. This arc, however, intersects the third side in two points F and G. Hence, two triangles, DEF and DEG, are both drawn from the given data, and only the triangle DEG is congruent with the triangle ABC. This is called the ambiguous case, and is studied more fully in Book II.

To prove that, in an isosceles triangle, the angles opposite the equal sides are equal.

Data: Triangle ABC in which $AC = BC$ (fig. 67).

It is required to prove that angle A = angle B.

Construction. Let D be the mid-point of AB. Join CD.

Proof. Compare the triangles ACD and CDB for congruency:

$$AC = CB \quad \text{(from the data)},$$

$$CD \text{ is common to both triangles},$$

$$AD = DB \quad \text{(by construction)}.$$

Therefore the triangles ACD and CDB are congruent.

In particular angle A = angle B. Q.E.D.

Similar triangles

Two triangles are similar if the angles of one are equal to the angles of the other. Fig. 68 shows two similar triangles in which angle A = angle D, angle B = angle E, angle C = angle F. The lengths of the sides are marked, showing that the ratios $\frac{AB}{DE}$, $\frac{AC}{DF}$, $\frac{BC}{EF}$ are all equal to 2.

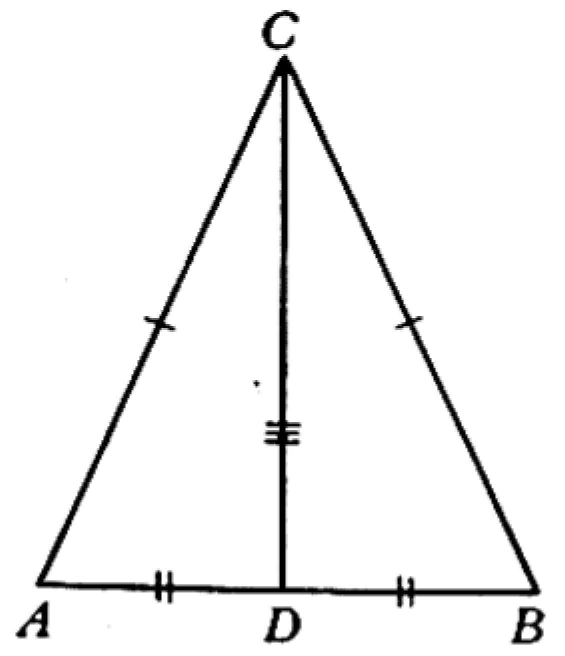


Fig. 67

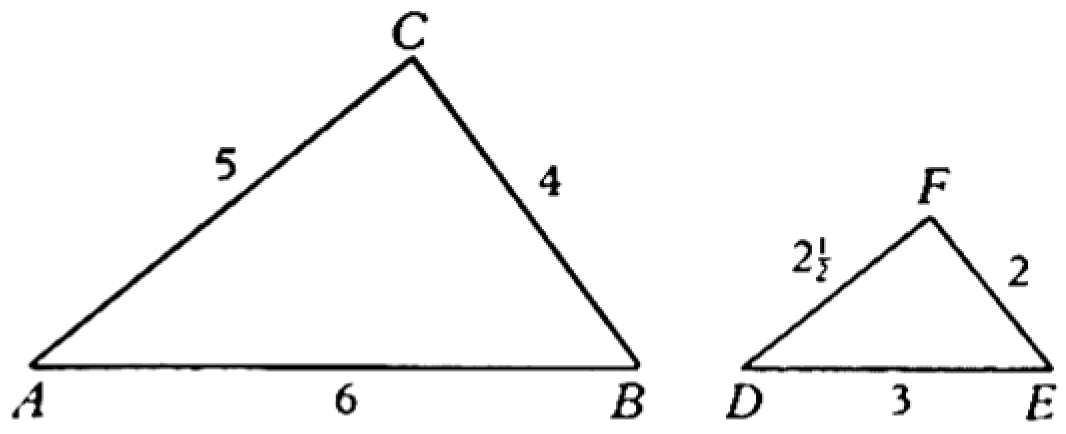


Fig. 68

Whatever lengths are drawn, it is always true that for similar triangles the ratios of the lengths of corresponding sides are equal. Moreover, the ratios of the lengths of corresponding sides of similar polygons are still the same. This fact will be evident from machine drawing, for if an edge of a casting is 1 ft. 6 in.

long, but drawn 9 in. long, then another edge of the same casting which is 12 in. long must be drawn 6 in. long. In short, the face of the casting, and the drawing, are similar figures.

Construction of scales

It is required to divide a line into a number of equal parts, when the appropriate divisions are not given on a ruler. The line *AB* (fig. 69) is to be divided into 6 equal parts.

From one end of the line, *A*, draw a line of length 6 units on any convenient scale, and arrange that the further end of this line is roughly vertically above *B*. This is the line shown as *AC*, with equal divisions at *D*, *E*, *F*, *G* and *H*. Join *CB* and, by means of set-square and ruler, as shown in Chapter 21, draw lines through *D*, *E*, *F*, *G* and *H*, parallel to *CB*, to cut *AB* in *J*, *K*, *L*, *M* and *N*. Then *AB* is divided into 6 equal parts. [When *C* is almost vertically above *B* the cutting points of the parallel lines and the line *AB* are easier to distinguish accurately than if the line *BC* made a small angle with *AB*.]

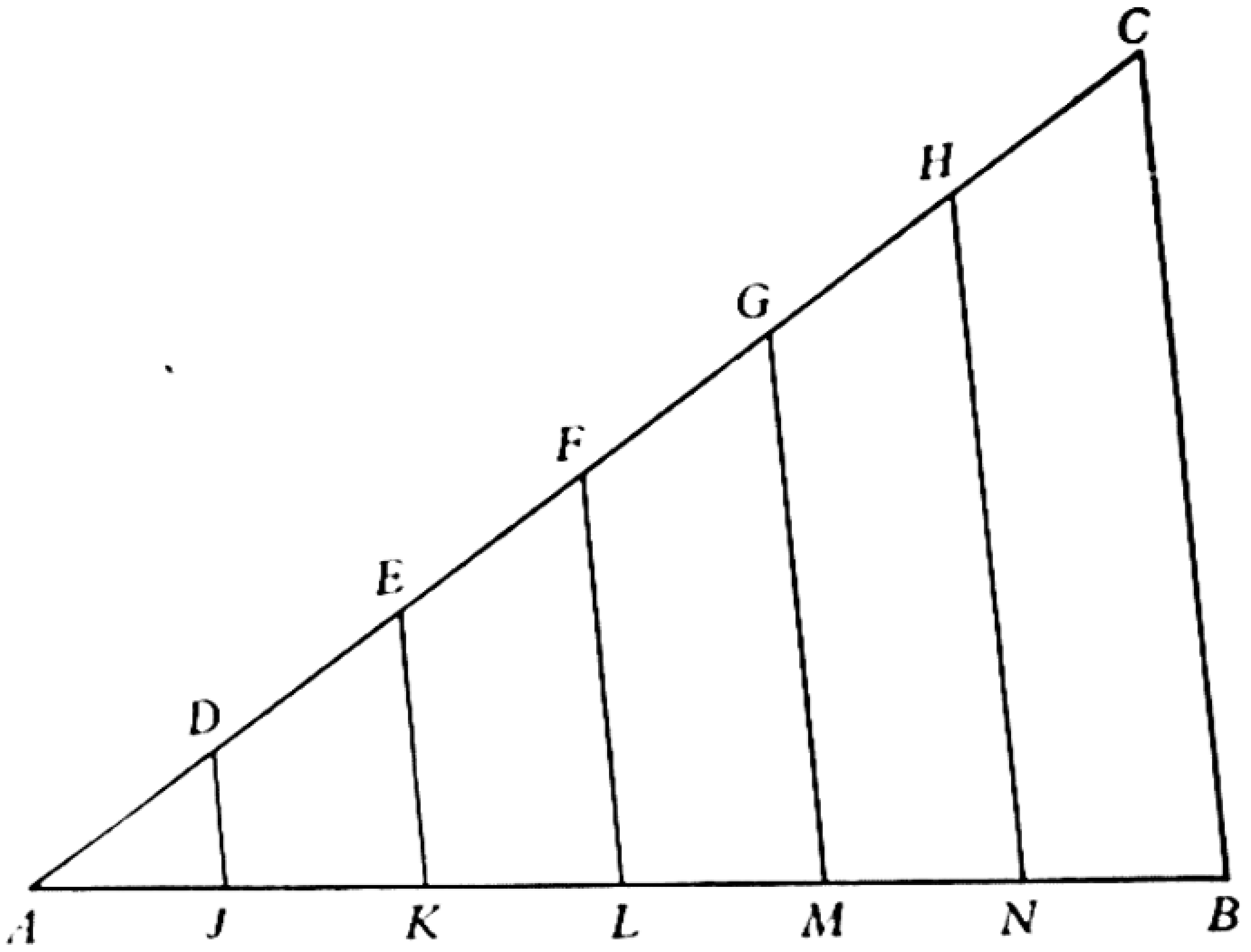


Fig. 69

Proof. The triangles *ADJ*, *AEK*, *AFL*, *AGM*, *AHN* and *ACB* are all similar because they all contain the same angle *A*, and also, by the facts of parallel lines (Chapter 21), the other two angles are equal in the corresponding triangles. Thus

$$\frac{AD}{AC} = \frac{AJ}{AB} = \frac{1}{6}.$$

Analytical problems in geometry require a definite arrangement, and practice in such problems will help the student to set out mathematical arguments generally in logical fashion. We have used a common method in one proof in this chapter. We shall adopt the same method in the following examples.

Example. Fig. 70 shows a girder framework in which $ABCD$ is a straight line. If $AE = ED$, and $AB = EC$, and angle AEB = angle DEC, prove that the triangle BEC is isosceles.

If angle $AEB = 40°$, find the angles of the triangle BEC.

Data. Figure $ABCDE$, in which $AE = ED$, $AB = EC$, and angle AEB = angle DEC.

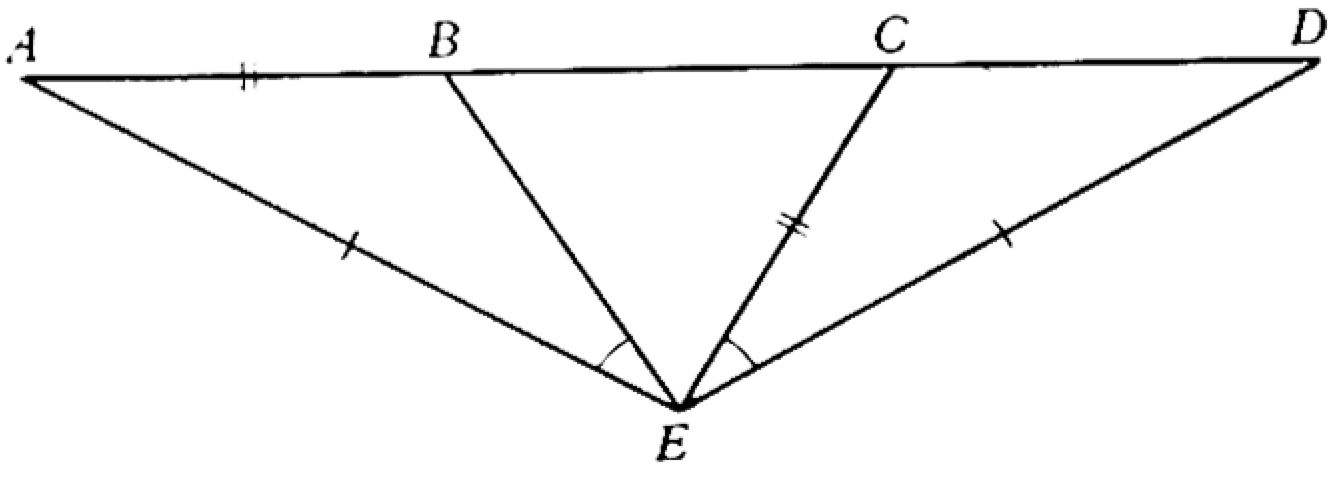


Fig. 70

It is required to prove that $BE = EC$.

Proof. AED is an isosceles triangle because $AE = ED$.

$\therefore$ angle EAB = angle EDC.

Compare the triangles AEB and ECD for congruency.

$AE = ED$ ($\because$ given),

angle AEB = angle CED,

angle EAB = angle EDC (Proved above).

$\therefore$ triangle AEB is congruent with triangle EDC.

In particular $BE = EC$.

Hence triangle BEC is isosceles. Q.E.D.

Also, since triangles AEB and ECD are congruent,

$$AB = CD.$$

But it is given that $AB = EC$.

$$\therefore AB = BE = EC = CD.$$

Hence triangles AEB and ECD are both isosceles.

$$\therefore \text{angle } A = \text{angle } D = 40°.$$

$$\therefore \text{angle } ABE = 180° - 2 \times 40° = 100°.$$

$$\therefore \text{angle } CBE = 180° - 100° = 80°.$$

Thus, the angles of the triangle BEC are 80°, 80° and 20°.

Example. AB and CD are two parallel lines, and EF is a transversal intersecting AB in G and CD in H. If J is the mid-point of GH, show that any other transversal which passes through J is also bisected there.

Data. AB and CD parallel, EF intersecting AB and CD at G and H respectively. $GJ = JH$. Similarly KL intersecting AB and CD at M and N respectively (fig. 71).

It is required to prove that $MJ = JN$.

Proof. Compare the triangles MJG and HJN for congruency.

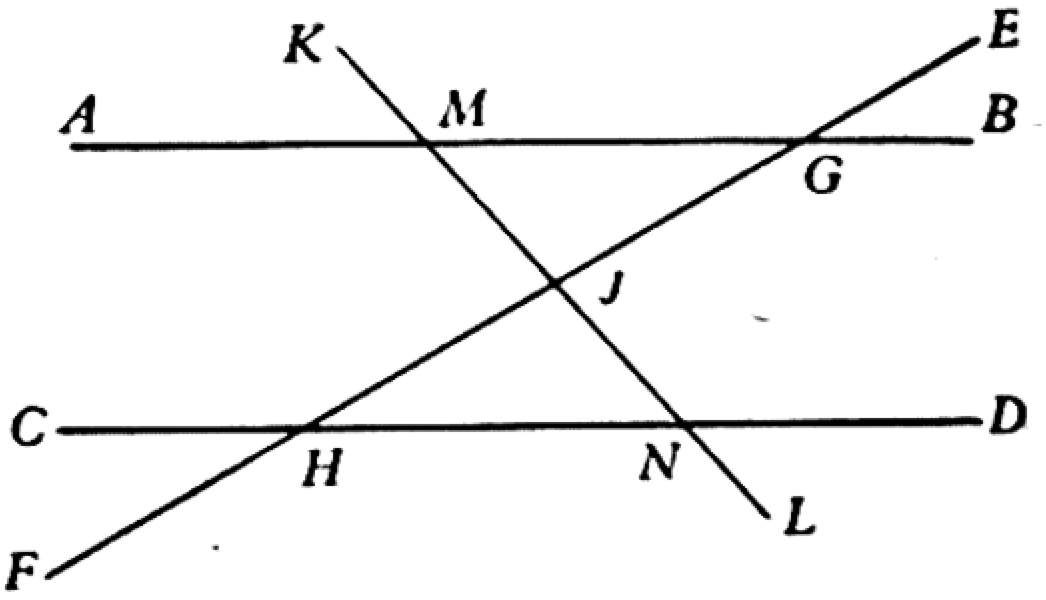


Fig. 71

Angle MGJ = angle NHJ

($\because$ AB parallel to CD).

Angle MJG = angle HJN

($\because$ vertically opposite),

$JG = JH$ (Given).

$\therefore$ triangles MJG and HJN are congruent.

In particular $MJ = JN$. Q.E.D.

Example. ABC is a triangle right-angled at A. AD is drawn perpendicular to BC. Prove the triangles BAD and DAC similar.

Data (fig. 72). Triangle ABC with angle $A = 90°$. AD perpendicular to BC.

It is required to prove that triangles BAD and DAC are similar.

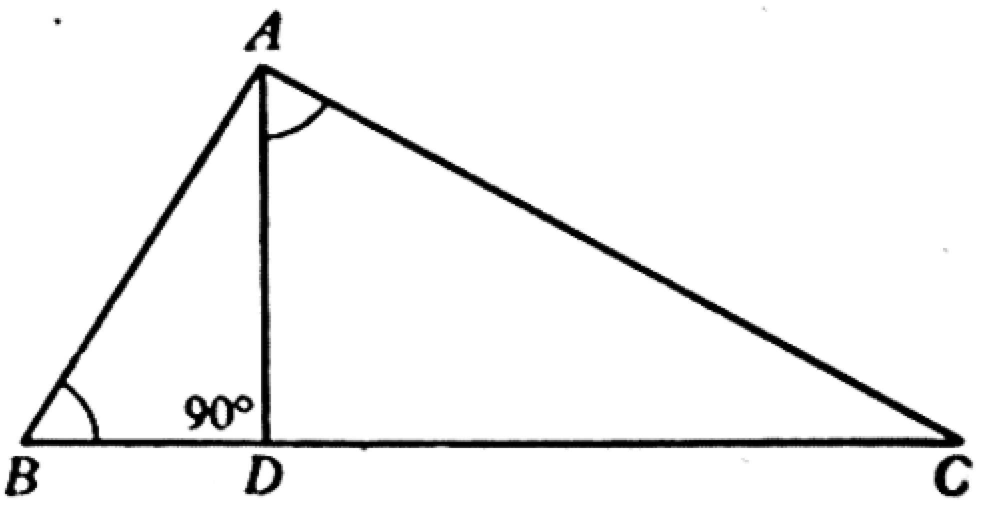


Fig. 72

Proof. Angle BAD + angle DAC = 90° (Given),

angle BAD + angle ABD = 90°

($\because$ angle BAD + angle ABD + angle BDA = 180°).

$\therefore$ angle DAC = angle ABD.

Since the triangles ABD and DAC contain an angle of 90° each, and another pair of equal angles, it follows that they are similar. (Q.E.D.)

This chapter concludes with a few constructions which will be used in Machine Drawing. The proofs are given also as examples of geometrical argument.

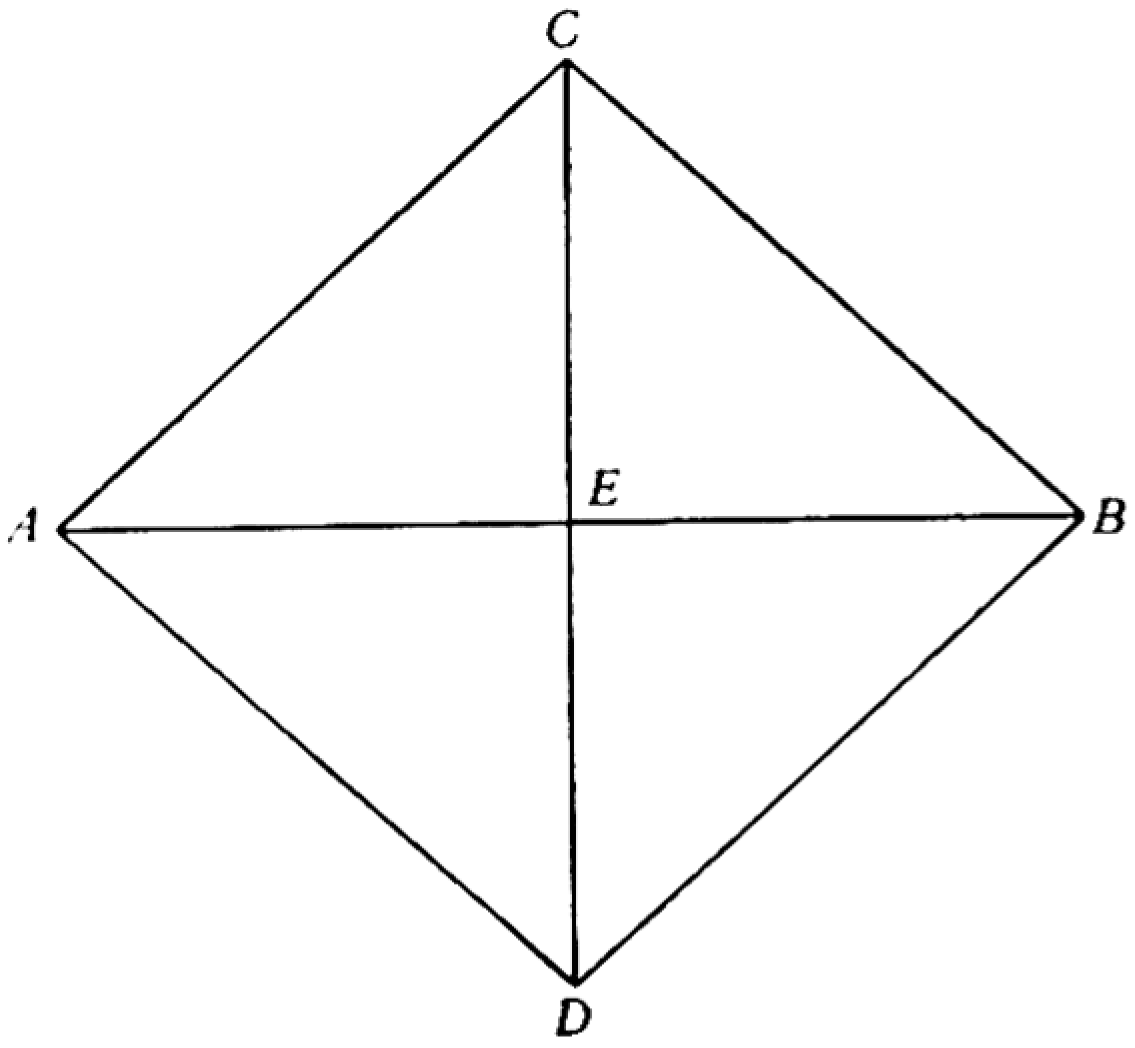


Fig. 73

To bisect a line (see fig. 73). AB is the line to be bisected. Centre A, using any radius greater than half the length of AB, and describe arcs above and below the line. Centre B, using the same radius as before, and describe arcs above and below. Let the two pairs of arcs intersect in C and D. Join CD. If CD intersects AB at E, then E is the mid-point of AB.

Proof. Join AC, CB, AD, DB.

Compare triangles ACD and CDB for congruency.

$AC = CB$ ($\because$ radii of equal circles),

$AD = DB$ ($\because$ radii of equal circles).

CD is common to both triangles.

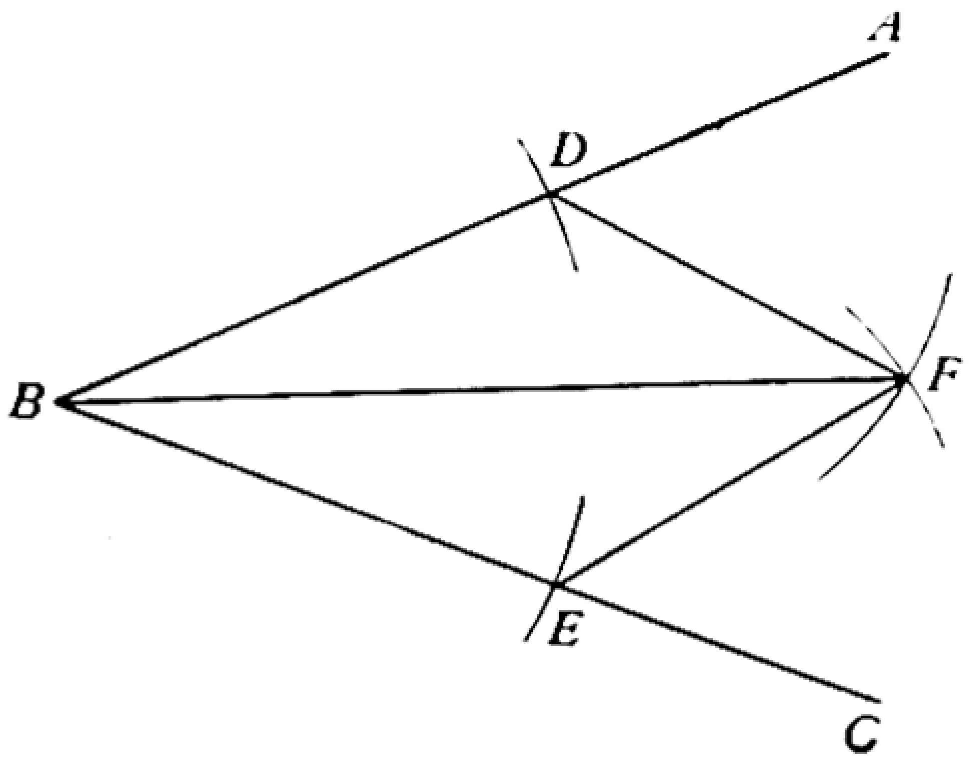


Fig. 74

Therefore the triangles are congruent.

In particular,

$$\angle ACE = \angle ECB.$$

Now comparing triangles ACE and ECB for congruency,

angle ACE = angle ECB (Proved).

Angle CAE = angle CBE

($\because$ ACB is an isosceles triangle).

$AC = CB$ ($\because$ radii of equal circles).

Therefore the triangles are congruent. In particular $AE = EB$.

To bisect an angle (see fig. 74, p. 223). ABC is the angle to be bisected. Centre A, any radius, and mark off equal distances BD and BE along BA and BC. Centre D, any radius greater than half the distance DE, and describe an arc. With the same radius describe an arc using E as the centre. Let these arcs intersect at F. Join BF. Then BF bisects the angle ABC.

Proof. Join DF and EF.

Compare the triangles BDF and BEF for congruency.

$$BD = BE \quad (\because \text{ radii of same circle}),$$
$$DF = EF \quad (\because \text{ radii of equal circles}).$$

BF is common to both triangles.

Therefore the triangles are congruent.

In particular angle DBF = angle EBF.

To draw a perpendicular to a given line from a given point outside the line (see fig. 75). XY is the given line, and P the given point. Centre P, any radius, to describe arcs cutting XY in A and B. Centre A with any radius greater than half AB, describe an arc below XY. Centre B, with the same radius as before, and describe an arc cutting the last arc at C. Then PEC is perpendicular to the line XY (E being the point in which PC intersects XY).

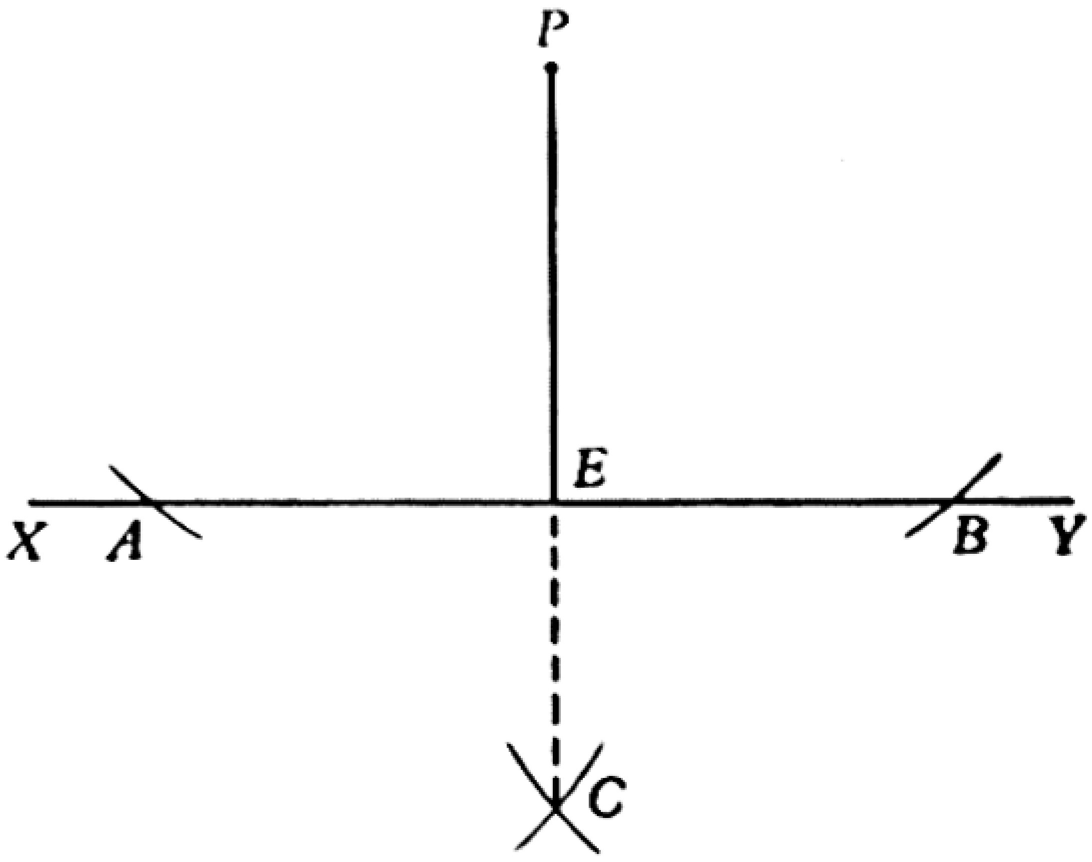


Fig. 75

Proof. As in the case of the bisection of a line, it may be proved that the triangles AEP and PEB are congruent. In particular, angle AEP = angle BEP. Now angle AEP + angle $BEP = 180°$.

$$\therefore \text{ angle } AEP = \text{angle } BEP = 90°.$$

Examples for Chapter 23

1. The following facts are given for a triangle: $AB = 8$, $BC = 6$, angle $BAC = 40°$. Show that two triangles satisfy these conditions by a scale drawing, and give the two values of angle ACB, and of AC. **(*Chap.* 23)**

2. Construct a perpendicular to the line XY from a point which lies on the line XY. (Start by marking off equal arcs on the line, using P as centre.)

Prove that your construction is correct. (*Chap.* 23)

3. (*a*) Measure an angle of 120°. Without further use of the protractor construct from it angles of 60°, 30° and 45°.

(*b*) Construct a triangle ABC in which angle $BCA = 90°$, angle $ABC = 36°$. Bisect the angle ABC and let the bisector meet AC in D. Mark off E on the line DB, such that $DE = AD$. Join AE. Prove that triangle ABE is isosceles. (*Chap.* 23)

4. ABC is an isosceles triangle (fig. 76) and D is the mid-point of BC. Prove that triangles BED and CFD are congruent, and that triangles AED and AFD are congruent. Find the values of the angles ABD and AED. (*Chap.* 23)

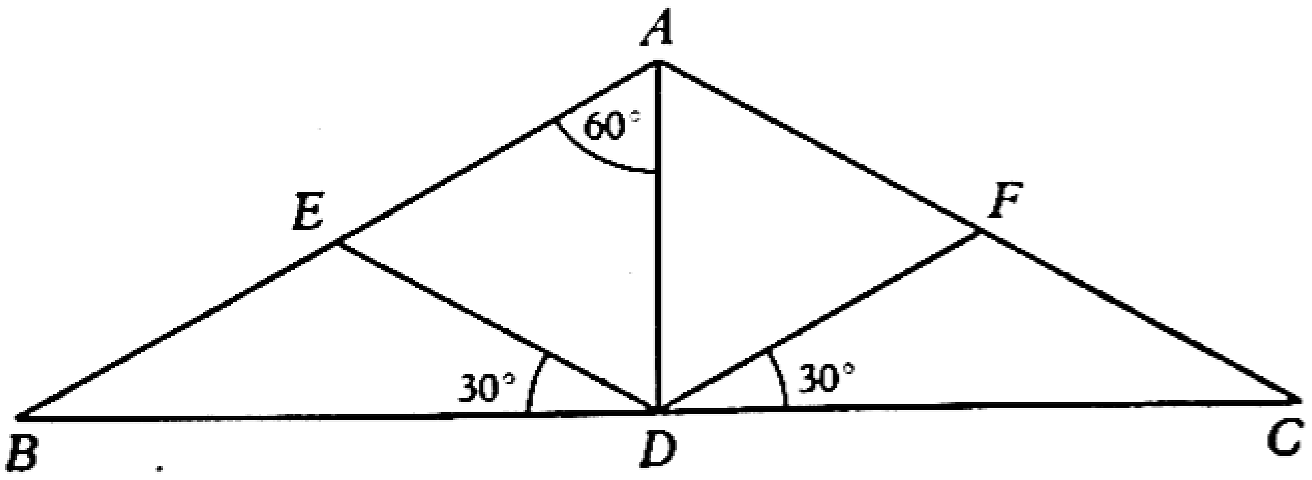


Fig. 76

5. (*a*) ABC is a triangle right-angled at A. From A, AD is drawn perpendicular to BC. Prove that the triangles ABC, ABD and ADC are all similar.

(*b*) If the lengths are as follows: $BC = 5$, $AB = 3$, $AC = 4$, calculate the length of AD. (*Chap.* 23)

6. (*a*) AB is a line 4 in. long, and E is a point on the line 1 in. from B. Another line CED is drawn. F is a point on the line CED, such that $AF = 2$ in. Through B a line is drawn parallel to AF, to cut CED in G. Prove that the triangles AFE and BGE are similar, and calculate the length of BG.

(*b*) If the point E had been 2 in. from B in the first place, prove that the triangles AFE and BGE would then be congruent. (*Chap.* 23)

7. (*a*) Calculate the angle of a regular pentagon.

(*b*) $ABCDE$ is a regular pentagon. Calculate the angles ACB, ADC, and prove that AC is parallel to ED. (*Chap.* 22)

8. (*a*) Construct a triangle in which $AB = 12$ units, $BC = 5$ units, $AC = 13$ units, and measure the angle ABC.

Show that $$\frac{AB}{AC} \times \frac{AB}{AC} + \frac{BC}{AC} \times \frac{BC}{AC} = 1.$$

(*b*) Repeat the problem when $AB = 24$ units, $BC = 7$ units, $AC = 25$ units. (*Chap.* 22)

9. A ship is 6 miles E. of a lighthouse and sailing on a course S. 30° W. at 18 m.p.h. Find its distance and bearing from the lighthouse after 20 min. It then changes its course to W. Find its distance and bearing from the lighthouse after a further 10 min. (*Chap.* 21)

10. Draw any triangle ABC. Through D, the mid-point of AB, and through E, the mid-point of BC, draw lines perpendicular to AB and to BC respectively. Let these perpendicular bisectors meet in O. With centre O, describe a circle radius OB. Hence show that the centre of a circle which passes through the three vertices (corners) of a triangle lies at the point of intersection of the perpendicular bisectors of the sides. (*Chap.* 21)

Answers to Examples for Chapter 23

1. Angle $ACB = 58°$ or $122°$; $AC = 3$ or $9{\cdot}3$.

4. Angle $ABD = 40°$; angle $AED = 70°$. 5. (*b*) $AD = 2\frac{2}{3}$.

6. (*a*) $BG = \frac{2}{3}$ in.

7. (*a*) 108°. (*b*) Angle $ACB = 36°$; angle $ADC = 72°$.

8. Angle $ABC = 90°$ in both cases.

9. 6 miles; S. 30° E.; 5·2 miles S.

CHAPTER 24

MENSURATION. SIMPLE AREAS

A few facts will be assumed:

(1) Areas are expressed in square measure.

(2) 144 sq.in. are equal to 1 sq.ft., and corresponding relations for other units of length and area.

(3) The area of a rectangle is equal to its length multiplied by its breadth.

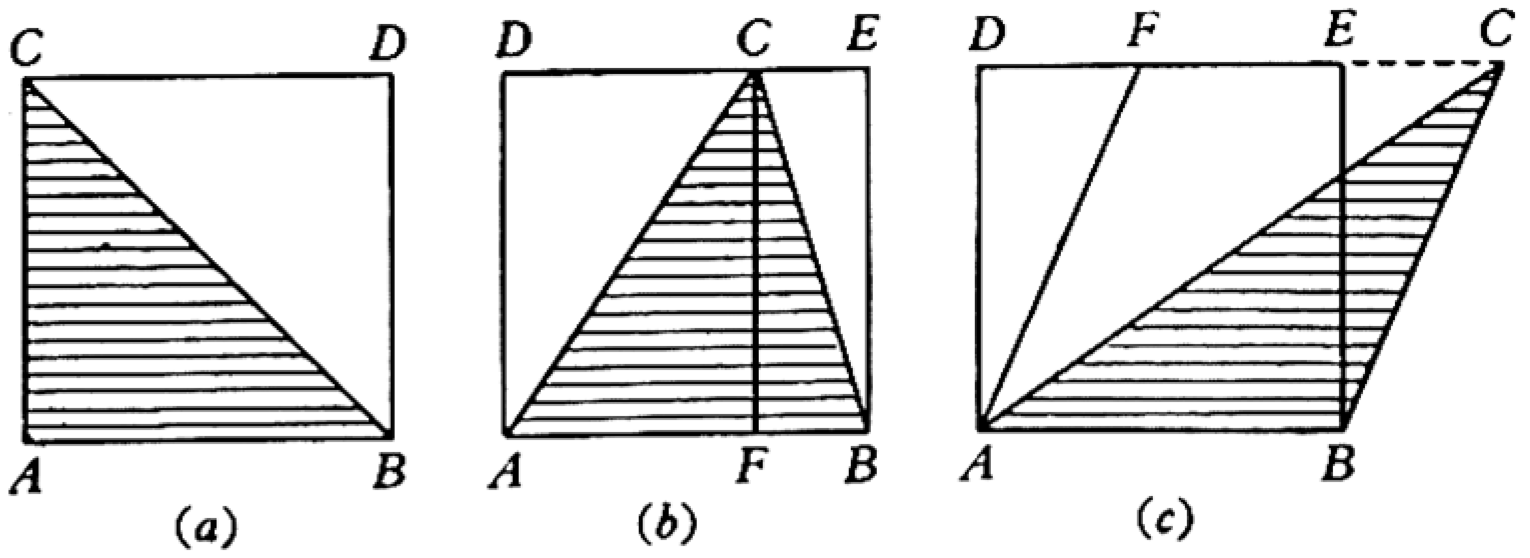
Area of a triangle

It is required to prove that the area of a triangle is equal to one-half the length of one side multiplied by the perpendicular distance from that side to the opposite corner. This may be expressed as: area $= \frac{1}{2}$ base × altitude.

Consider fig. 77 (*a*). It is obvious that the area of the triangle ABC is half the area of the rectangle $ABDC$, thus

$$\text{triangle } ABC = \frac{AB \times AC}{2}.$$

In fig. 77 (*b*) the triangle ACF is half the rectangle $AFCD$, and the triangle BCF is half the rectangle $FBEC$. Thus the triangle ABC is half the rectangle $ABED$.



(*a*) (*b*) (*c*)

Fig. 77

The case of fig. 77 (*c*) is not so obvious. AF is drawn parallel to BC. Comparing triangles ABC and AFC for congruency:

AC is common.

Angle FCA = angle BAC ($\because$ DC is parallel to AB),

angle FAC = angle BCA ($\because$ AF is parallel to BC).

$\therefore$ triangles AFC and ABC are congruent.

Thus triangle ABC is half the area of the figure $AFCB$ (which is called a parallelogram). It remains to prove that the parallelogram $AFCB$ is equal in area to the rectangle $ABED$. Now the triangle ECB is congruent with the triangle DFA because, (1) $DA = EB$, (2) angle $D =$ angle E, (3) angle $DAF =$ angle EBC. Hence the triangle ABC is half the rectangle $ABED$.

The area of any triangle is therefore $\frac{1}{2}$ base × altitude.

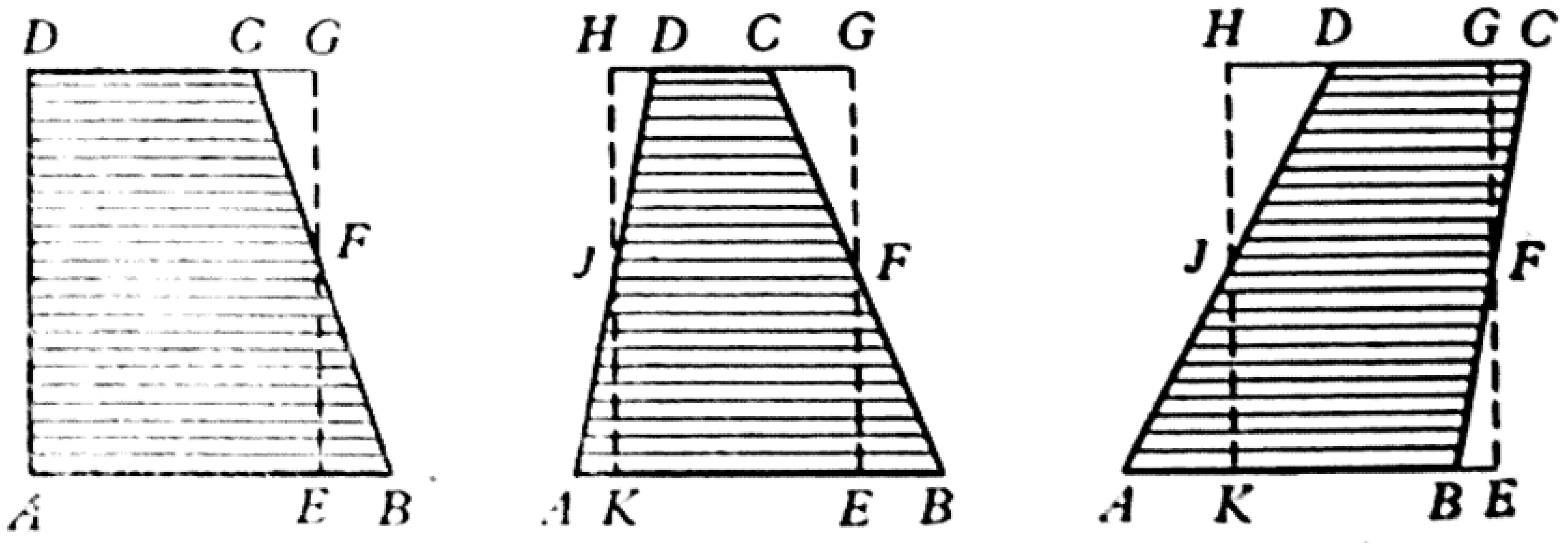
Area of a parallelogram

A parallelogram is a quadrilateral with its opposite sides parallel. It is easily proved that its opposite sides are also equal in length. The area of a parallelogram is equal to the length of one side multiplied by the perpendicular distance to the opposite side. This has actually been proved already from fig. 77 (*c*). Since the opposite sides of squares and rectangles are parallel they come within the category of parallelograms.

Area of a trapezium

A trapezium is a quadrilateral with two of its sides parallel. If the triangle EBF is removed from the trapezium $ABCD$ in fig. 78 (*a*), and fitted into the space CGF, the area is seen to be that of the rectangle $AEGD$. Now AE is a length midway between that of DC and that of AB, i.e. AE is the average of the parallel sides AB and CD. The area of the trapezium $ABCD$ is thus equal to the average of the parallel sides multiplied by the distance between the parallel sides.

In fig. 78 (*b*) the triangles AKJ and EBF are removed and placed in positions HDJ and CGF respectively. Then the rectangle $KEGH$ has the same area as the trapezium $ABCD$. And, as before, KE is the average of AB and DC. It is obvious



(*a*) (*b*) (*c*)

Fig. 78

that the same procedure may be adopted in fig. 78(*c*). The area of a trapezium is thus equal to the average of the parallel sides multiplied by the distance between those sides.

The *perimeter* of a figure is the distance round the figure, the sum of the lengths of its sides, whether straight lines or curves.

Use of trapezia in obtaining approximate areas. The shape of fig. 79 represents a piece of land whose area is required approximately. A direction AB is chosen to obtain the longest possible straight line. From points along this line are measured 'offsets' perpendicular to AB, so that when the extremities of the offsets are joined by straight lines, the resulting triangles and trapezia give the shape of the plot approximately. The readings are entered in the field-book in a particular manner. Starting from A, entries are made in a centre column showing at what distances from A the offsets occur. Columns at either side give the lengths of the offsets.

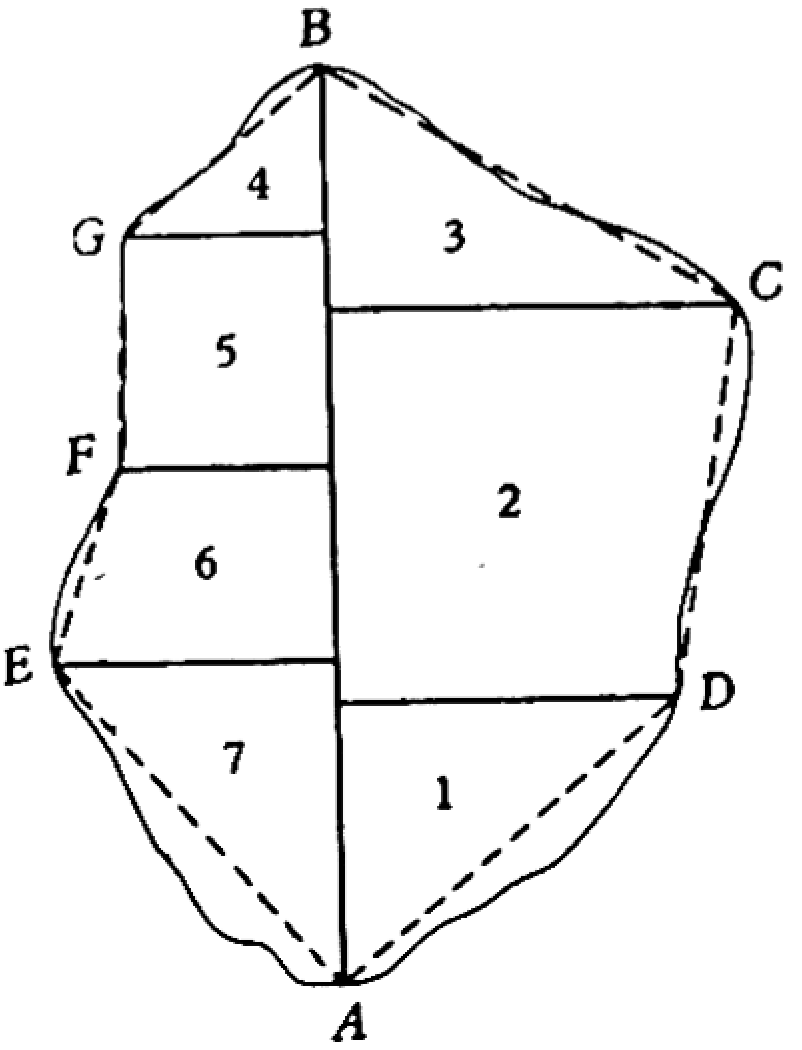


Fig. 79

	Yards to B	
	200	
To G, 40	184	
	148	80 to C
To F, 44	115	
To E, 56	72	
	64	68 to D
	From A	

The table corresponds to fig. 79. Thus, for instance, it is 64 yards from A to the point where the offset to D intersects the line AB.

The entries enable the surveyor to complete a figure the same shape as the dotted polygon in fig. 79.

Total area:

Area of triangle (1) $= \frac{1}{2} \times 64 \times 68 = 32 \times 68 =$ 2176 sq.yd.

Area of trapezium (2) $= \frac{68+80}{2} \times 84 = 74 \times 84 =$ 6216 sq.yd.

Area of triangle (3) $= \frac{1}{2} \times 80 \times 52 = 40 \times 52 =$ 2080 sq.yd.

Area of triangle (4) $= \frac{1}{2} \times 40 \times 16 = 20 \times 16 =$ 320 sq.yd.

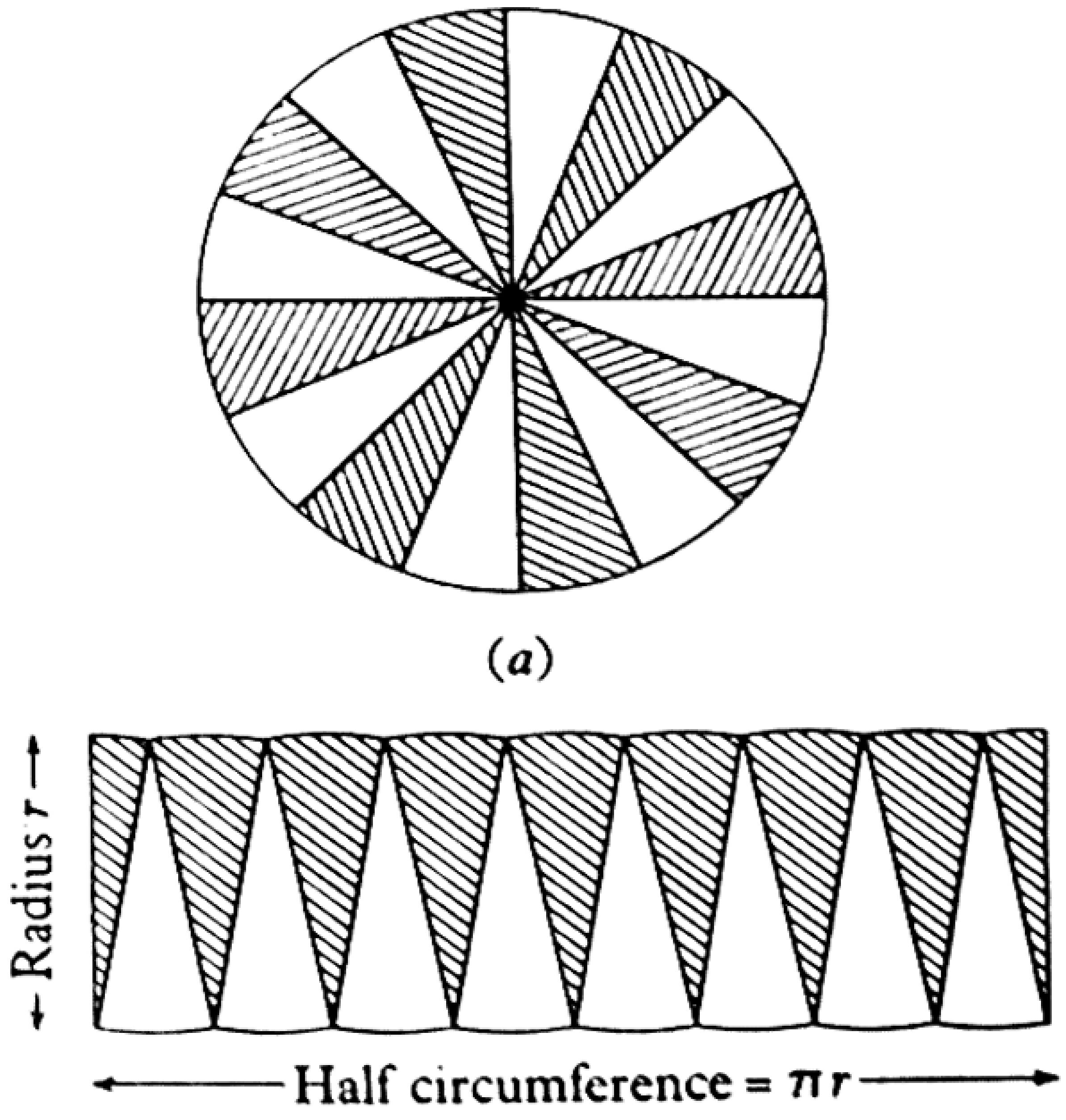
Area of trapezium (5) $= \frac{40+44}{2} \times 69 = 42 \times 69 =$ 2898 sq.yd.

Area of trapezium (6) $= \frac{44+56}{2} \times 43 = 50 \times 43 =$ 2150 sq.yd.

Area of triangle (7) $= \frac{1}{2} \times 72 \times 56 = 36 \times 56 =$ 2016 sq.yd.

Total area = 17856 sq.yd.

$$\text{Area} = \frac{17{,}856}{4{,}840} \text{ acres} = \underline{3 \cdot 69 \text{ acres (approx.).}}$$



(a)

(b)

Fig. 80

Frequently links are used as the units of length, since the measurements are performed with a chain 22 yards in length, consisting of 100 links. Areas are then obtained in square links, where 100×100 square links are equal to 1 square chain, and 10 square chains equal 1 acre. Thus 10^5 square links are equal to 1 acre.

The circle

The ratio between the circumference of a circle and its diameter is denoted by π (pi), where π cannot be determined accurately. Its value is 3·14159 to five places of decimals, and is usually taken as 3·142 when decimals are used, or $\frac{22}{7}$ when fractions are used.

Thus circumference $= \pi \times$ diameter

$$= 2\pi \times \text{radius}.$$

Area of a circle. The proof of the formula for the area of a circle is beyond the scope of this book, but we can illustrate how the area may be calculated approximately.

Fig. 80(a) is a circle divided into sixteen equal sectors. (A sector of a circle is an area bounded by two radii and an arc of the circumference.) The sectors are then cut out and arranged as in fig. 80(b); one sector is cut into two equal parts to fit the ends of the figure. This figure is approximately a rectangle, and, in fact, becomes more nearly a rectangle as the number of sectors is increased. Hence the area is approximately the product of the length and breadth given, viz. $\pi r \times r$ or πr^2.

The area of a circle is $\pi \times (\text{radius})^2$.

It is convenient to memorise the same formula expressed in terms of the diameter of the circle, because diameters are more frequently measured in engineering work than radii:

$$\begin{aligned} \text{Area} &= \pi\left(\frac{\text{diameter}}{2}\right) \times \left(\frac{\text{diameter}}{2}\right) \\ &= \frac{\pi d^2}{4} \\ &= \frac{3{\cdot}142 d^2}{4} \\ &= \underline{0{\cdot}786 d^2}. \end{aligned}$$

Area of an annulus. An annulus is a ring, that is, the area between two concentric circles. The area is obtained by subtraction (see fig. 81).

$$\begin{aligned} \text{Area of larger circle} &= \pi R^2, \\ \text{Area of smaller circle} &= \pi r^2. \\ \therefore\ \text{Area of annulus} &= \pi R^2 - \pi r^2. \end{aligned}$$

This may be written, Area $= \pi(R^2 - r^2)$.

It is not necessary to square R and r and subtract. The value of R^2-r^2 is the same as $(R+r)\times(R-r)$, so the area may be written as $\pi(R+r)(R-r)$.

(This particular identity $R^2-r^2=(R+r)(R-r)$ is discussed fully in Chapter 11.)

Example. An annulus has inner and outer radii 4·5 in., and 5·4 in. Find its area.

$$\begin{aligned}\text{Area} &= \pi(R+r)(R-r)\\ &= \pi(5{\cdot}4+4{\cdot}5)(5{\cdot}4-4{\cdot}5)\\ &= \pi\times 9{\cdot}9\times 0{\cdot}9\\ &= 3{\cdot}142\times 9{\cdot}9\times 0{\cdot}9\\ &= 28\text{ sq.in. (approx.).}\end{aligned}$$

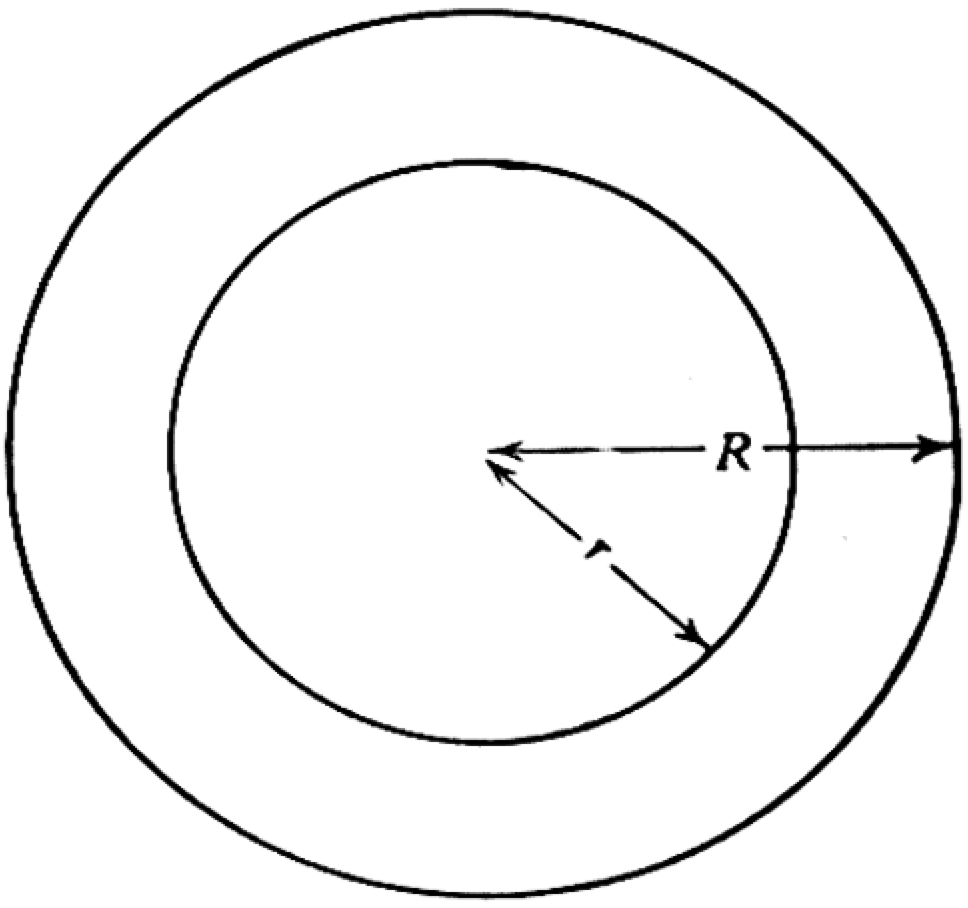


Fig. 81

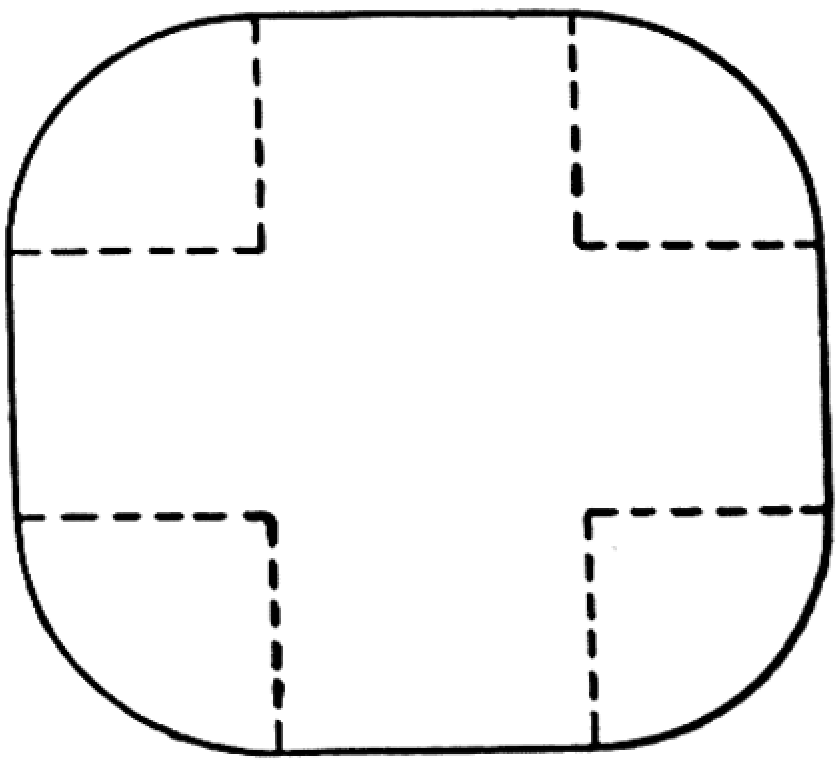


Fig. 82

Example. Find the area of the metal plate shown in fig. 82, given that the square from which it was cut had sides 3 in., and that the corners are quarter circles of radius 1 in.

Dividing by dotted lines as shown,

$$\text{Area of cross} = 5\times 1\times 1 = 5\text{ sq.in.}$$

Total area of the four corner pieces is equal to that of a circle radius 1 in., i.e. $\pi\times 1\times 1 = 3{\cdot}142$ sq.in.

Thus, area of plate $= 8{\cdot}142$ sq.in.

Example. Find the area of the face of the plate shown in fig. 83. (The circular holes are punched out.)

Divide the area into a rectangle and two trapezia as shown:

$$\text{Area of rectangle} = 8{\cdot}5 \times 5{\cdot}5 = 46{\cdot}75\,\text{sq.in.} \qquad (1)$$

$$\text{Area of two trapezia} = \frac{5{\cdot}5 + 3{\cdot}5}{2} \times 1{\cdot}75 \times 2 = 4{\cdot}5 \times 1{\cdot}75 \times 2$$

$$= 15{\cdot}75\,\text{sq.in.} \qquad (2)$$

$$\text{Area of two circles} = 2 \times 3{\cdot}142 \times \frac{1{\cdot}5}{2} \times \frac{1{\cdot}5}{2}$$

$$= 3{\cdot}142 \times 1{\cdot}125 = 3{\cdot}535\,\text{sq.in.} \qquad (3)$$

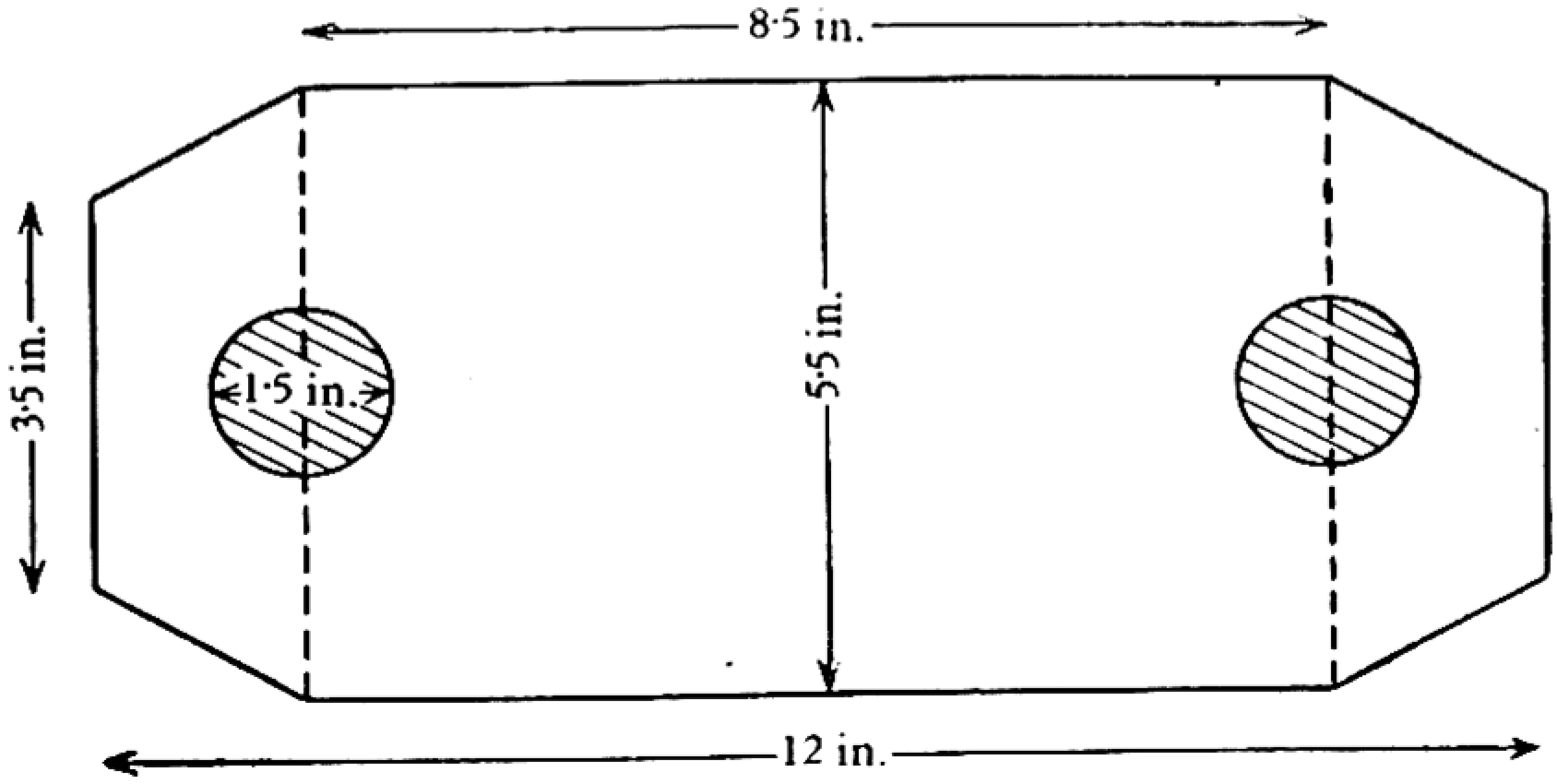


Fig. 83

Adding (1) and (2) and subtracting (3), we have

$$\text{Area of plate} = 58{\cdot}965\,\text{sq.in.}$$

$$= 58{\cdot}97\,\text{sq.in., correct to two places.}$$

Example. Find the area of the figure in fig. 84. The curved ends are semicircles.

The figure consists of (*a*) a sector of an annulus, (*b*) two semicircles.

Now, area of complete annulus with outer and inner radii 4 in. and $1\frac{1}{2}$ in. $= \pi(4 + 1\frac{1}{2})\ (4 - 1\frac{1}{2})$ sq.in.

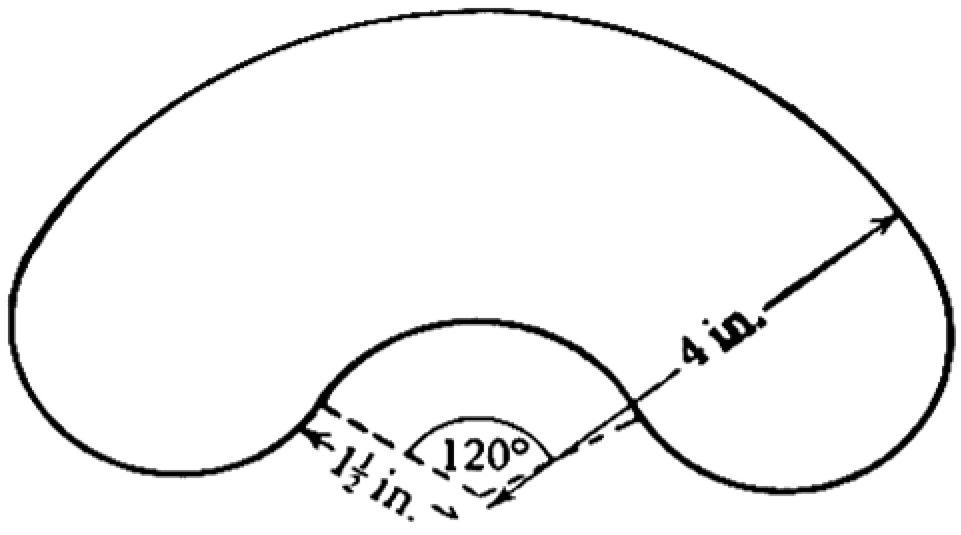


Fig. 84

$\therefore$ Area of sector whose angle is 120°,

$$= \tfrac{120}{360}\pi(4 + 1\tfrac{1}{2})\,(4 - 1\tfrac{1}{2})\,\text{sq.in.}$$

$$= \tfrac{1}{3} \times 3{\cdot}142 \times 5{\cdot}5 \times 2{\cdot}5\,\text{sq.in.}$$

$$= 1{\cdot}047 \times 13{\cdot}75\,\text{sq.in.}$$

$$= 14{\cdot}39\,\text{sq.in.}$$

Area of complete circle, diameter 2·5 in.

$$= 3{\cdot}142 \times 2{\cdot}5 \times 2{\cdot}5 \text{ sq.in.}$$

$$= 4{\cdot}91 \text{ sq.in.}$$

$$\therefore \text{ Area of figure} = 14{\cdot}39 + 4{\cdot}91 \text{ sq.in.}$$

$$= \underline{19{\cdot}3 \text{ sq.in.}}$$

Examples for Chapter 24

1. (*a*) A railway cutting is 32 ft. wide at rail level, and 50 ft. wide at the top, and its depth is 48 ft. Find its area of cross-section.

(*b*) Sketch the piece of land represented by the field-book entry given, and calculate its area in square yards:

	Yards to *B*	
	300	
	230	110 to *C*
To *G*, 60	150	
	120	125 to *D*
To *F*, 120	85	
	40	40 to *E*
	From *A*	

(*Chap.* 24)

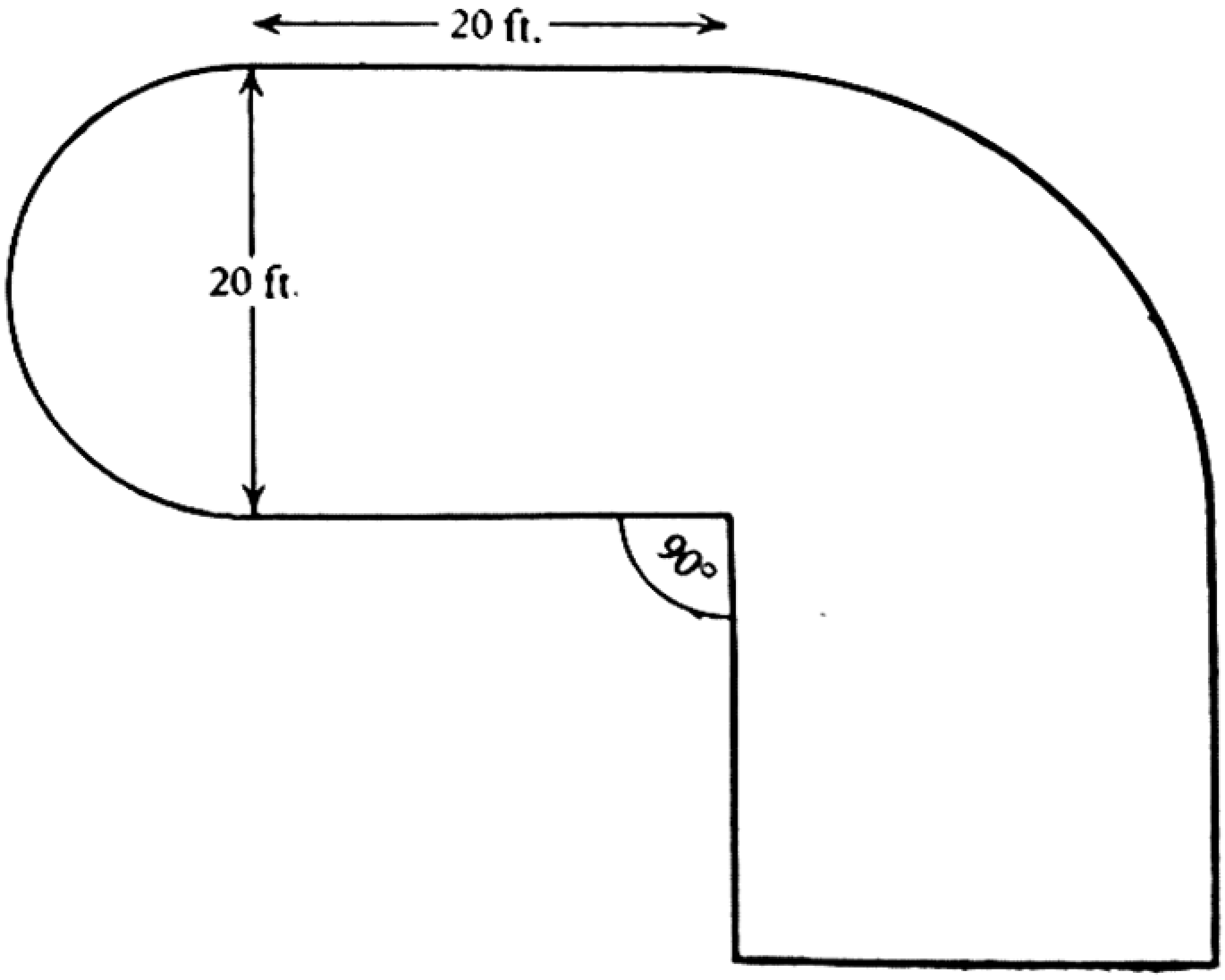


Fig. 85

2. Fig. 85 represents the plan of a sports pavilion. Find the area of its floor space and its perimeter. (*Chap.* 24)

3. (*a*) Find the area of the face of the metal plate in fig. 86.

(*b*) If the plate is stamped from a square as shown, express the waste as a percentage of the area of the plate, correct to three significant figures. (See Chapter 3 for percentages.) (*Chap.* 24)

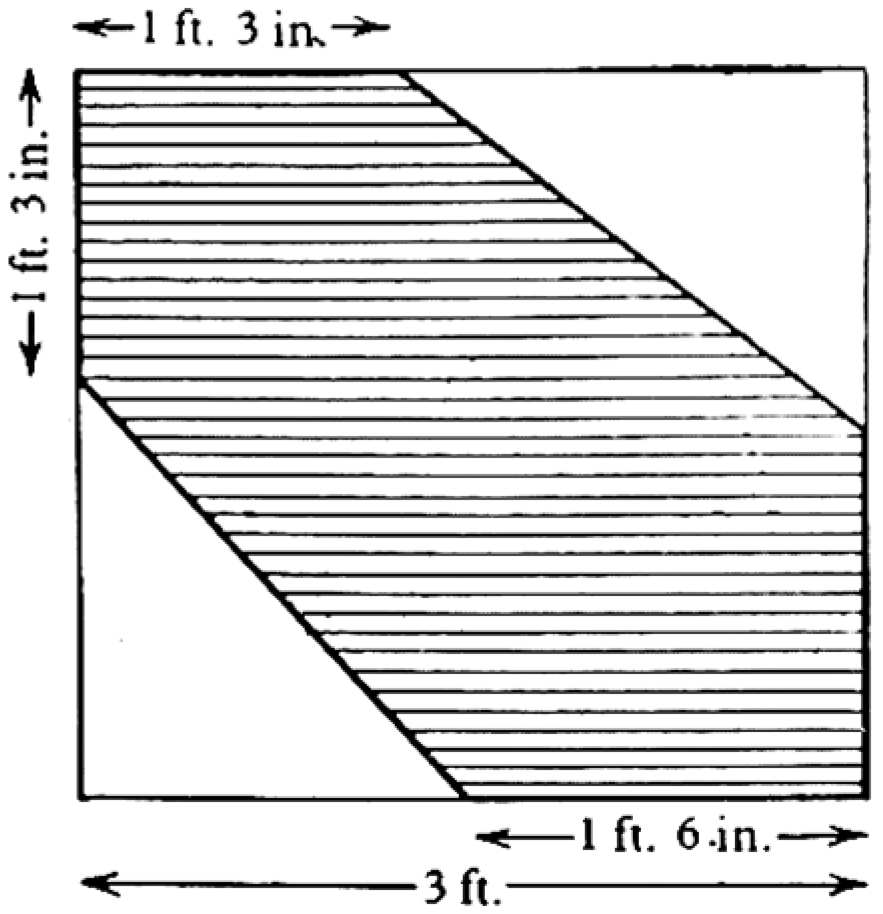


Fig. 86

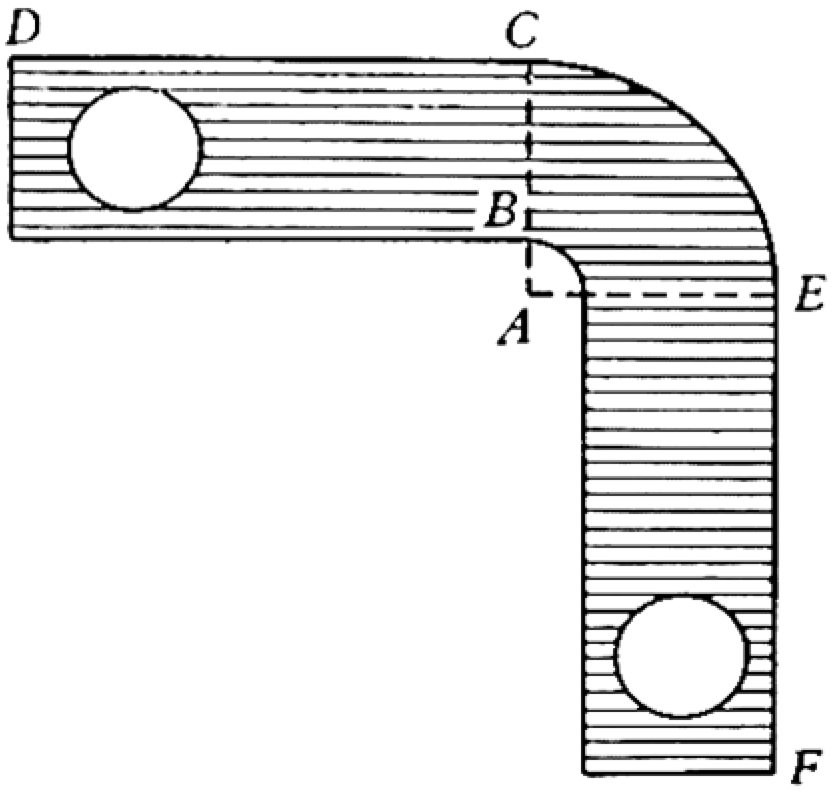


Fig. 87

4. (*a*) Calculate the area of an annulus whose inner and outer radii are $\frac{1}{4}$ in. and 1 in. respectively.

(*b*) Find the area of the face of the plate in fig. 87, given that $AC = 1$ in., $AB = \frac{1}{4}$ in., $\angle CAE = 90°$, $DC = EF = 2$ in., and the circles have radii of $\frac{1}{4}$ in. (Take $\pi = \frac{22}{7}$.) (*Chap.* 24)

5. (*a*) Find the area of metal in the electric fan in fig. 88. The radius of the outer circle is 6 in. and that of the centre circle is 1 in. (Take $\pi = \frac{22}{7}$.)

(*b*) Calculate the area of a fan with the same linear dimensions, but whose blades have an angle of 30°. (*Chap.* 24)

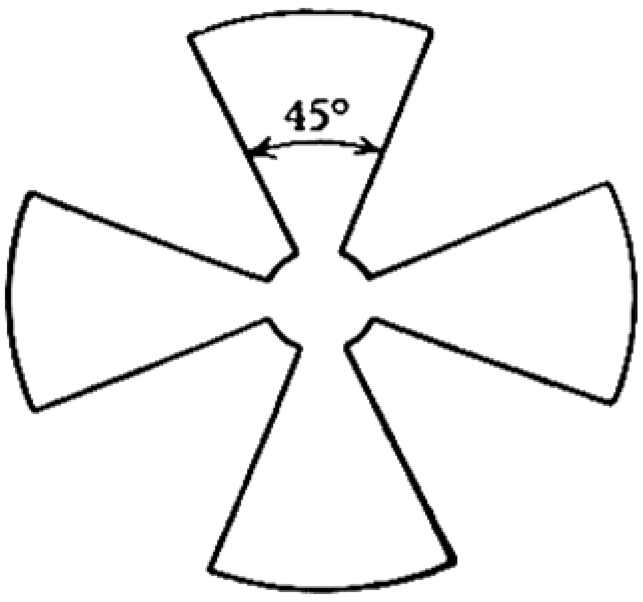


Fig. 88

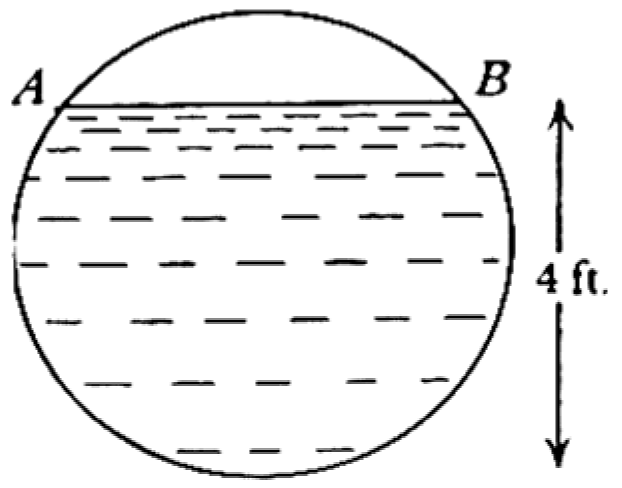


Fig. 89

6. Fig. 89 represents the end of a cylindrical water tank, radius 2 ft. 6 in. Draw the figure to scale and measure the distance AB.

Measure the angle AOB, where O is the centre of the circle, and hence calculate the cross-sectional area of the water-space. (*Chap.* 24)

7. (*a*) Prove that the diagonals of a parallelogram bisect one another.

(*b*) If the four sides of the parallelogram are equal in length, the figure is called a rhombus. Prove that the diagonals of a rhombus are perpendicular. (*Chaps.* 23 *and* 21)

8. Show, by a scale drawing, that two different triangles may be constructed in which angle $A = 50°$, $AC = 60$, $BC = 50$, and measure the possible values of angle B to the nearest degree. (*Chap.* 23)

9. Draw any triangle and bisect the three interior angles. Show that these bisectors meet in a point. From this point draw a perpendicular to any one of the three sides, and with centre at the point of intersection and radius equal to the length of the perpendicular just drawn, describe a circle. This circle is called the inscribed circle of the triangle, and should touch all three sides. (*Chap.* 23)

10. A cyclic quadrilateral is a quadrilateral drawn inside a circle so that its four corners lie on the circumference. By drawing several such quadrilaterals in a circle prove the truth of the statement: 'The opposite angles of a cyclic quadrilateral are supplementary.'

Arrange that in one case two opposite corners of the quadrilateral lie at opposite ends of a diameter. Show that in this case two angles are equal to each other. (*Chap.* 21)

Answers to Examples for Chapter 24

1. (*a*) 1968 sq.ft. (*b*) 39,625 sq.yd.
2. (*a*) 1271 sq.ft. (*b*) 163 ft.
3. (*a*) $6\frac{3}{8}$ sq.ft. (*b*) 41·2 %.
4. (*a*) $2\frac{53}{56}$ sq.in. (*b*) $3\frac{11}{32}$ sq.in.
5. (*a*) $58\frac{1}{7}$ sq.in. (*b*) $39\frac{17}{21}$ sq.in.
6. $AB = 4$ in.; angle $AOB = 106°$; 13·86 sq.ft.
8. Angle $B = 113°$ or $67°$.

CHAPTER 25

THE MID-ORDINATE RULE

The mid-ordinate rule is a method of determining, approximately, the area of an irregular figure. Fig. 90 represents a plot of land bounded by the straight line AL, the straight line AM, and the curved line ML. Divide AL into any convenient number of equal parts AB, BC, CD, ..., KL, and at the points chosen for division erect ordinates (i.e. straight lines perpendicular to AL). The ordinates are AM, BN, CO, ..., KW. Now it is almost true to consider each of the figures so formed as a trapezium, and in this case the average length of the two parallel sides of each trapezium will be given by the length of a *mid-ordinate*, that is, an ordinate erected to the curve in the middle of each trapezium. These mid-ordinates are numbered 1 to 11 in the diagram. Thus

Area of figure $ABNM = AB \times$ mid-ordinate 1 (approx.).

Area of figure $BCON = BC \times$ mid-ordinate 2 (approx.).

....

Area of figure $WKL = KL \times$ mid-ordinate 11 (approx.).

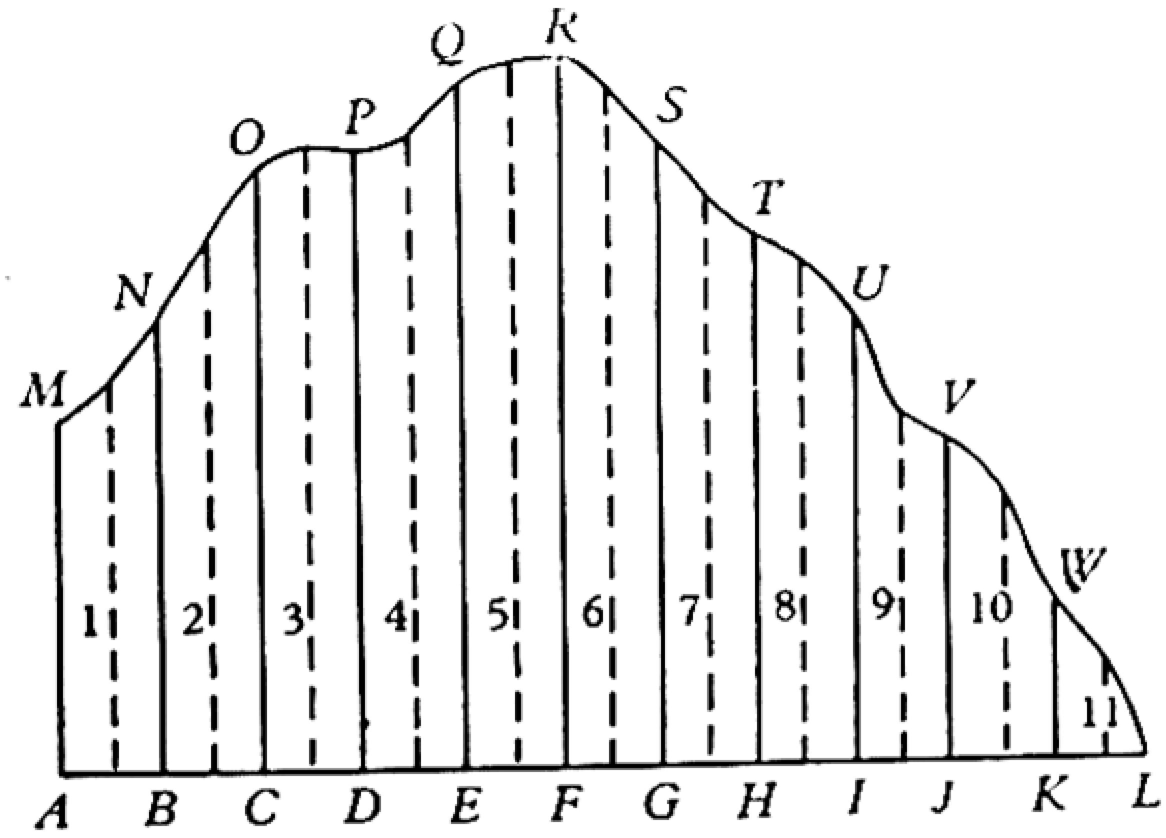


Fig. 90

But since all the distances AB, BC, CD, ..., KL are equal, the total area of the figure is equal to the sum of the products of each mid-ordinate and $\frac{1}{11}$ of AL.

Thus,

$$\text{Area} = \frac{\text{sum of all mid-ordinates} \times AL}{11}$$

$$= \text{average of length of mid-ordinates} \times AL$$

$$= \text{average of mid-ordinates} \times \text{length of diagram.}$$

Example. A plot of land is bounded by a straight river bank on one side, and a curved boundary touching this bank at both ends. Distances of the boundary from the bank are given in the table. Find the area of the plot approximately:

Distances along bank (yd.)	0	5	10	15	20	25	30	35	40	45	50
Distances from bank to boundary (yd.)	0	8	11	14	18	20	17	13	9	6	0

There is no need to draw the figure to scale, since the distances along the river bank (corresponding to distances AB, BC, CD, etc., in fig. 90) are all equal, and as we have no information about more detailed points, we should have to join the tops of the ordinates with straight lines. Since this is the case, we may calculate the lengths of the mid-ordinates. There are eleven ordinates given, including the end ordinates, therefore there will be ten mid-ordinates. The mid-ordinate between the first two ordinates given must be of length $(0+8) \div 2 = 4$ yd. Similarly the others may be calculated:

Lengths of mid-ordinates (yd.):

4, 9·5, 12·5, 16, 19, 18·5, 15, 11, 7·5, 3.

Average of mid-ordinates $= \frac{116}{10} = 11{\cdot}6$ yd.

Area of plot of land = average of mid-ordinates × length of plot

$= 11{\cdot}6 \times 50$ sq.yd.

$= \underline{580 \text{ sq.yd. (approx.)}}$.

Example. The depths of a river at different distances from one bank are given in the table. Find the area of cross-section of the bed:

Distances from bank (ft.)	1	3	6	9	11	13	16	17	19	21	24	25
Depths (ft.)	$2\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	$5\frac{1}{2}$	5	$3\frac{1}{2}$	2	1	$\frac{1}{2}$	0

The problem gives unequal distances along the straight edge of the figure, i.e. along the surface of the water. It will be necessary to make a drawing to scale (fig. 91), and to join the ends of the ordinates freehand. The surface of the water may now be divided into a number of equal lengths, the greater the number, the greater the accuracy in general, but in this case we cannot increase the accuracy indefinitely since we are ignorant of the contour of the bed of the river between the points given. We choose to divide the surface into 10 equal parts, and to erect 10 *mid*-ordinates numbered 1 to 10 in the diagram. The lengths of these mid-ordinates are marked on the figure.

Sum of mid-ordinates

$$= 2{\cdot}6 + 4{\cdot}1 + 4{\cdot}6 + 5 + 5{\cdot}5 + 5{\cdot}4 + 4{\cdot}6 + 2{\cdot}2 + 0{\cdot}9 + 0{\cdot}5 \text{ ft.}$$

$$= 35{\cdot}4 \text{ ft.}$$

Average of mid-ordinates $= 3{\cdot}54$ ft.

Area = average of mid-ordinates × length of diagram

$$= 3{\cdot}54 \times 25.$$

$\therefore$ $\underline{\text{Area of cross-section of river bed} = 88{\cdot}5 \text{ sq.ft.}}$

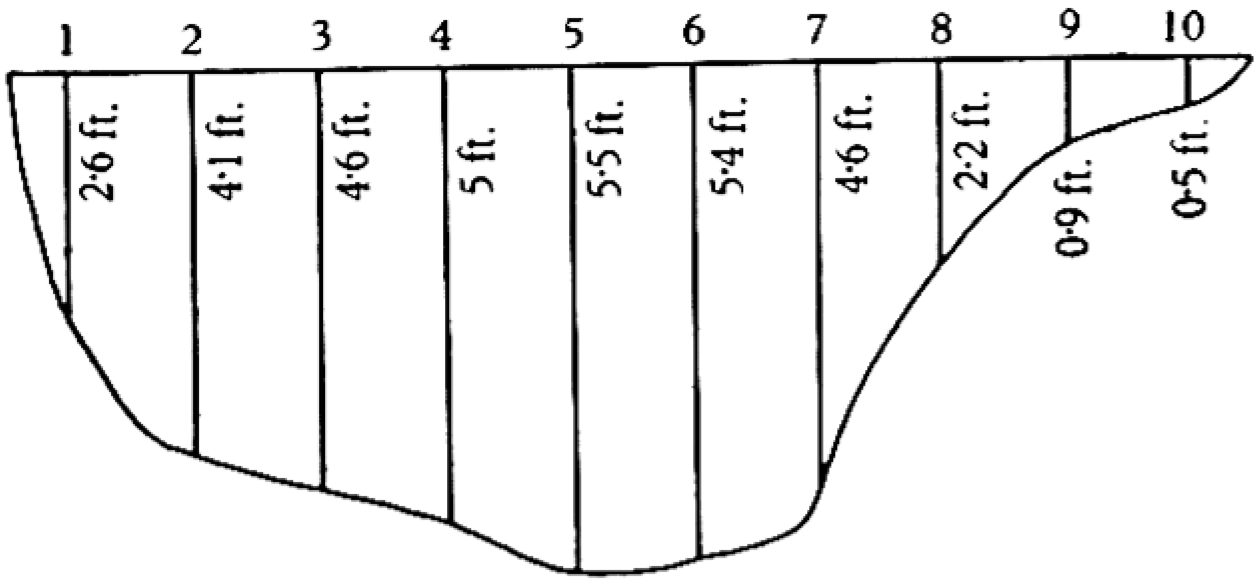


Fig. 91

Indicator diagrams

When an instrument called an indicator is fitted to the cylinder of an engine, the pressure of the gas in the cylinder for different positions of the piston may be read from a diagram which the indicator makes during the actual movement of the piston. The shapes of these indicator diagrams differ with details of construction of different engines, and, in addition, the two scales for horizontal and vertical distances must be calculated before a diagram has any meaning. The indicator itself carries a small piston whose vertical movement depends on the pressure of gas in the cylinder, and on the stiffness of a control spring calibrated to allow a given movement for a given pressure. (In many cases 1 in. vertical movement corresponds to a pressure of gas of 80 lb. per sq.in.) The full horizontal distance represents the 'stroke' of the piston. Such diagrams allow the calculation of the mean, or average, pressure, and the work done by the engine, from which the indicated horse-power may be calculated.

Fig. 92 shows an indicator diagram. The length has been divided into 10 equal parts, and mid-ordinates erected, numbered 1 to 10.

$$\text{Sum of mid-ordinates} = 1{\cdot}93 + 1{\cdot}76 + 1{\cdot}55 + 1{\cdot}35 + 1{\cdot}2 + 1{\cdot}05 + 0{\cdot}9 + 0{\cdot}77 + 0{\cdot}68 + 0{\cdot}5 \text{ in.}$$

$$= 11{\cdot}69 \text{ in.}$$

$$\text{Average of mid-ordinates} = 11{\cdot}69 \div 10 = 1{\cdot}169 \text{ in.}$$

Area of diagram = average of mid-ordinates × length of diagram, 5 in.

$$= 1{\cdot}169 \times 5 \text{ sq.in.}$$

$$= 5{\cdot}845 \text{ sq.in.}$$

$$= 5{\cdot}85 \text{ sq.in. to two decimal places.}$$

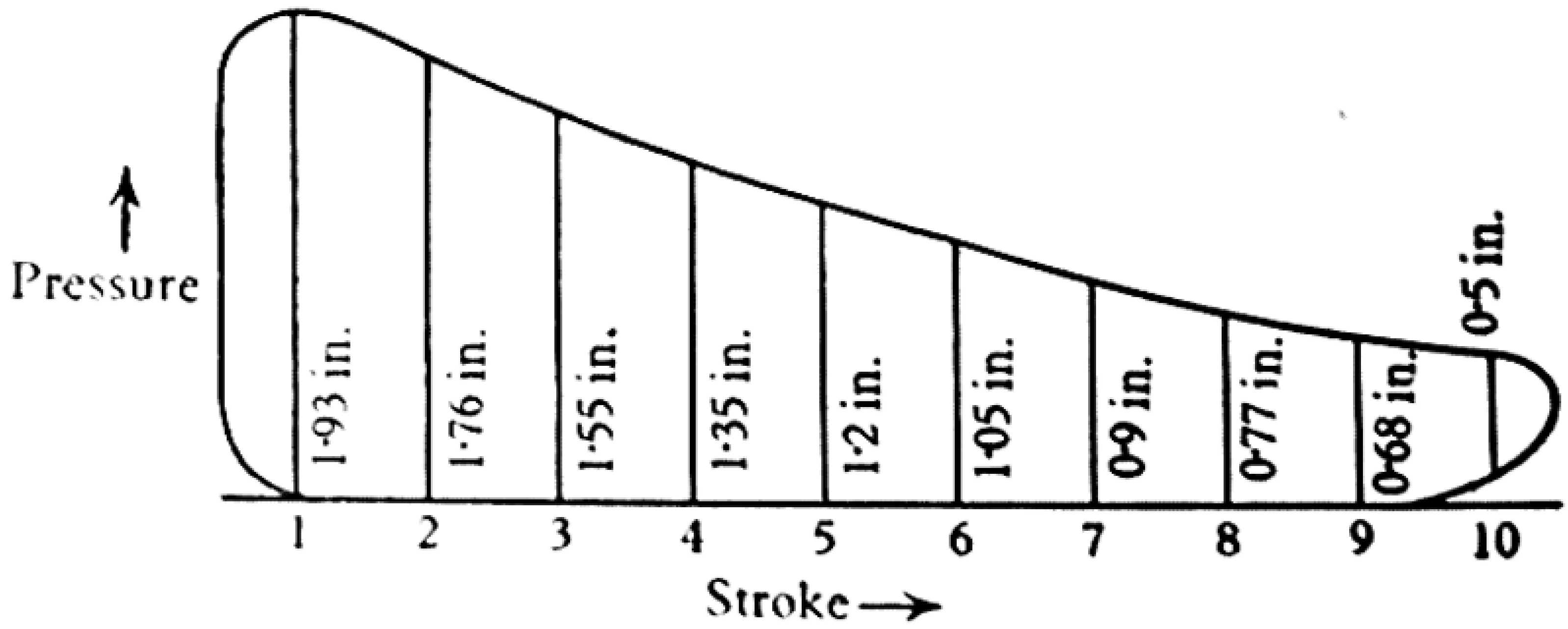


Fig. 92

Suppose the scale of pressure is 80 lb. per sq.in. to 1 in. measured vertically, then the average pressure is the average length of the mid-ordinates multiplied by 80:

$$\text{Average pressure} = 1{\cdot}17 \times 80 \text{ lb. per sq.in.}$$

$$= \underline{93{\cdot}6 \text{ lb. per sq.in.}}$$

Examples for Chapter 25

1. A river is 60 ft. wide. The table gives depths at equal distances from one bank, the ordinates being 5 ft. apart. Find the area of cross-section of the bed of the river.

Distance from left bank (ft.)	0	5	10	15	20	25	30	35	40	45	50	55	60
Depth (ft.)	0	0·6	1·6	3·7	5·6	6	5·2	3·5	2·5	1·5	1	0·5	0

If the river is flowing at n ft. per sec., the volume of water passing any point per second is given by:

area of cross-section × n cu.ft.

Find the volume of water passing per second in the case above, if the average rate of flow is 2 ft. per sec. (*Chap.* 25)

2. A plot of land bordered by a straight road on one side is measured as follows:

Distances from one end (yd.)	0	12	18	25	32	40	48	55	65	75	85	90
Distances perpendicular to road (yd.)	15	25	31	35	35	37	45	42	40	35	30	25

From a drawing obtain 9 mid-ordinates and find the area of the plot approximately in square yards. (*Chap.* 25)

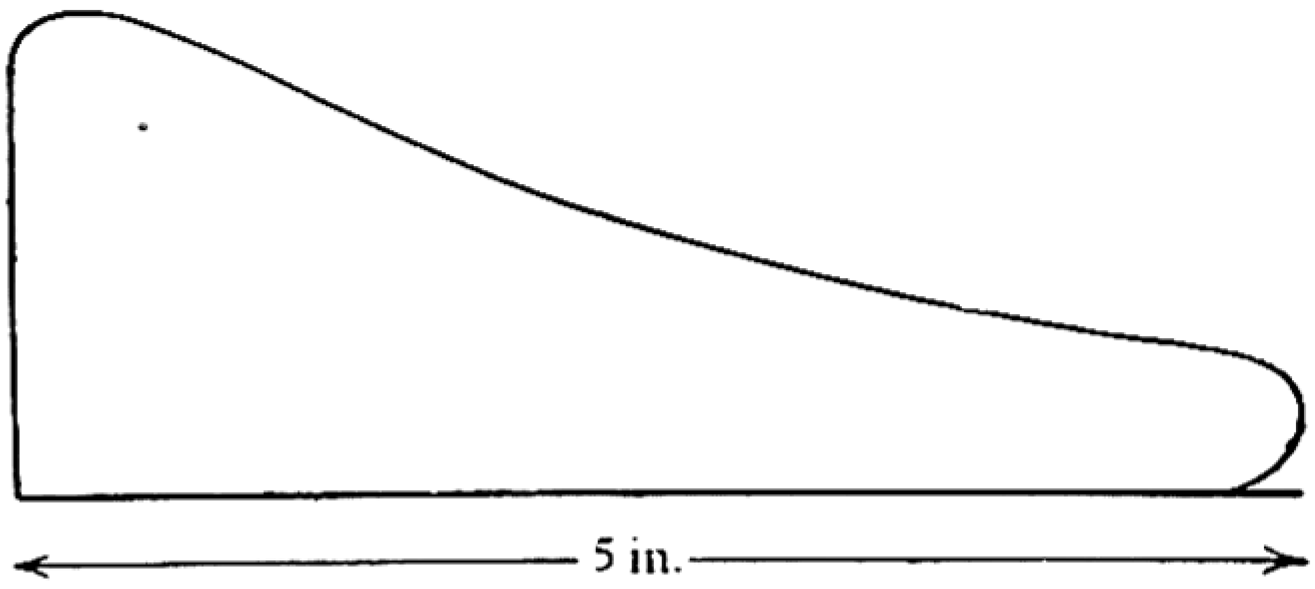


Fig. 93

3. Fig. 93 represents an indicator diagram. Using 10 ordinates, find the area of the diagram, and find the mean pressure in the cylinder, if the scale of pressure is 60 lb. per sq.in. to 1 in. (*Chap.* 25)

4. An indicator diagram is 3·6 in. long. Eleven evenly spaced ordinates, including the end ordinates, are as follows:

1·7, 1·8, 1·6, 1·4, 1·2, 1·0, 0·85, 0·73, 0·6, 0·5, 0·35 in.

Find the area of the diagram and the mean pressure in the cylinder, if the pressure scale is 80 lb. per sq.in. to 1 in. (*Chap.* 25)

5. Draw a semicircle, radius 6 in. Find its area by means of the mid-ordinate rule, using 12 mid-ordinates. Compare with the area found by calculation, taking $\pi = 3{\cdot}142$. (*Chap.* 25)

6. The length of the deck of a ship is 50 ft. The widths of the deck, measured at equal distances of 5 ft., are as follows:

0, 7, 12, 15, 17, 17, 16·4, 15, 13, 10, 6 ft.

Make a sketch to scale, join the ends of the ordinates by curved lines, and find the area of the deck using 10 mid-ordinates. (*Chap.* 25)

7. Fig. 94 shows part of the motion of a steam engine. The crank is 1 ft. long, and the connecting rod is 5 ft. long. The cross-head, C, slides along the line OC. Find by a scale drawing the distance moved by the cross-head between the instant when the $\angle BOC$ is 30°, and the instant when it is 120°. (*Chap.* 21)

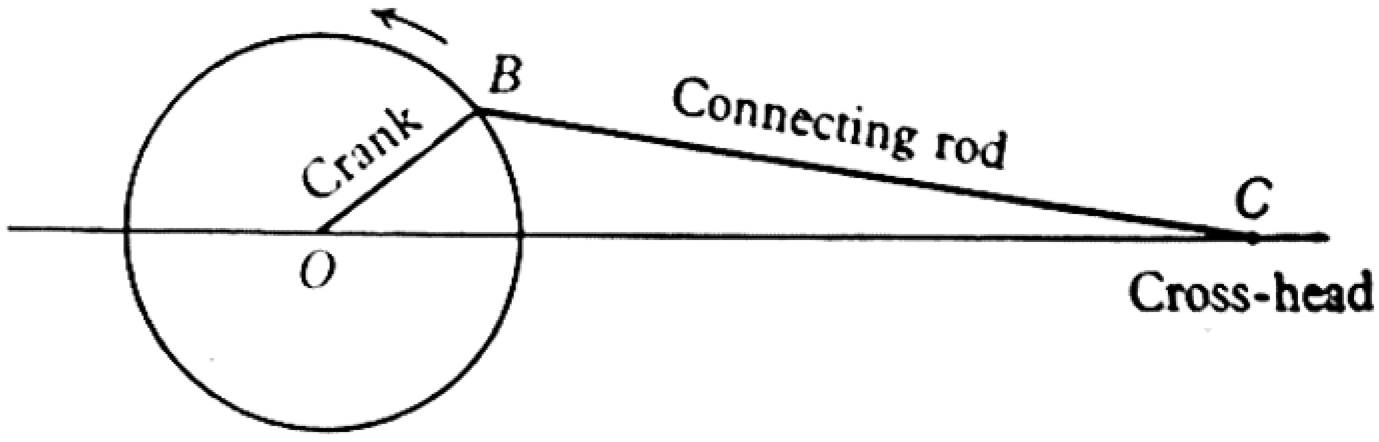


Fig. 94

8. Construct a regular octagon inside a circle, using ruler and compasses only. (*Chap.* 23)

9. A shed whose length is 15 ft. and width 8 ft. has a simple gable roof, the highest point of which is 11 ft. from the ground. The vertical side walls are 8 ft. high. Draw one end of the shed to scale and measure the length of the sloping edges of the roof. Hence calculate the wall and roof area, allowing no overlap. (*Chap.* 24)

10. A straight road runs N.–S., and a pylon lies to the west of this road. From two points on the road 13 miles apart the distances to the pylon are 5 miles and 12 miles. Make a scale drawing and measure the angle at the pylon. Then, without measurement, prove that the distance of the pylon from the road is $4\frac{8}{13}$ miles. (*Chaps.* 23 *and* 22)

Answers to Examples for Chapter 25

1. 158·4 sq.ft. 316·8 cu.ft.

2. 3060 sq.yd.

3. 5·84 sq.in. 70 lb. per sq.in.

4. 3·85 sq.in. 85·6 lb. per sq.in.

6. 630 sq.ft.

7. 1·42 ft.

9. 530 sq.ft.

10. Use similar right-angled triangles.

CHAPTER 26

GRAPHS. INTERPOLATION

A graph is a mathematical picture of a relationship between two variable quantities.

Difference between diagrams and graphs

Fig. 95 represents the distance covered per day, over a period of a fortnight, by a delivery lorry. Distances are measured vertically and days marked horizontally. The only use of such a diagram is that it enables comparisons to be made quicker than by the examination of a table of figures. Nothing is gained by joining the extremities of the vertical lines; for instance, the straight line joining the first two points has no meaning. There is obviously no simple mathematical relationship between distances and days, so the diagram is not a graph in the sense of the definition above.

The following figures are taken from a test on a car:

Time from start of test (sec.)	0	1	2	3	4	5	6	7	8
Distance from starting-point (ft.)	0	12	28	48	72	100	132	168	208

To illustrate the figures graphically, we choose two *axes* (or directions) perpendicular to each other. In this case, distances are chosen vertically, and times horizontally. The point at which the two axes intersect is called the *origin* of the graph. It is convenient in this example that the origin should be zero for both scales, but this is not always either necessary or convenient. Fig. 96 shows the points plotted with respect to the axes chosen. Thus, point A represents a distance of 12 units vertically and a distance of 1 unit horizontally. Similarly, point B represents 28 vertically and 2 horizontally. It is evident that a smooth curve could be drawn to pass through all these points, and such a curve would have a meaning in that we should be able to find from it the distances from the starting-point at times other than those given in the table. There is some mathematical relationship

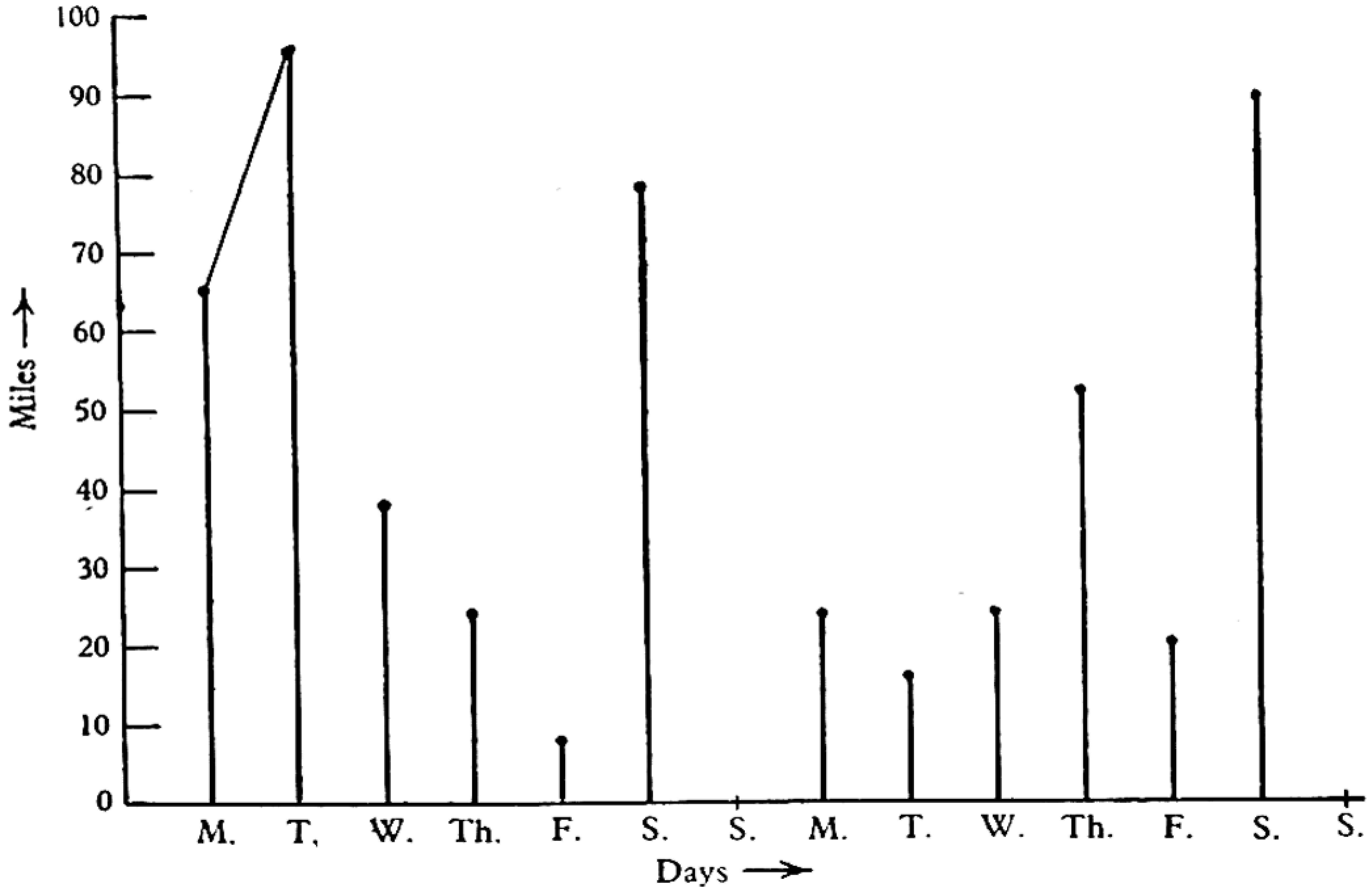

100
90
80
70
60
50
40
30
20
10
0
Miles →
M.
T.
W.
Th.
F.
S.
S.
M.
T.
W.
Th.
F.
S.
S.
Days →

Fig. 95

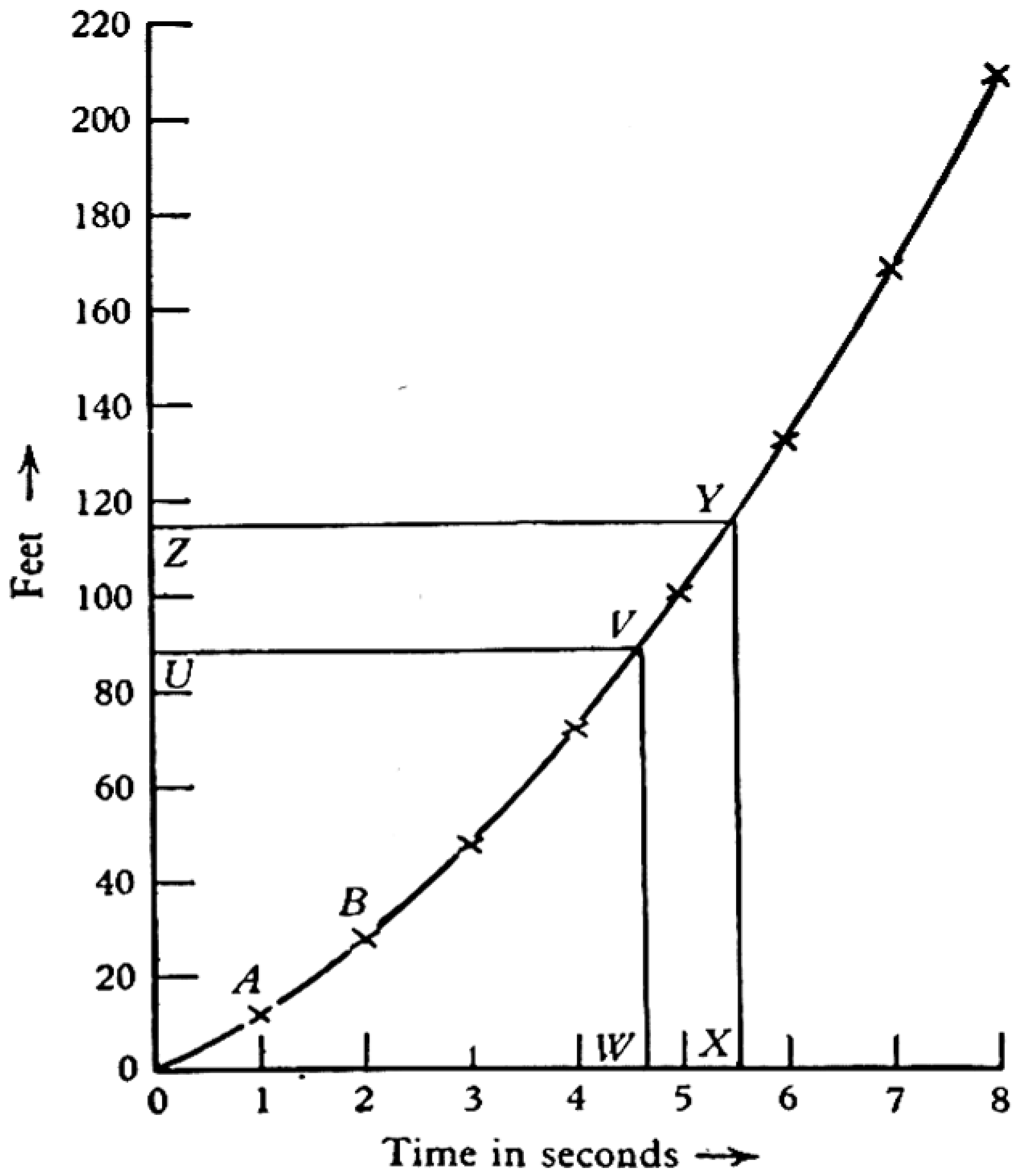

220
200
180
160
140
120
100
80
60
40
20
0
Feet →
Y
Z
V
U
B
A
W
X
0
1
2
3
4
5
6
7
8
Time in seconds →

Fig. 96

between distance covered and time taken under these conditions, and a graph will supply us with facts which could not be obtained from a table of figures without further calculations.

To find the distance covered by the car in $5\frac{1}{2}$ sec.: Measure $5\frac{1}{2}$ units along the time axis, and from that point erect a vertical line to cut the curve. From this point read off on the vertical scale the distance covered. (Lines XY and YZ represent these operations.) The answer, as accurately as can be read from the scales used, is 115 ft.

To find how long the car takes to cover 84 ft.: Through 84 on the vertical axis, draw a horizontal line to cut the curve, and from this point draw a vertical line to read off the time. (Lines UV and VW represent these operations.) The answer is 4·6 sec.

An experiment to show the relation between temperature and time, when a quantity of methylated spirit was allowed to cool, gave the following figures:

Time in min.	0	5	10	15	20	25	30	35
Temperature in ° C.	78	61	48	37	29	22·5	17	13

Fig. 97 is the cooling curve plotted from these results, the points being joined by a smooth curve.

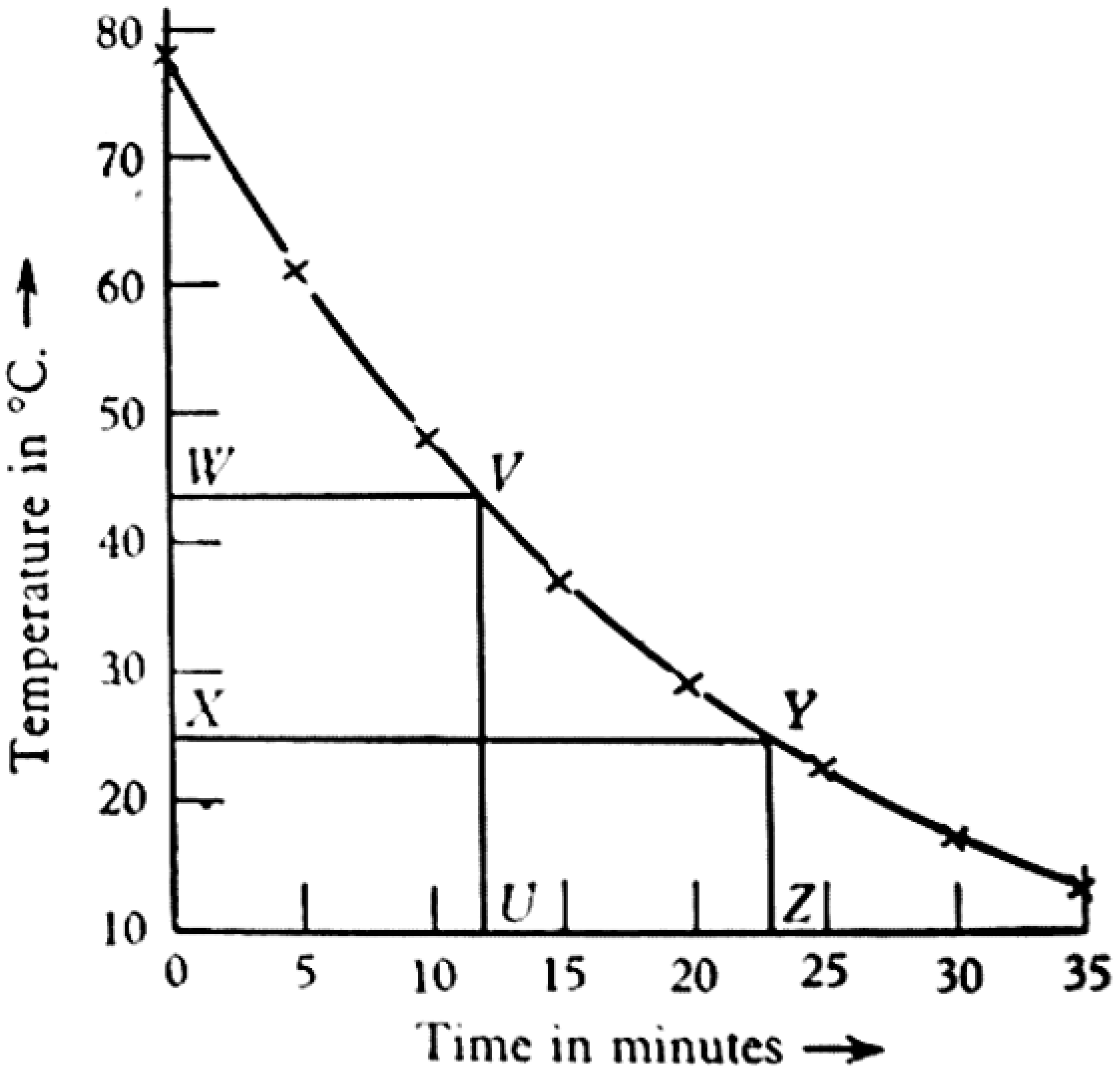


Fig. 97

To find when the temperature was 25° C.: The line XY is drawn horizontally through 25° C., cutting the curve in Y. Then the foot of the vertical line YZ gives the answer, 23 min.

To find the temperature after 12 min.: The vertical line *UV* is drawn through 12 min. on the time axis, cutting the curve in *V*. A horizontal line *VW* gives the answer, $43\frac{1}{2}$° C.

The table below gives figures for the self-induction of a coil of wire when the number of turns is varied:

Number of turns	50	75	100	125	150	175	200
Self-induction	5	11	18	32	43	60	84

The curve is shown in fig. 98, and it is found that a smooth curve cannot be drawn to include all the points given. This frequently happens when plotting the results of experiments, due to various experimental errors. A smooth curve is drawn to pass through (or near) some of the points, while others, which are obviously incorrect, are missed. Points *A* and *B* are off the curve, the self-induction for 100 turns should be nearly 20, and that for 125 turns should be 30 if the other points are correct.

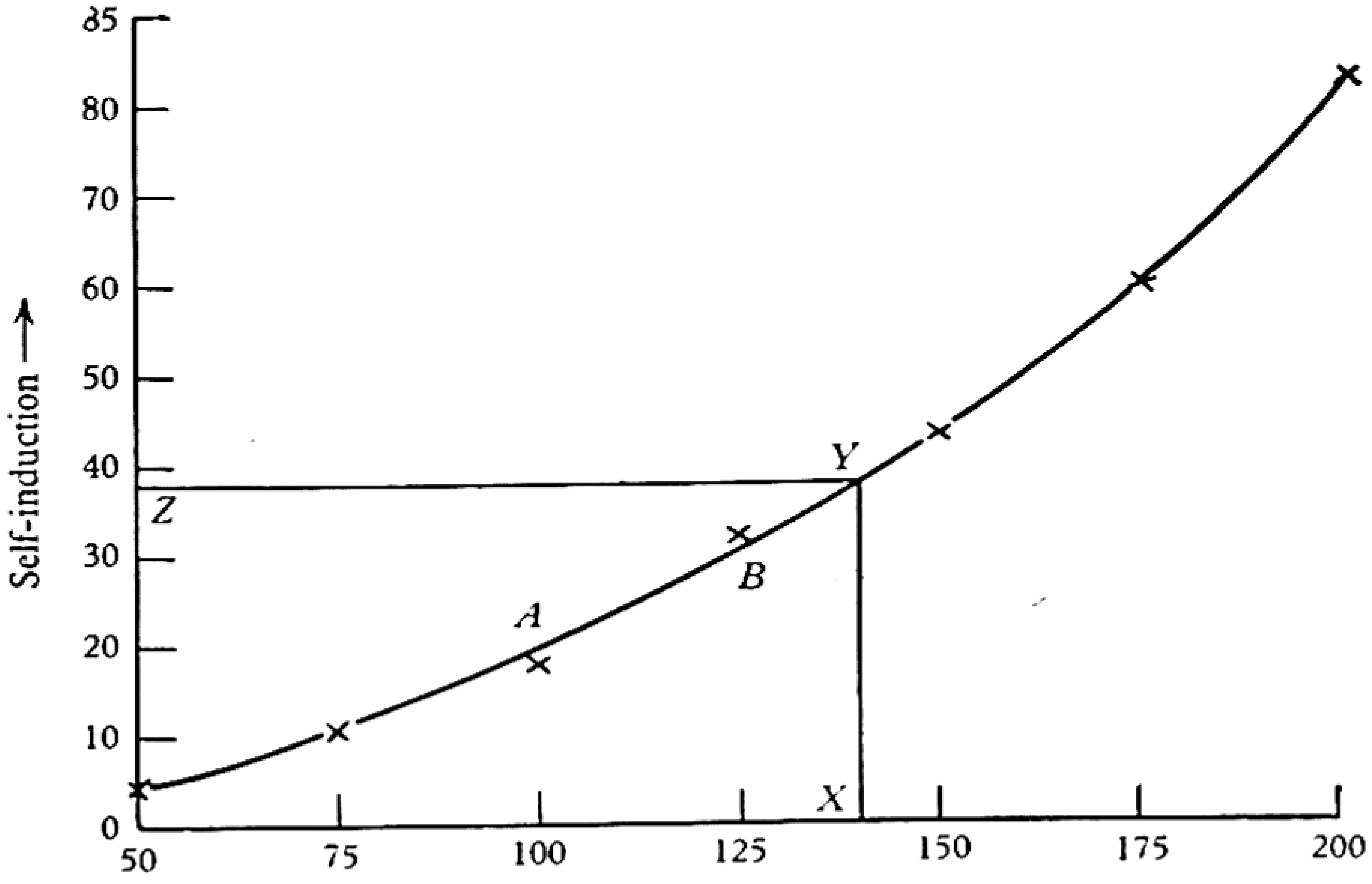


Fig. 98

To find the self-induction for 140 turns, draw lines *XY* and *YZ* to give the answer 38.

The process of obtaining numerical answers from a graph is called *interpolation*, and may be performed only when the graph is a straight line or a smooth curve. To obtain an answer by calculation from a table of figures is a much longer process.

A study of the examples completed so far will help in drawing up rules for graphical work:

(1) Choose scales so that the graph may occupy as large a space as possible on the sheet of graph paper to be used.

(2) Avoid awkward fractions in the scales, so that interpolation is comparatively easy (that is, do not choose 7 spaces to represent 10 units, but rather 10 or 5 spaces).

(3) The scales should not start at zero if this will result in crowding points in one corner of the page (that is, if figures range from 100 to 200, start at 100, not at 0).

(4) Mark the plotted points in an obvious manner. Possibly crosses are preferable to dots.

(5) Make a statement along each axis explaining what the numbers represent.

(6) Write sufficient numbers along each axis to help in reading, but not so many as to cramp the figures.

(7) Make a statement on the graph to explain what relationship is illustrated by it.

Examples for Chapter 26

1. The table gives distances in feet at intervals of 1 sec., covered by a body falling vertically. Plot distance (vertically) against time (horizontally), and from the graph find (*a*) the distance fallen in $2\frac{1}{2}$ sec., (*b*) the time taken to fall 128 ft.:

Time in sec.	0	1	2	3	4
Distance in ft.	0	16	64	144	256

(*Chap.* 26)

2. A circuit consists of a battery of cells, a resistance box, and an ammeter. Readings are given of the current (in amperes) and the corresponding resistance (in ohms) of the resistance box. Plot current (I amperes) against R ohms resistance, the latter horizontally, and from the graph find (*a*) the current when the resistance is 16 ohms, (*b*) the resistance when the current is 0·5 ampere:

Resistance (R ohms)	2	5	10	12	20	26
Current (I amperes)	1	0·67	0·43	0·38	0·25	0·20

(*Chap.* 26)

3. The tension in lb. wt. required to cause slipping in a belt when the angle of contact of the belt is varied is given by the following figures:

Angle of contact of belt and pulley ($\theta°$)	0	90	180	270	360	450
Tension to cause slipping (T lb. wt.)	5	12	30	74	183	450

Plot tension against angle of contact and from the graph find (*a*) the tension for slipping when the angle of contact is 300°, (*b*) the angle of contact when the tension for slipping is 37 lb. wt. (*Chap.* 26)

4. A cylinder of gas was compressed slowly and readings taken of the volumes and corresponding pressures:

Pressure (P lb. per sq.in.)	14	20	25	30	35	40
Volume (V cu.in.)	35·7	25	20	18·5	14·3	12·5

Plot P vertically against V horizontally. One point is off the curve. Give the correct figure for the volume corresponding to the pressure at this point, and find also the volume when the pressure is 32 lb. per sq.in. (*Chap.* 26)

5. The heat produced, in calories, in a coil of wire in a given time by varying currents (amperes) was measured approximately:

Current (amperes)	0·1	0·2	0·3	0·4	0·5
Heat (calories)	60	240	540	960	1500

Plot heat vertically against current horizontally, and from the graph determine the heat produced by a current of 0·25 ampere. (*Chap.* 26)

6. The deflexion ($\theta°$) of a galvanometer needle was read for different values of current (I amperes):

Deflexion ($\theta°$)	30	35	40	45	50	55
Current (I amperes)	0·116	0·140	0·168	0·200	0·240	0·286

Plot current against deflexion, and hence find the deflexion produced by a current of 0·15 ampere. (*Chap.* 26)

7. Half-ordinates at distances of 5 ft. for the deck of a ship 45 ft. long are given in the table:

Distance from bows (ft.)	0	5	10	15	20	25	30	35	40	45
Half-ordinates (ft.)	0	3·5	5·5	6·5	7	7	6·2	5·2	4	2

The deck contains two hatches, one 6 ft. square, and the other 4 ft. square. Find the area of the deck. (Half-ordinates are measured from the centre line of the deck.) (*Chap.* 25)

8. Calculate the area of one face of the cam shown in fig. 99. (*Chap.* 24)

9. Two ships leave port together. One ship *A* steers a course N. 30° W. at 6 m.p.h., and the other ship *B* steers a course N. 40° E. at 10 m.p.h. Find by a scale drawing their distance apart, and the bearing of *B* from *A* after 1 hr. Calculate their distance apart after 3 hr., and give the bearing then of *B* from *A*. (*Chaps.* 21 *and* 23)

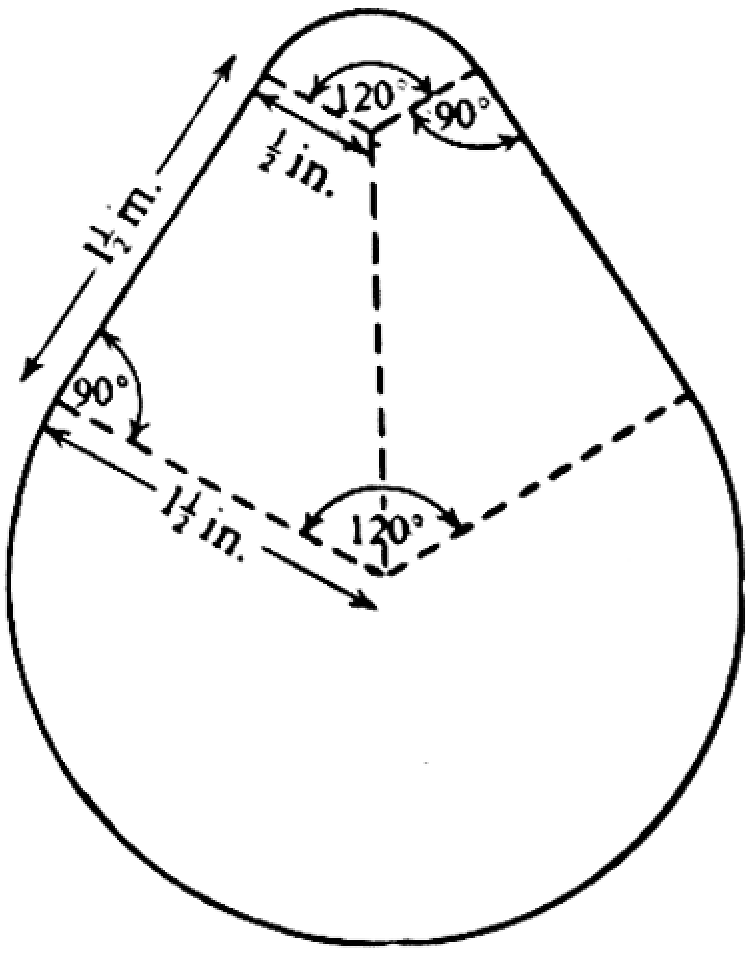


Fig. 99

10. Calculate the area of the piece of land represented by the field-book entry given:

	To *E*, Links	
	2000	
	1400	750 to *D*
To *F*, 1050	1200	
	1050	900 to *C*
To *G*, 650	800	
	350	450 to *E*
	From *A*	

(*Chap.* 24)

Answers to Examples for Chapter 26

1. (*a*) 100 ft. (*b*) 2·83 sec.

2. (*a*) $I = 0{\cdot}3$ amp. (*b*) $R = 8$ ohms.

3. (*a*) $T = 100$ lb. wt. (*b*) $\theta = 200°$.

4. Corrected point $P = 30, V = 16\cdot7$. When $P = 32, V = 15\cdot7$.

5. 375 calories.

6. 37° (approx.).

7. 408 sq.ft. (approx.).

8. 7·98 sq.in.

9. 9·75 miles, N. 75° E. (approx.); 29·1 miles, N. 75° E.

10. 20·85 acres.

CHAPTER 27

GRAPHS WITH DIRECTED NUMBERS. THE STRAIGHT LINE

Use of directed numbers on graphs

Alternating current is so named because it is changing its direction continually. Fig. 100 shows the magnitude and direction of a simple case of alternating voltage taken over a period of $\frac{1}{8}$ sec. In the first thirty-second part of a second the voltage drops from 100 V. to zero, and then increases in the opposite direction, until its magnitude at the end of one-sixteenth of a second is equal to that at the beginning. It is now, however, in the opposite direction from the one it had at the beginning of the measurement, and in consequence, the current is travelling in the opposite direction along the wires. In fact, the voltage has become a directed quantity, having positive and negative values. To represent directed numbers graphically, the axes are continued beyond the origin, as has been done with the vertical axis in this case.

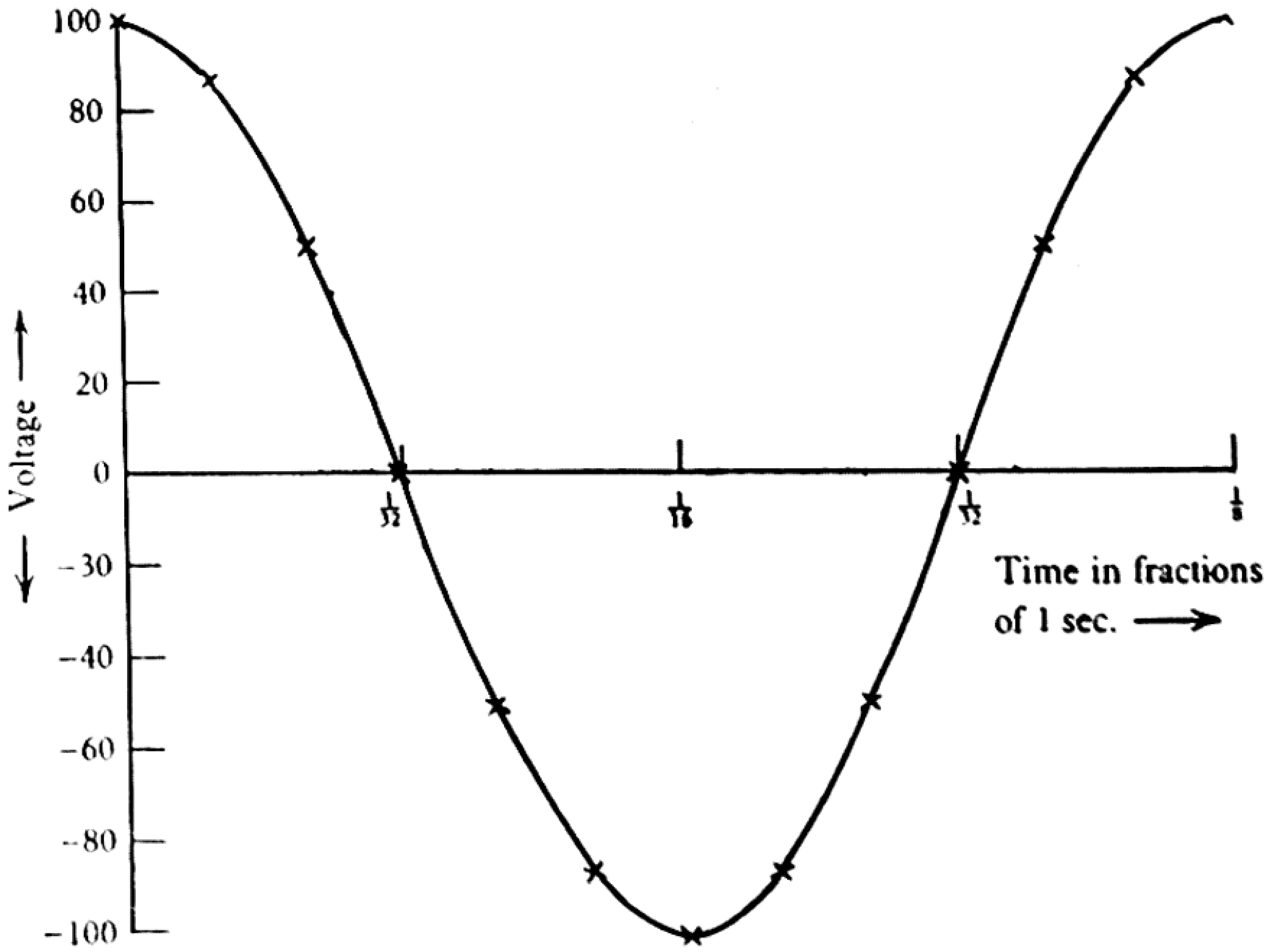


Fig. 100

Algebraic graphs

The method of illustrating algebraic relations graphically is due to Descartes, a French mathematician of the seventeenth century. Values of x are measured horizontally, positive to the right of the origin and negative to the left, while values of y are measured vertically, positive above the origin, and negative below. The x distance of a point is called its 'abscissa' (plural abscissae), and the y value, the 'ordinate'. The two figures representing the position of a point on a graph are called the 'co-ordinates' of the point, the abscissa being written first. To distinguish from other methods of representing the positions of points in two-dimensional space, co-ordinates measured in this manner are called Cartesian co-ordinates, after Descartes. Thus the point (2, 3) is situated 2 units to the right of the origin, and 3 units above the origin, while the point $(-3, -7)$ is situated 3 units to the left of the origin, and 7 units below the origin.

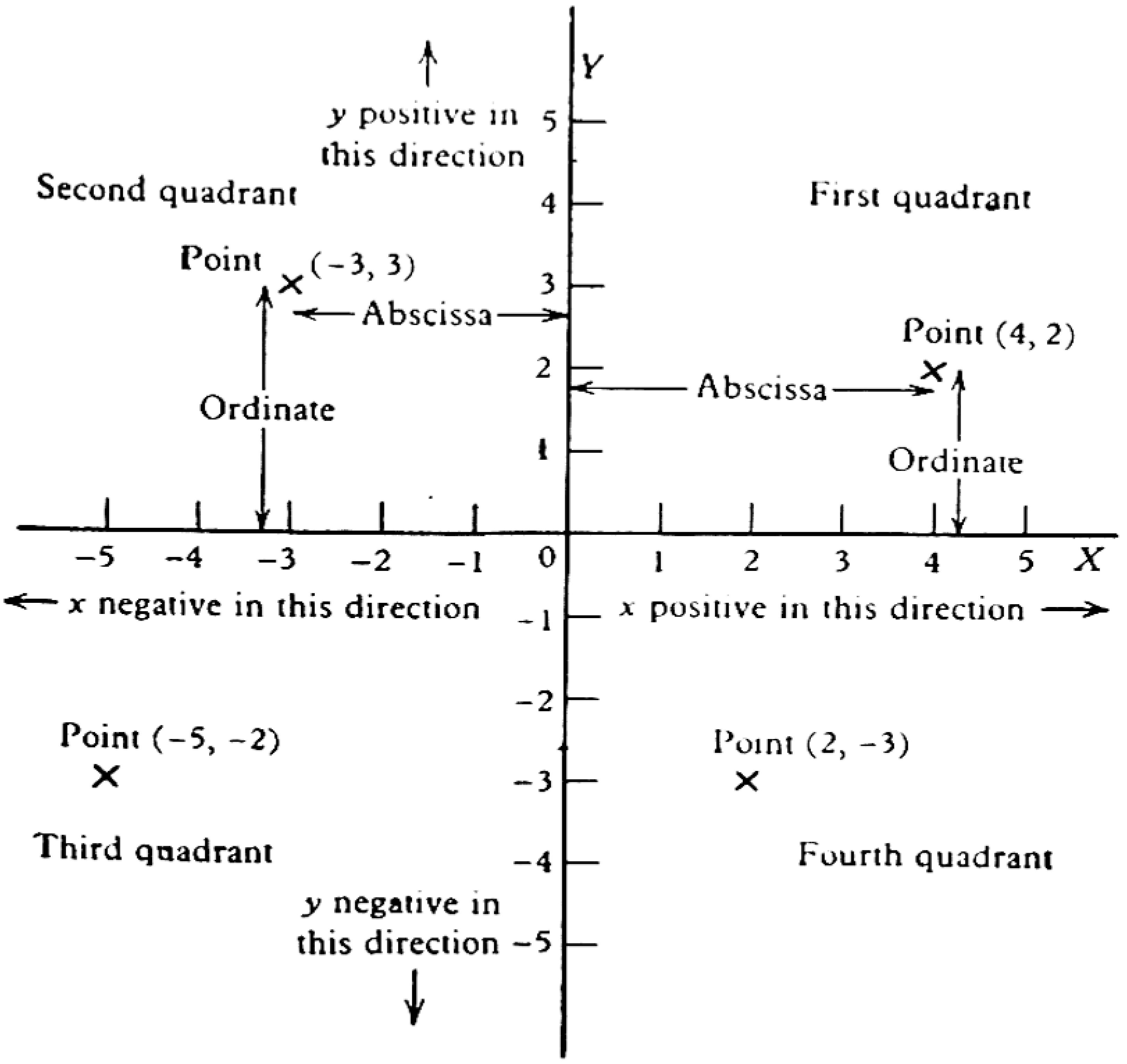


Fig. 101

It will be seen from fig. 101 that the method adopted divides two-dimensional space into four parts. These are called 'quadrants'. The first quadrant contains all points whose abscissae and ordinates are both positive, the second quadrant contains points with negative abscissae and positive ordinates, the third those whose abscissae and ordinates are both negative, while the points in the fourth quadrant have only their ordinates negative.

The straight line

Fig. 102 shows three lines (1), (2) and (3). Consider line (1) and note that the points A, B, C, D, E, F are the points (1, 1), (2, 2), (3, 3), $(-1, -1)$, $(-2, -2)$ and $(-3, -3)$ respectively. Thus the abscissa of every point on the line is equal to the ordinate of the same point. The relation (or equation) connecting x and y is thus $y = x$ for line (1).

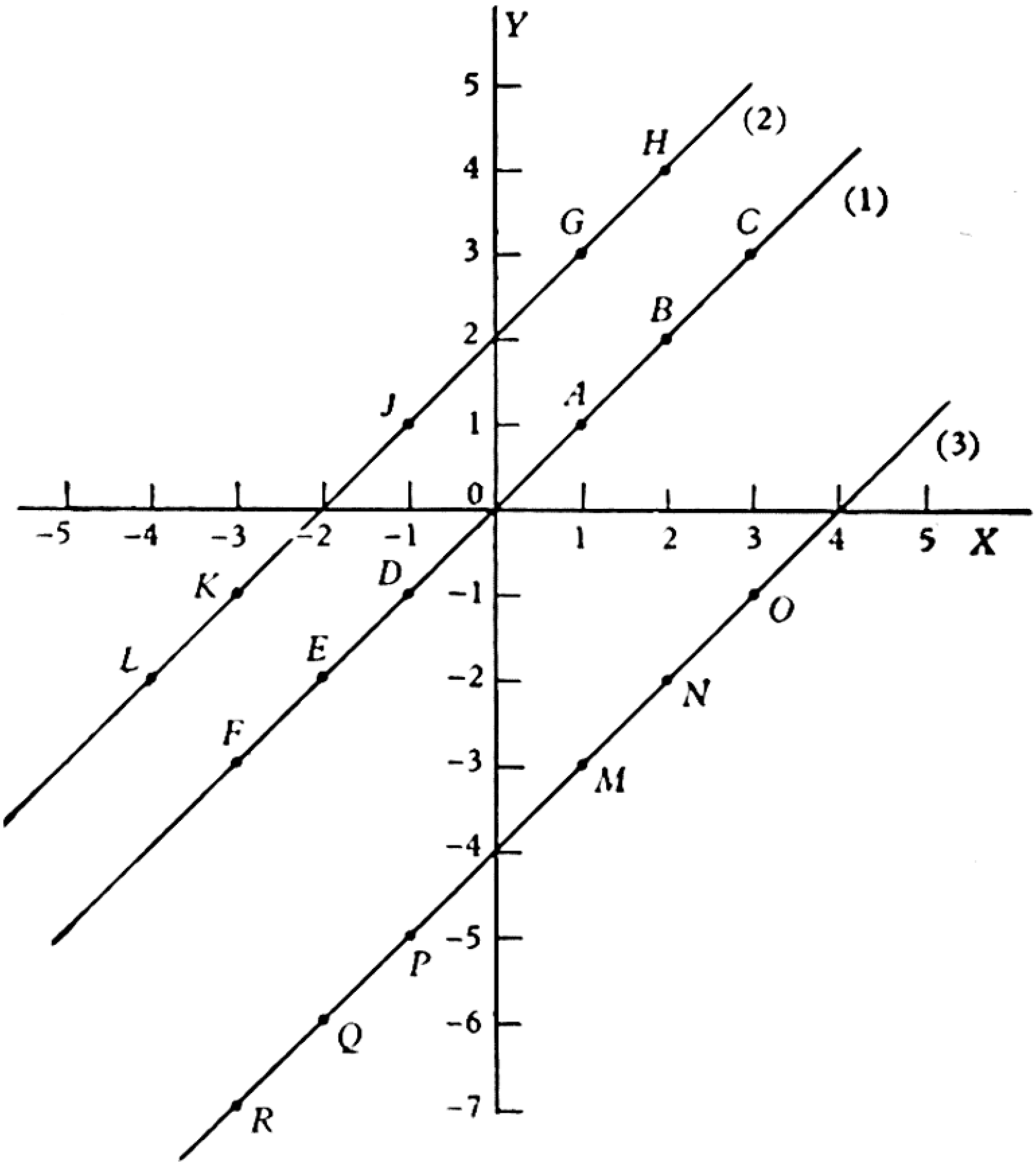


Fig. 102

On line (2) the points G, H, J, K, L are respectively (1, 3), (2, 4), $(-1, 1)$, $(-3, -1)$ and $(-4, -2)$. In every case the ordinate of the point is equal to the abscissa of the point plus 2. All points on line (2), therefore, satisfy the equation $y = x + 2$.

On line (3) the points M, N, O, P, Q, R are respectively $(1, -3)$, $(2, -2)$, $(3, -1)$, $(-1, -5)$, $(-2, -6)$ and $(-3, -7)$. In every case the ordinate of the point is equal to the abscissa of the point minus 4. All points on line (3) therefore satisfy the equation $y = x - 4$.

Notice again that line (1) cuts the axis of y at 0, line (2) cuts the axis of y at $+2$, and line (3) cuts that axis at -4, i.e.

$y = x + 0$ [line (1)] cuts y axis at 0,

$y = x + 2$ [line (2)] cuts y axis at 2,

$y = x - 4$ [line (3)] cuts y axis at -4.

Again, all three lines are parallel, and all have the same coefficient, namely, 1, for x.

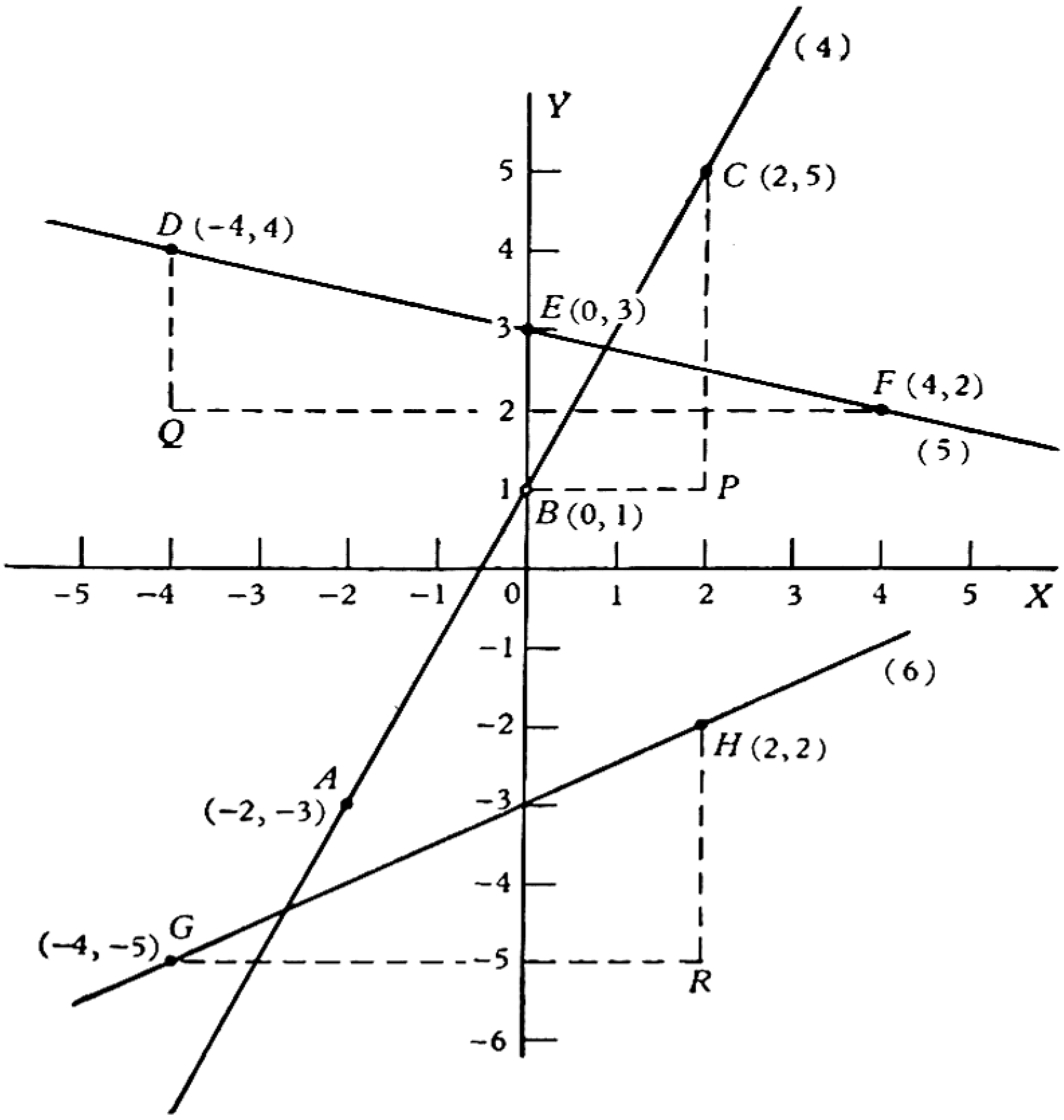


Fig. 103

Now consider line (4) on fig. 103. Three points A, B and C are chosen, namely, $(-2, -3)$, $(0, 1)$ and $(2, 5)$. The ordinate in every case is twice the abscissa plus 1. Thus the equation $y = 2x + 1$ is satisfied by all points on the line.

By examination of the points D, E and F, namely, $(-4, 4)$, $(0, 3)$ and $(4, 2)$ on line (5), we see that the equation $y = -\frac{1}{4}x + 3$ is represented by this line. It is left as an exercise for the student to check that points on line (6) satisfy the equation $y = \frac{1}{2}x - 3$. These last three lines intersect the axis of y at $+1$, $+3$ and -3 respectively. They are not parallel, and the coefficients of x differ in consequence.

Now let us choose two points B and C on line (4). Draw a horizontal through B and a vertical through C, and let these intersect in P. The length CP is 4 units, and the length BP is 2 units. The quotient CP/BP is 2, the coefficient of x in the equation of the line. It is obvious that we should have obtained the same quotient if we had chosen points A and C, because the vertical distance of C above A is 8, and the horizontal distance between C and A is 4. If we choose any two points on line (4) and measure the vertical and horizontal distances between them according to the scales of y and x on the graph, the quotient will always be 2.

Applying the same process to line (5) using D and F as the points, and calling the point of intersection of the vertical and horizontal lines Q, we have $DQ/FQ = \frac{2}{8} = \frac{1}{4}$. The slope of this line, however, is downwards from the left, while the slope of line (4) was downwards from the right. This fact is shown in the coefficient of x in the equation of line (5) being negative, i.e. $y = -\frac{1}{4}x + 3$. Applying the method to line (6), the quotient HR/GR is $\frac{3}{6}$ or $\frac{1}{2}$, and the equation we have found to be $y = \frac{1}{2}x - 3$.

If a vertical and a horizontal line are drawn through any two points, then the quotient obtained by dividing the vertical distance to the point of intersection of the lines, by the horizontal distance (measured in the scales of the axes), is called the gradient or slope of the straight line joining the two points. The gradient is positive if the line joining the two points slopes downwards from the right, and negative if it slopes downwards from the left.

The reason for the algebraic sign of the gradient may be shown in another way.

Consider points C $(2, 5)$ and B $(0, 1)$ on line (4). The distance CP represents, ordinate of C minus ordinate of B, i.e. $5 - 1 = 4$, and the distance BP represents, abscissa of C minus abscissa of B, i.e. $2-0=2$. Thus

$$\frac{CP}{BP}=\frac{4}{2}=2.$$

Similarly for points D and F on line (5): DQ represents ordinate of D minus ordinate of F, i.e. $4-2=2$, and QF represents abscissa of D minus abscissa of F, i.e. $-4-4=-8$. Thus

$$\frac{DQ}{QF}=\frac{2}{-8}=-\frac{1}{4}.$$

The same result will be obtained if we choose

$$\frac{\text{Ordinate of } F-\text{ordinate of } D}{\text{Abscissa of } F-\text{abscissa of } D}=\frac{2-4}{4-(-4)}=\frac{-2}{8}=-\frac{1}{4}.$$

In general, then, to find the gradient of a line, choose any two points on the line. The gradient is equal to

$$\frac{\text{Ordinate of first point minus ordinate of second point}}{\text{Abscissa of first point minus abscissa of second point}}.$$

The following conclusions may be drawn from our investigation:

(1) Any equation of the form $y=mx+c$ (where m and c are constants) represents a straight line.

(2) When expressed in this form, the constant which stands alone, namely, c, is the distance from the origin at which the line intersects the axis of y.

(3) The coefficient of x, namely, m, is the gradient of the line.

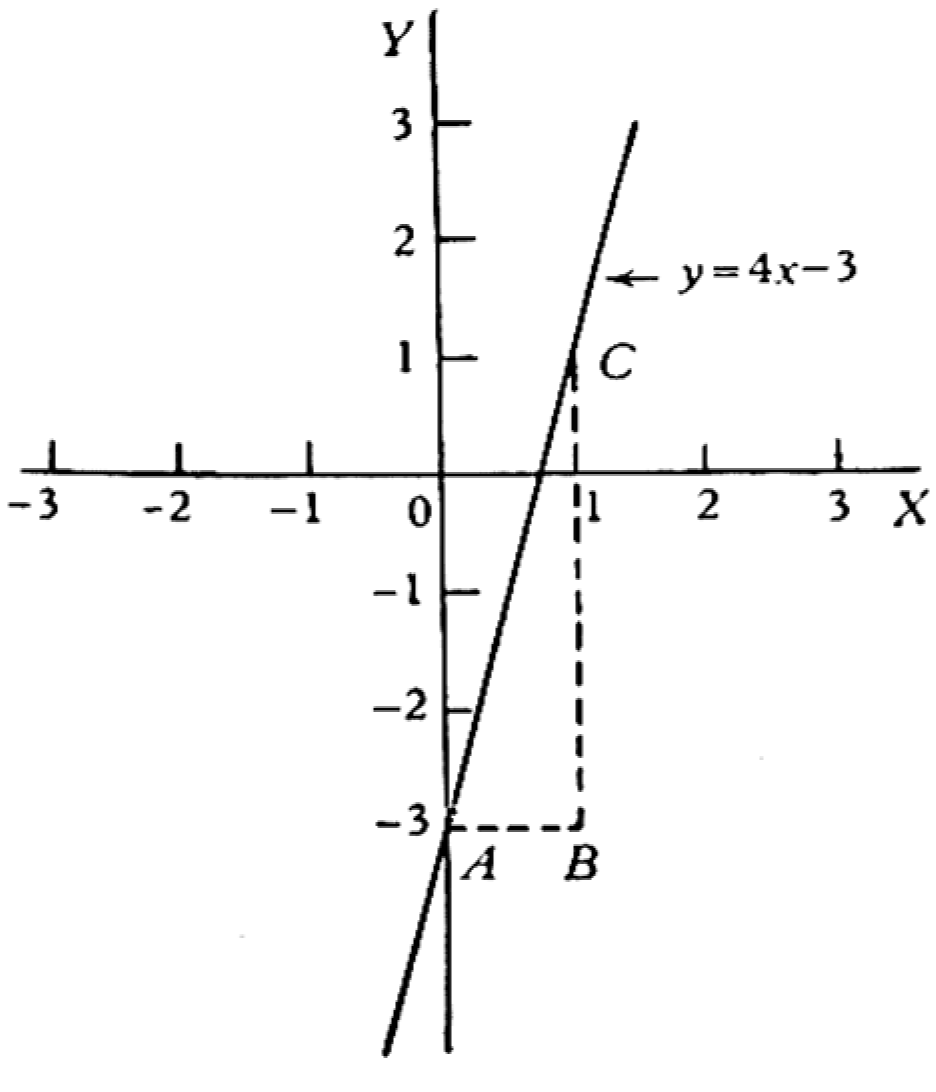


Fig. 104

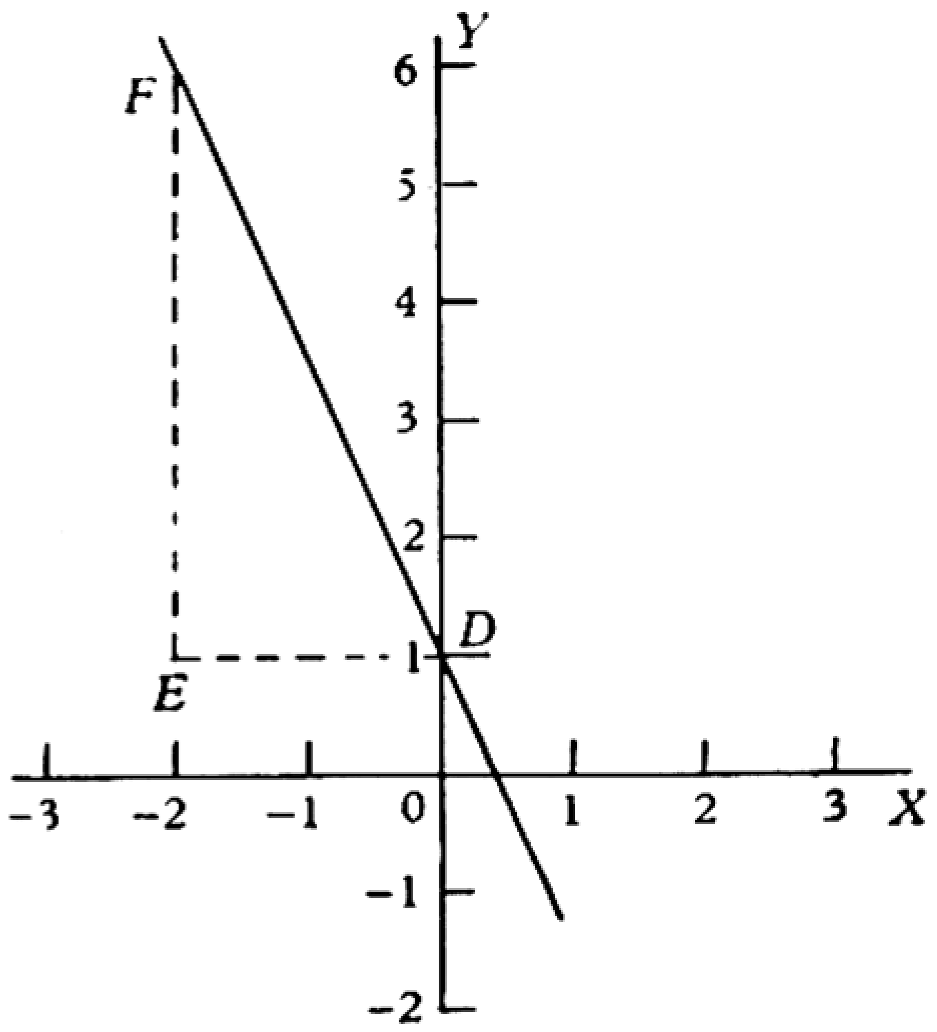


Fig. 105

To draw a straight line given its equation

Method 1. To draw the line whose equation is $y = 4x - 3$. The constants m and c are $+4$ and -3 respectively. Starting at the point where the line intersects the y axis, viz. $(0, -3)$, measure a horizontal distance of 1 unit to the right (line AB, fig. 104). From B measure a vertical distance of 4 units, BC. Join CA. Then the line passing through A and C is the required line. (If it is not convenient to have the same scales for the x and the y axes, then the distances 1 and 4 are measured according to the scales chosen for the two axes.)

To draw the line whose equation is $y = -\frac{5x}{2} + 1$. The constants m and c are $-\frac{5}{2}$ and $+1$. Starting at the point $(0, 1)$ draw a horizontal line of length 2 units to the *left* (DE, fig. 105). Through E draw a vertical line 5 units in length to F. Then the line through FD is the required line.

Method 2. To draw the line whose equation is $y = 4x - 3$. Let x have any three values, let us say $x = 1$, $x = 0$, $x = -1$. Then calculate the corresponding values of y from $y = 4x - 3$.

When $x = 1$, $y = 4 \times 1 - 3 = 1$.

When $x = 0$, $y = 4 \times 0 - 3 = -3$.

When $x = -1$, $y = 4x(-1) - 3 = -7$.

Tabulate, as follows:

x	1	0	-1
y	1	-3	-7

The three numerical columns in the table give us three points on the straight line, viz. $(1, 1)$, $(0, -3)$ and $(-1, -7)$. Draw a line through these points. It is not really necessary to choose three values of x, two would suffice, because two points are sufficient to fix a straight line. It is advisable, however, to choose three, so that any error in a calculation may be observed when the points are plotted (see fig. 106). It will be seen that figs. 104 and 106 give the same line by the two methods.

To draw the line whose equation is $y = -\frac{5x}{2} + 1$. Choose values 1, 0 and -1 for x as before.

When $x = 1$, $y = -\frac{5}{2} \times 1 + 1 = -1\frac{1}{2}$.
When $x = 0$, $y = -\frac{5}{2} \times 0 + 1 = +1$.
When $x = -1$, $y = -\frac{5}{2} \times -1 + 1 = 3\frac{1}{2}$.

Tabulating:

x	1	0	-1
y	$-1\frac{1}{2}$	1	$3\frac{1}{2}$

Fig. 107 shows a straight line drawn through $(1, -1\frac{1}{2})$, $(0, 1)$ and $(-1, 3\frac{1}{2})$. This is the same line as that in fig. 105.

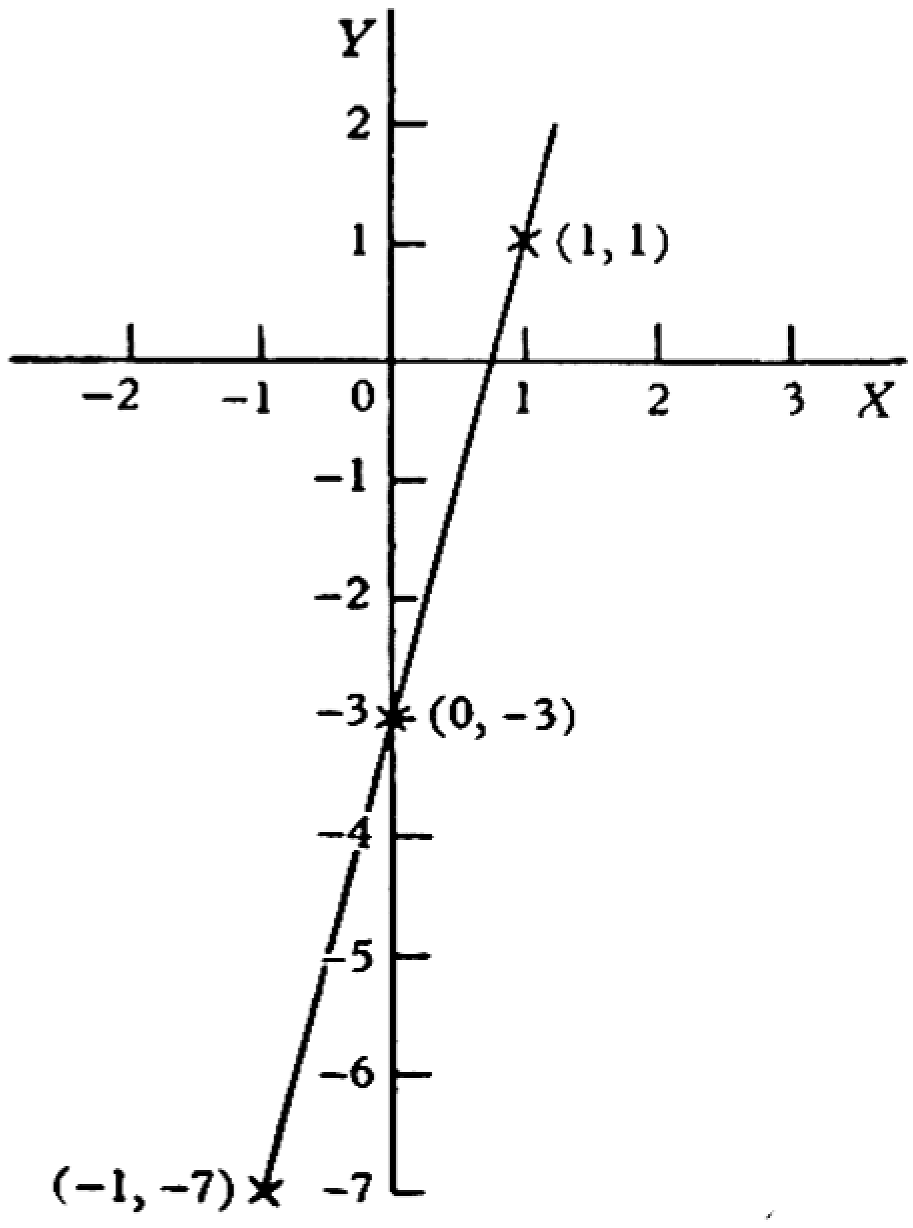


Fig. 106

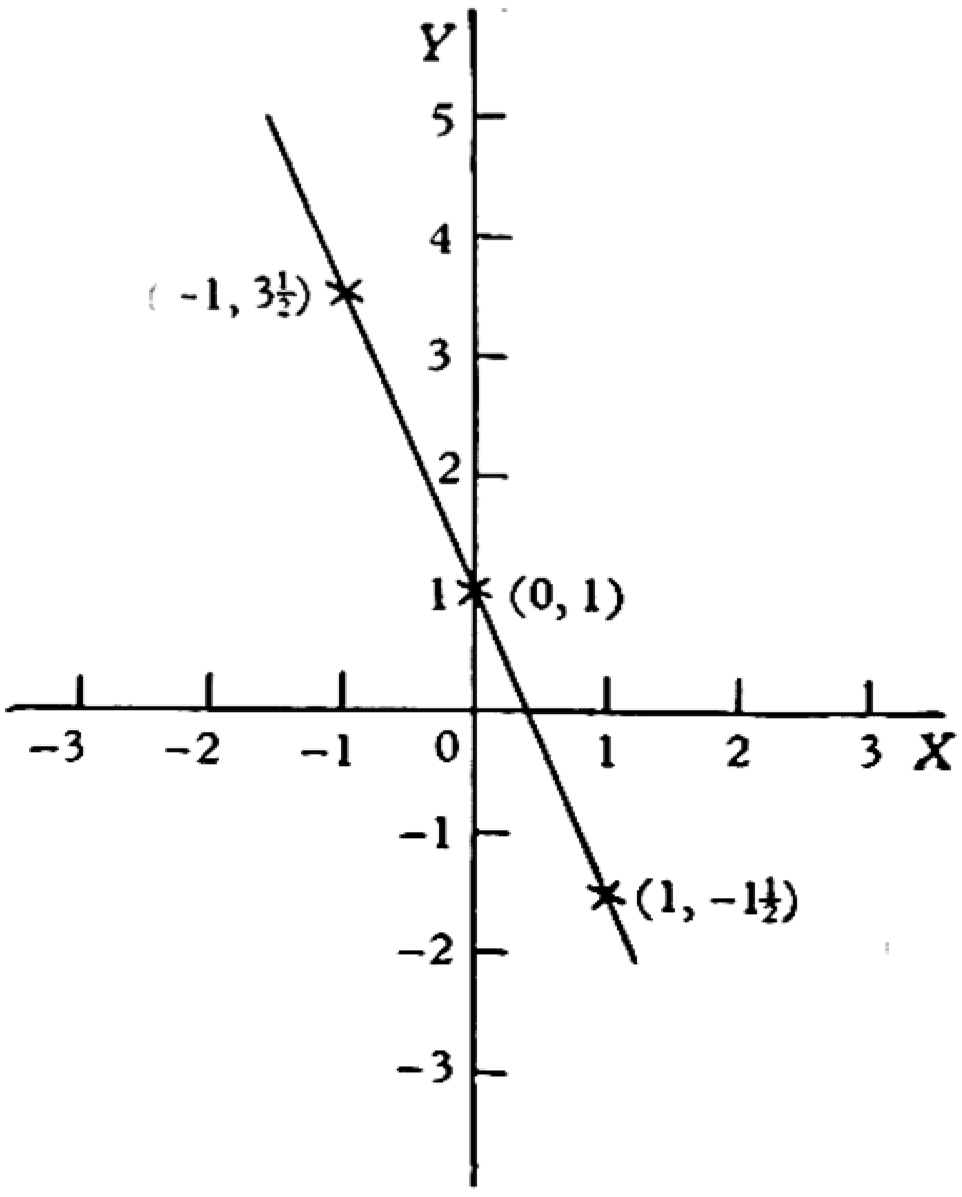


Fig. 107

To find the equation of a straight line which passes through two given points

To find the equation of the line passing through $(-3, 6)$ and $(1, \frac{2}{3})$.

Gradient of the line

$$= \frac{\text{Ordinate of first point minus ordinate of second point}}{\text{Abscissa of first point minus abscissa of second point}}$$

$$= \frac{6 - \frac{2}{3}}{-3 - 1}$$

$$= \frac{5\frac{1}{3}}{-4} = -\frac{16}{3} \times \frac{1}{+4} = -\frac{4}{3}.$$

Thus the equation is $y = -\frac{4x}{3} + c$, where c is the y intercept.

Plotting the points $(-3, 6)$ and $(1, \frac{2}{3})$ in fig. 108 we find that the y intercept is $+2$.

Hence the equation of the line passing through $(-3, 6)$ and $(1, \frac{2}{3})$ is $y = -\frac{4x}{3} + 2$.

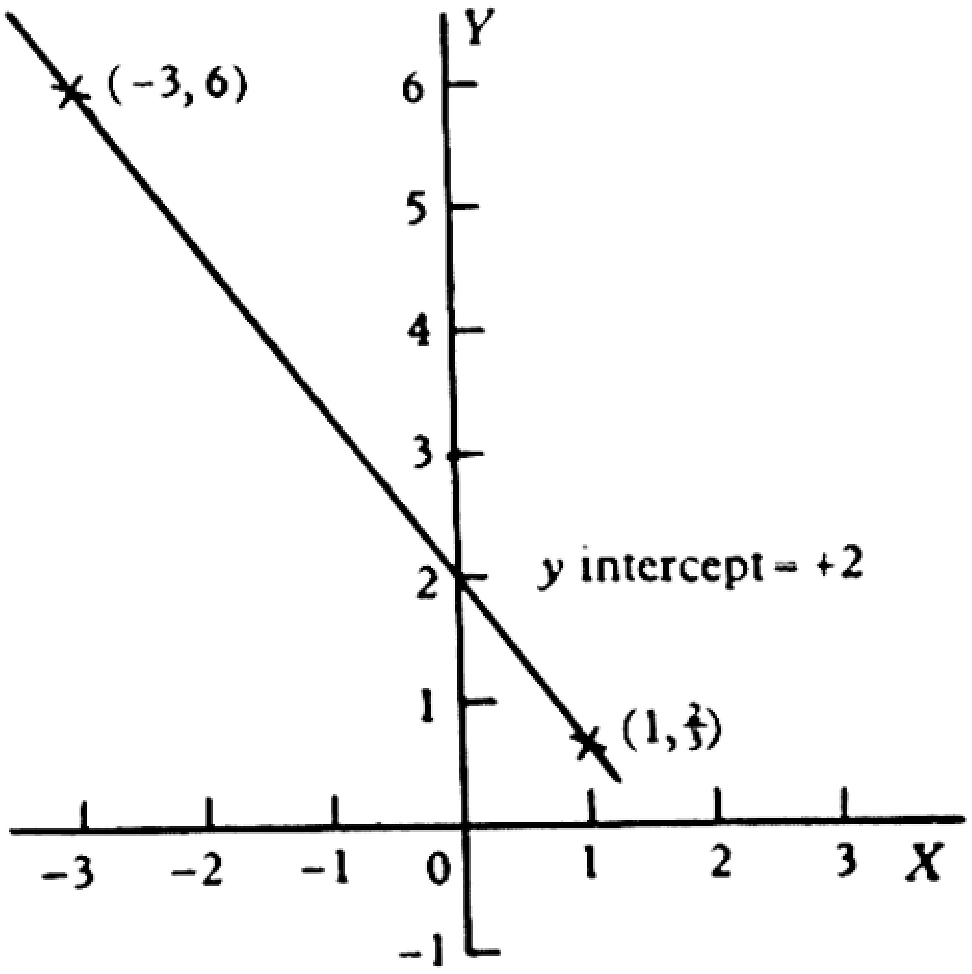


Fig. 108

Equations of the axes and lines parallel to the axes

Every point on the x axis has zero for its ordinate, so the equation of the x axis is $y = 0$. Similarly, every point on the y axis has zero for its abscissa, so the equation of the y axis is $x = 0$.

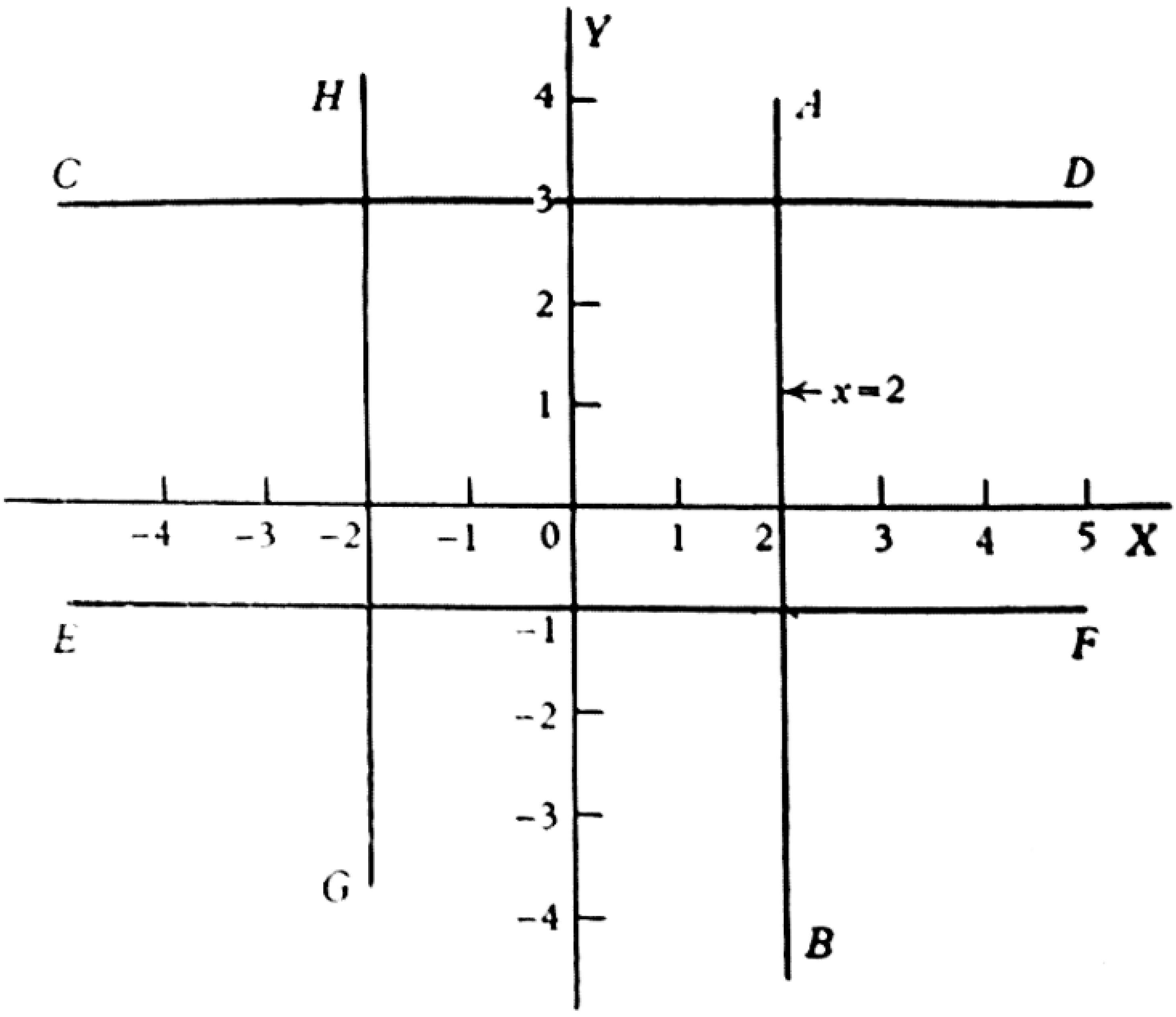


Fig. 109

By the same reasoning, it will be agreed that the equations of the lines in fig. 109 are

AB has the equation $x = 2$,
CD has the equation $y = 3$,
EF has the equation $y = -1$,
GH has the equation $x = -2$.

Any line parallel to the axis of x has an equation $y =$ constant. Any line parallel to the axis of y has an equation $x =$ constant.

Examples for Chapter 27

1. Using the same scales for the axes of x and y, plot the following points:

$$A(4, 3),\quad B(4, 0),\quad O(0, 0),\quad C(0, -3),\quad D(-5, -3),$$
$$E(0, 9),\quad F(2\tfrac{1}{2}, 6),\quad G(5, 9).$$

(*a*) Measure the angle ABO, and the length OB.

(*b*) Measure the length DE, and the length CG.

(*c*) State what is similar about the lines DE and OF.

(*d*) Calculate the areas of the triangles OAB and OAE.

(*e*) Calculate the area of the figure $DCGE$. (*Chap.* 27)

2. Using the same scale for both axes, draw the straight line joining the points (1, 2) and (− 1, − 4). Write the ordinates of points on this line which have abscissae 0, 2, 3, − 2.

Calculate the gradient of the line and hence write its equation. Write the equation of a line parallel to this one, but passing through the origin. (*Chap.* 27)

3. Plot the following lines on the same axes:

$$(a)\ y = \frac{2x}{3} - 1,\quad (b)\ x = -3,\quad (c)\ y = -x + 4.$$

If A is the point of intersection of (*a*) and (*b*), B the point of intersection of (*b*) and (*c*), and C the point of intersection of (*a*) and (*c*), write the co-ordinates of A, B and C. Calculate the area of the triangle ABC and state what fraction of this triangle lies in the first and fourth quadrants. (*Chap.* 27)

4. Plot on the same diagram:

$$(1)\ y = \frac{4x}{3} - 2, \quad (2)\ y = -\frac{3x}{4} - 4,$$

$$(3)\ y = \frac{4x}{3} + 5, \quad (4)\ y = -\frac{3x}{4} + 3.$$

Show that (1) and (3) are parallel, (2) and (4) are parallel, and that the figure formed by the four lines is a rectangle.

Hence show that when two lines are perpendicular, and drawn to the same scales on both axes, the product of their gradients is -1. Show by plotting and by calculation that the straight line which intersects the y axis at $(0, 1)$ and is perpendicular to the line $y = \frac{1}{2}x - 3$ has the equation $y = -2x + 1$.

(*Chap.* 27)

5. Plot the lines $y = -\frac{5x}{4} + 7\frac{1}{4}$, $y = \frac{4x}{5} - 3$, $y = -\frac{5x}{4} - 3$ and $y = \frac{4x}{5} + 5\frac{1}{5}$ on the same axes.

Show that the figure enclosed by these lines is a square, and find the equation of the diagonal with a positive gradient.

(*Chap.* 27)

6. Consider the following lines:

$$(a)\ y = -\frac{x}{3} + 2, \quad (b)\ y = -3x + 2, \quad (c)\ y = x,$$

$$(d)\ y = x - 2, \quad (e)\ y = 1, \quad (f)\ x = 3, \quad (g)\ y = -\frac{x}{3}.$$

State which of these lines pass through $(3, 1)$.
State which are parallel.
State which pass through the origin.
State which pass through $(0, 2)$. (*Chap.* 27)

7. Mark the following points on graph paper:

$$A(2, 3), \quad B(3, 2), \quad C(3, 0), \quad D(2, -2), \quad E(-2, -2),$$

$$F(-3, 0), \quad G(-3, 2), \quad H(-2, 3).$$

Join BC, CD, DE, EF, FG and HA with straight lines. Join AB and HG by quarter circles, radius 1 unit, centred at $(3, 3)$ and $(-3, 3)$ respectively. The figure represents a metal plate, scale 1 unit = 3 in. Find the area of its face. (*Chaps* 24 *and* 27)

8. Solve the two triangles completely by scale drawings:

(*a*) $BC = 4$, $AC = 5$, $AB = 6$.

(*b*) $AB = 4$, $AC = 5$, angle $A = 76°$.

Measure one altitude in each, and calculate their areas.

(*Chaps.* 22 *and* 24)

9. ABC is a triangle. D is the mid-point of BC. A line is drawn through B parallel to AC. Join AD and produce to intersect the line through B at the point E. Prove that the triangles ACD and BDE are congruent, and hence show that AE is bisected at D. Prove also that CE is parallel to AB.

(*Chap.* 23)

10. The figures refer to an experiment on resilience; s = height of drop (feet), h = height of rebound (feet):

s (ft.)	2·5	3	3·5	4	4·5	5
h (ft.)	0·80	1·08	1·26	1·44	1·62	1·80

Plot h vertically against s horizontally, and from the graph find the height of rebound when the height of drop is 3·8 ft.

(*Chap.* 26)

Answers to Examples for Chapter 27

1. (*a*) Angle $ABO = 90°$, $OB = 5$. (*b*) $DE = 13$, $CG = 13$.
(*c*) DE parallel to OF.
(*d*) Triangle OAB = 6 units, triangle OAE = 18 units.
(*e*) $DCGE$ = 60 units.

2. $-1, 5, 8, -7$; $y = 3x - 1$; $y = 3x$.

3. $A(-3, -3)$; $B(-3, 7)$; $C(3, 1)$. Triangle ABC = 30 units; $\frac{1}{4}$.

5. $y = 9x - 3$.

6. (*a*), (*d*), (*e*) and (*f*) pass through (3, 1); (*c*) and (*d*) are parallel; and (*a*) and (*g*) are parallel; (*c*) and (*g*) pass through origin; (*a*) and (*b*) pass through (0, 2).

7. 237·86 sq.in.

8. (*a*) Angles 41°, 83° and 56° (approx.).
(*b*) $BC = 5·6$, angle $B = 60°$, angle $C = 44°$; area (*a*) 9·92 units; area (*b*) = 9·7 units.

10. $h = 1·37$ ft.

CHAPTER 28

THE PARABOLA AND THE RECTANGULAR HYPERBOLA. CUBIC CURVES

The parabola

To plot the graph of $y = x^2$. The table is calculated as in the previous chapter:

x	-4	-3	-2	-1	0	1	2	3	4
$y(=x^2)$	16	9	4	1	0	1	4	9	16

The graph is shown in fig. 110. It will be noticed that y cannot be negative as it is always the square of x, and no ordinary number can give a negative square. It has been convenient also to choose different scales for the two axes, but this will lead to no difficulties if the axes are clearly marked. The curve is called a parabola, and its shape should be noted because parabolic relationships are met with frequently in engineering work. Any parabola can be folded about a straight line so that both parts will fit over one another exactly. A figure which may be so folded is said to be *symmetrical* about the line, and in the above

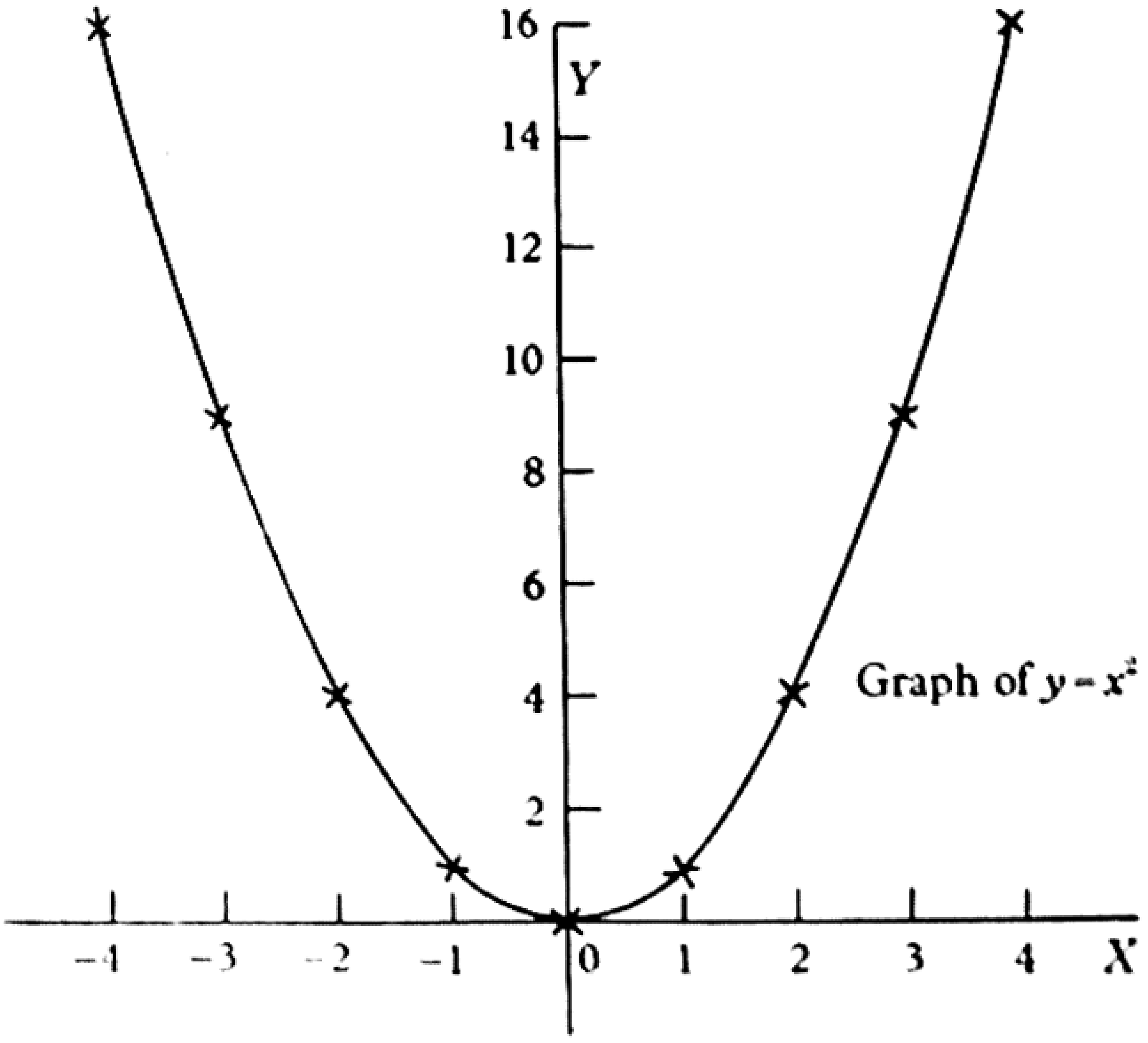


Fig. 110

case it is obvious that the line about which this curve is symmetrical is the y axis. Now the line about which a parabola is symmetrical is called the *axis* of the parabola, so the axis of the parabola $y = x^2$ is the axis of y. We shall find that in most cases the axis of a parabola does not coincide with one of the coordinate axes. The point at which the parabola turns is called its *vertex* (plural vertices). In this case the vertex is the origin.

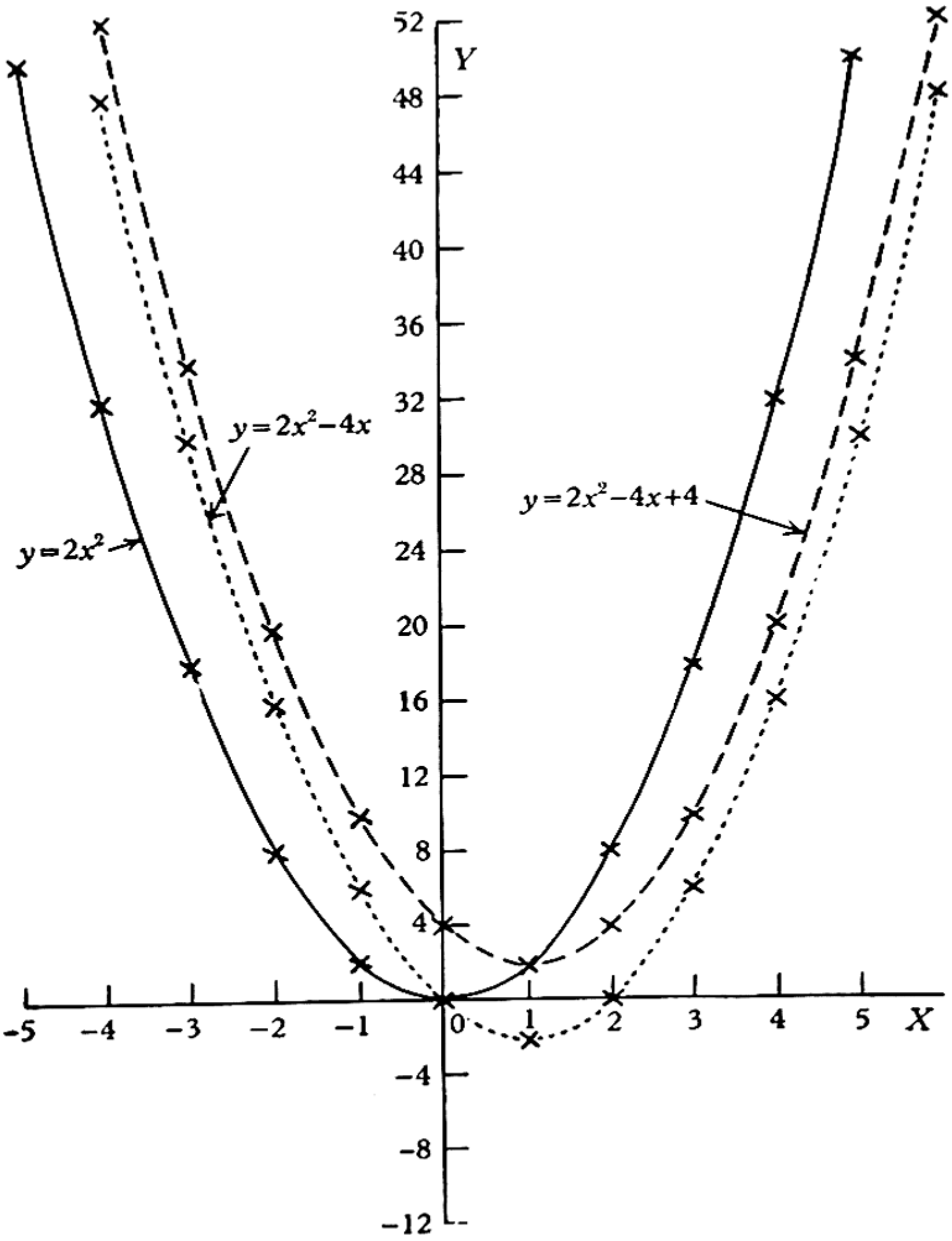


Fig. 111

A student should be able to recognise the equation of a parabola at sight, just as he recognises a straight line in the form

$$y = mx + c,$$

so we shall proceed to plot other parabolas and note the similarity in their equations.

Fig. 111 shows the graphs of three equations, $y = 2x^2$,

$y = 2x^2 - 4x$ and $y = 2x^2 - 4x + 4$, all three of which are plotted from appropriate lines in the following table:

x	-4	-3	-2	-1	0	1	2	3	4	5	A
x^2	16	9	4	1	0	1	4	9	16	25	
$2x^2$	32	18	8	2	0	2	8	18	32	50	B
$-4x$	16	12	8	4	0	-4	-8	-12	-16	-20	C
$+4$	4	4	4	4	4	4	4	4	4	4	D
$y = 2x^2 - 4x$	48	30	16	6	0	-2	0	6	16	30	$E = B + C$
$y = 2x^2 - 4x + 4$	52	34	20	10	4	2	4	10	20	34	$F = B + C + D$

To plot $y = 2x^2$, take values of line A horizontally, and line B vertically.

To plot $y = 2x^2 - 4x$, take values of line A horizontally, and line E vertically.

To plot $y = 2x^2 - 4x + 4$, take values of line A horizontally, and line F vertically.

All three graphs are parabolic but have different vertices:

The graph $y = x^2$ has the y axis for its axis, and the origin for its vertex.

The graph $y = x^2 - 4x$ has $x = 1$ for its axis, and the point $(1, -2)$ for vertex.

The graph $y = x^2 - 4x + 4$ also has $x = 1$ for its axis, and its vertex is at $(1, 2)$.

Next, let us consider three cases of parabolas in which the coefficient of x^2 is negative.

Fig. 112 shows the curves $y = -\frac{1}{2}x^2$, $y = -\frac{1}{2}x^2 + 2x$, and $y = -\frac{1}{2}x^2 + 2x - 3$, all of which are plotted from appropriate lines in the following table:

x	-4	-3	-2	-1	0	1	2	3	4	5	A
x^2	16	9	4	1	0	1	4	9	16	25	
$-\frac{1}{2}x^2$	-8	$-4\frac{1}{2}$	-2	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-2	$-4\frac{1}{2}$	-8	$-12\frac{1}{2}$	B
$+2x$	-8	-6	-4	-2	0	2	4	6	8	10	C
-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	D
$y = -\frac{1}{2}x^2 + 2x$	-16	$-10\frac{1}{2}$	-6	$-2\frac{1}{2}$	0	$1\frac{1}{2}$	2	$1\frac{1}{2}$	0	$-2\frac{1}{2}$	$E = B + C$
$y = -\frac{1}{2}x^2 + 2x - 3$	-19	$-13\frac{1}{2}$	-9	$-5\frac{1}{2}$	-3	$-1\frac{1}{2}$	-1	$-1\frac{1}{2}$	-3	$-5\frac{1}{2}$	$F = B + C + D$

The graph of $y = -\frac{1}{2}x^2$ is plotted from lines A and B, that of $y = -\frac{1}{2}x^2 + 2x$ from lines A and E, and that of $y = -\frac{1}{2}x^2 + 2x - 3$ from lines A and F. Again all three graphs are parabolas.

$y = -\frac{1}{2}x^2$ has the y axis for its axis and its vertex is the origin.

$y = -\frac{1}{2}x^2 + 2x$ has $x = 2$ for axis and the point (2, 2) for vertex.

$y = -\frac{1}{2}x^2 + 2x - 3$ has $x = 2$ for axis and the point (2, -1) for vertex.

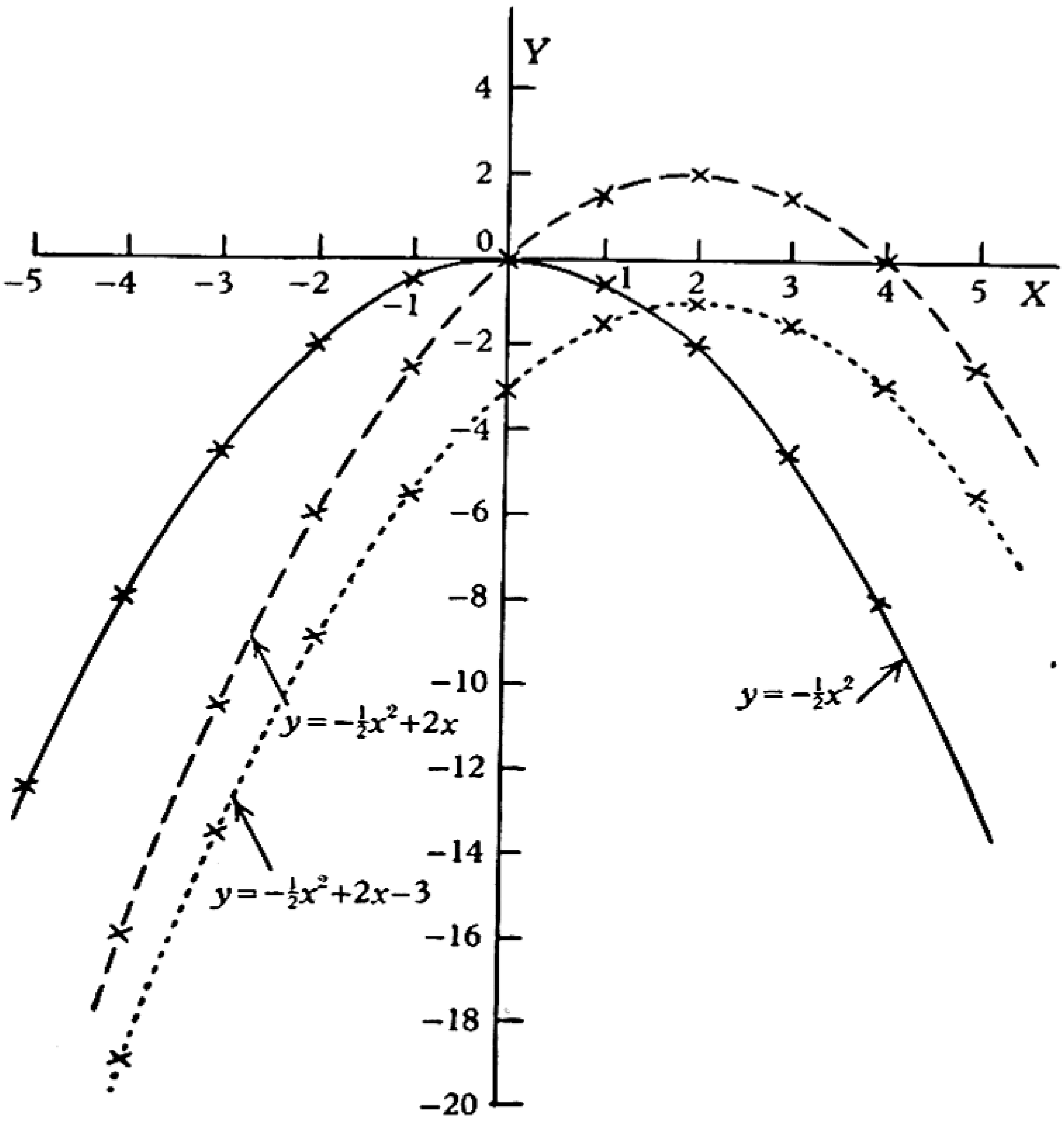


Fig. 112

These graphs differ from the previous ones in having their vertices uppermost.

Conclusions. (1) All graphs of the forms

$$y = ax^2, \quad \text{or} \quad y = ax^2 + bx, \quad \text{or} \quad y = ax^2 + bx + c,$$

where a, b and c are constants, are parabolas. This statement may be simplified by saying that all graphs of the form

$$y = ax^2 + bx + c$$

are parabolas, where a is any constant, and b and c are constants or zero.

(2) When the coefficient of x^2 (i.e. a) is positive, the vertex is at the bottom of the curve, and when the coefficient of x^2 is negative, the vertex is at the top of the curve.

The *ordinate* of the vertex is called the *maximum* value of y if the vertex is at the top, and the *minimum* value of y if the vertex is at the bottom.

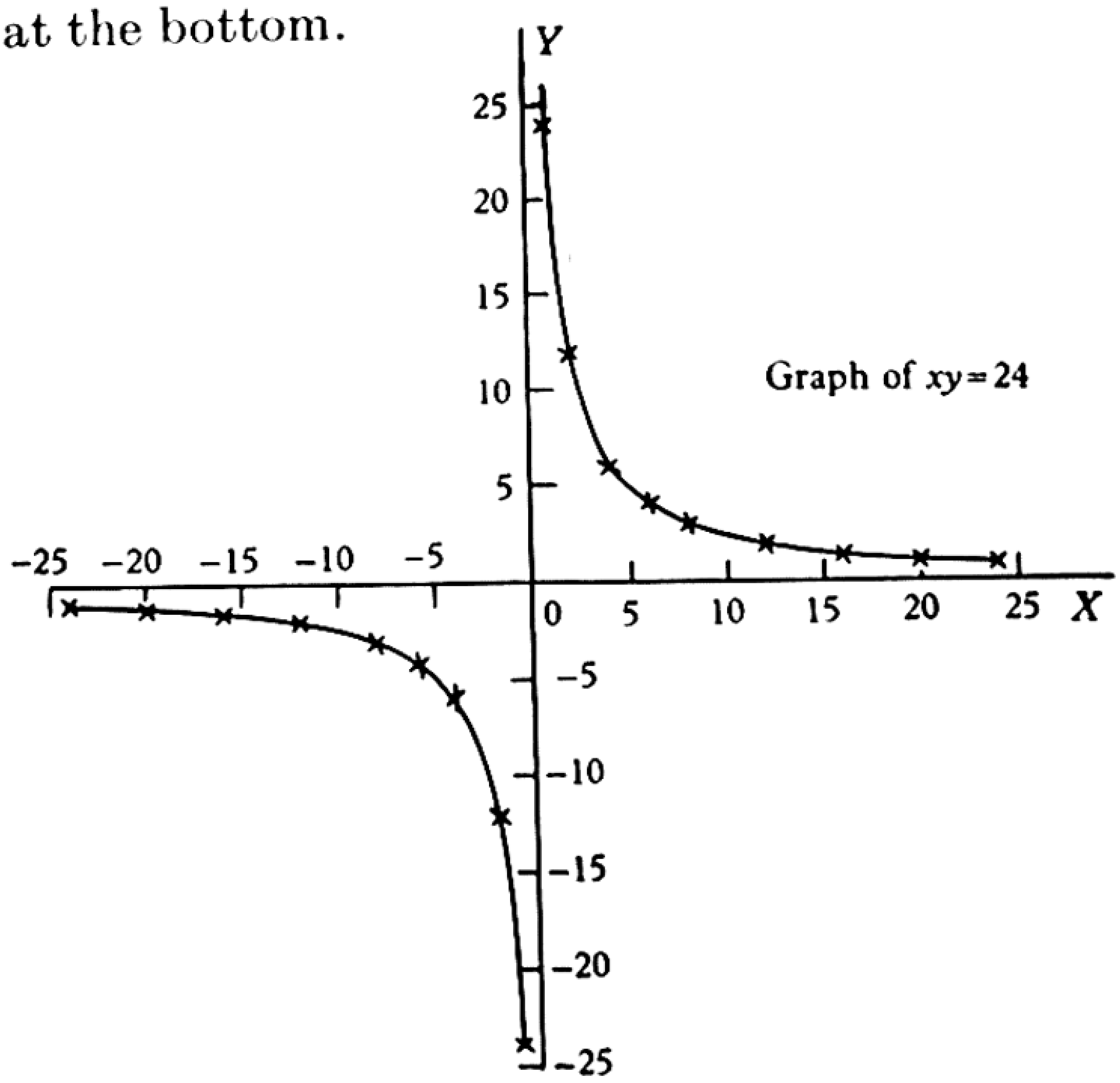


Fig. 113

The equation xy = constant (rectangular hyperbola)

Consider the equation $xy = 24$. To make the table, choose a series of values of x, then divide each of these into 24 to obtain the corresponding values of y, for if $xy = 24$, then $y = 24/x$:

x	24	20	16	12	8	6	4	2	1
y	1	$1\frac{1}{5}$	$\frac{3}{2}$	2	3	4	6	12	24

x	-24	-20	-16	-12	-8	-6	-4	-2	-1
y	-1	$-1\frac{1}{5}$	$-\frac{3}{2}$	-2	-3	-4	-6	-12	-24

The curve is seen to consist of two equal parts (fig. 113), one in the first quadrant and one in the third quadrant, and is called a rectangular hyperbola. The part which lies in the first quadrant is the same shape as the curve produced by plotting pressure of a gas against volume for constant temperature. In this case the equation is $PV =$ constant (Boyle's Law).

An interesting fact about the rectangular hyperbola is that the ends of the two parts move closer to the axes the further they are from the origin. But, however great x becomes, the value of $24/x$ will never be absolutely zero, although it becomes smaller and smaller as x increases. We express this by saying that the curve touches the axes at infinity. A straight line which touches a curve at infinity is said to be an *asymptote* of the curve, or to be *asymptotic* to the curve. In this rectangular hyperbola the co-ordinate axes are the asymptotes.

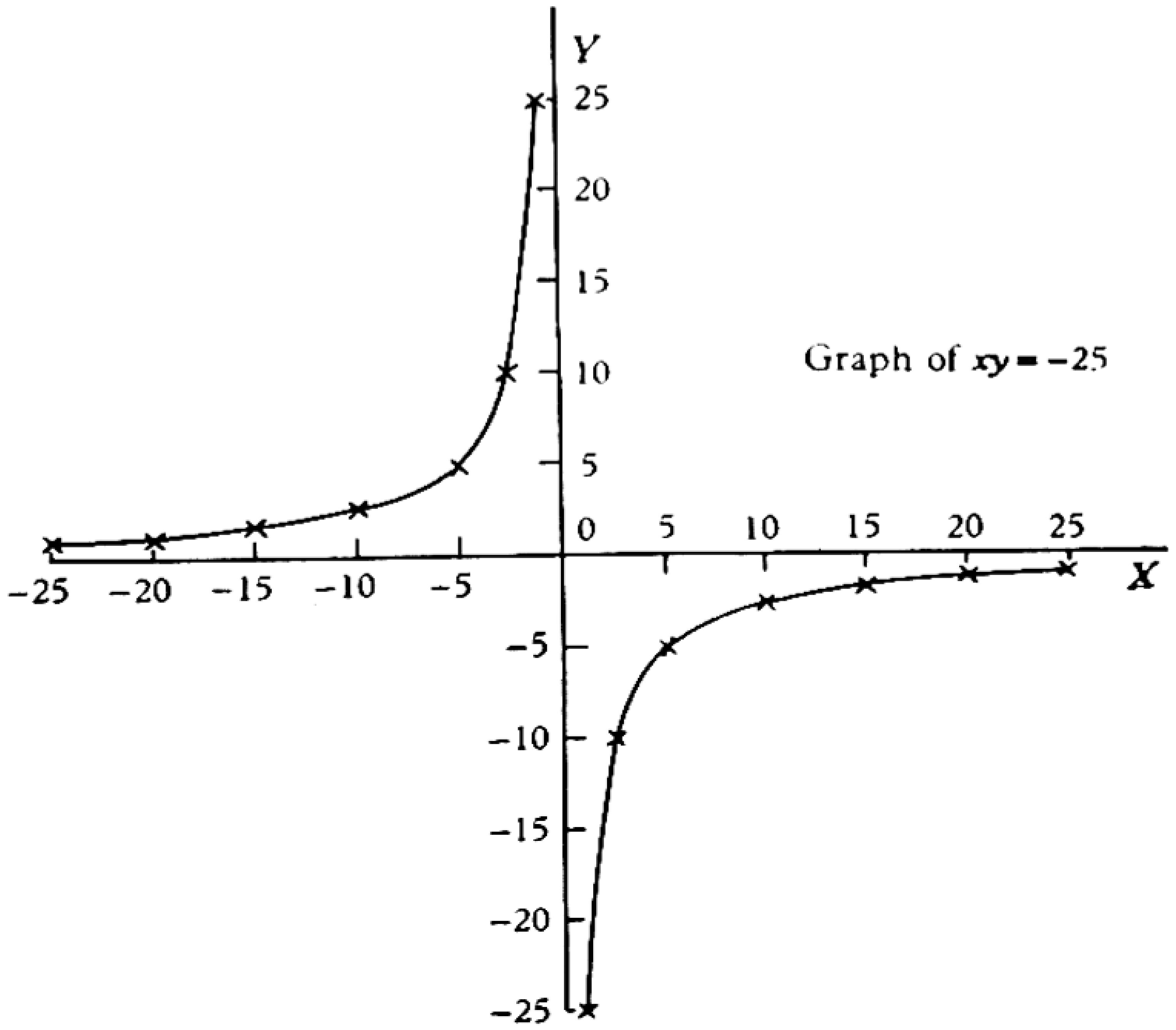


Fig. 114

Next, consider the equation $xy =$ constant when the constant is negative. Take the equation $xy = -25$.

As before, to find values of y, divide corresponding values of x into -25:

x	± 25	± 20	± 15	± 10	± 5	$\pm 2\frac{1}{2}$	± 1
y	∓ 1	$\mp \frac{5}{4}$	$\mp \frac{5}{3}$	$\mp \frac{5}{2}$	∓ 5	∓ 10	∓ 25

[Note that $+25$ for x corresponds to -1 for y, etc.]

The curve is seen to be a rectangular hyperbola again (fig. 114), but the two parts occur in the second and fourth quadrants.

Conclusion. Curves of the form $xy =$ constant are rectangular hyperbolas. If the constant is positive the two parts occur in the first and third quadrants, while if the constant is negative they fall in the second and fourth quadrants.

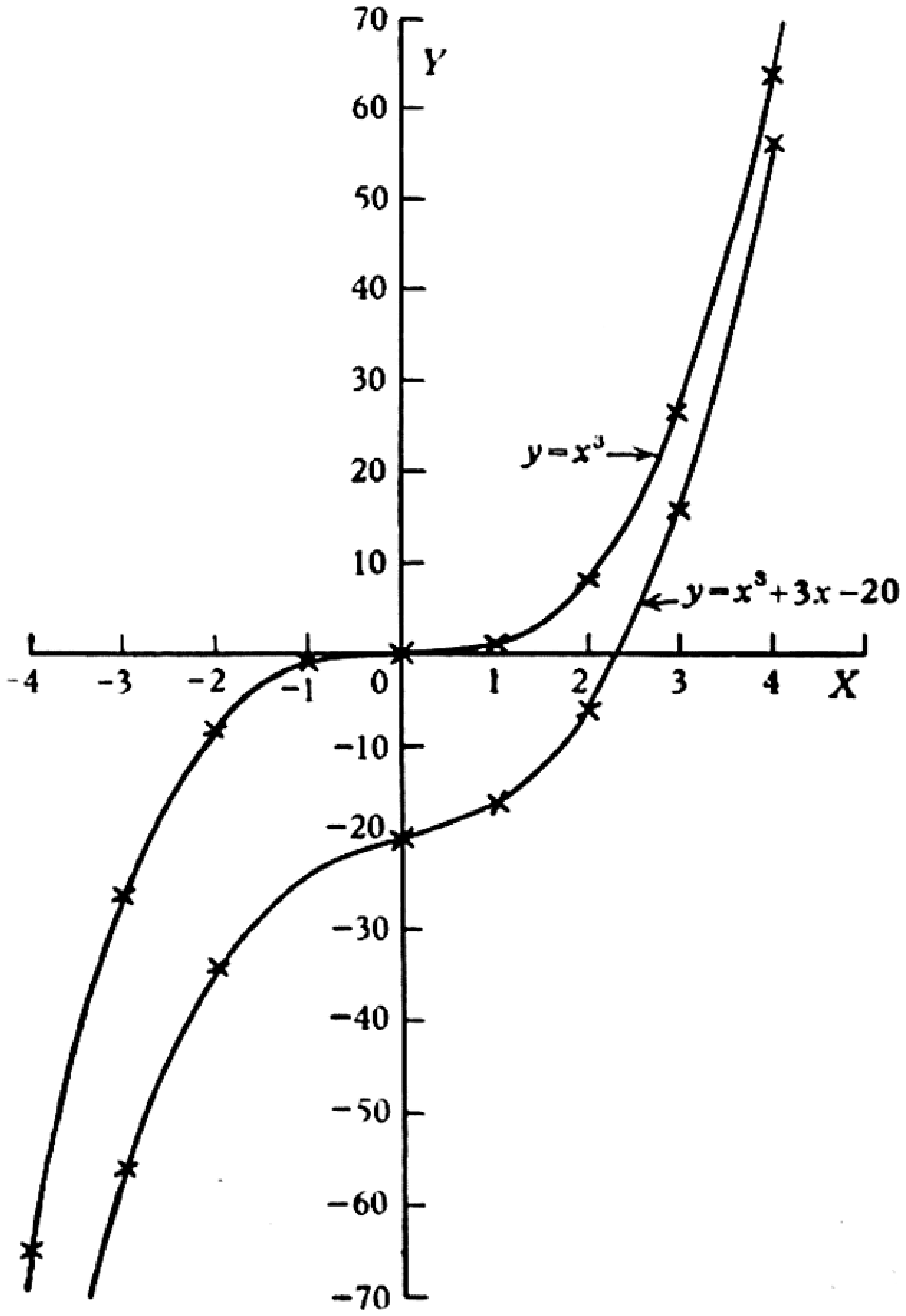


Fig. 115

Cubic curves

Curves of the form $y = ax^3 + bx^2 + cx + d$, where a is any constant, and b, c and d are constants or zero, are called cubic curves. The curves $y = x^3$ and $y = x^3 + 3x - 20$ are shown in fig. 115, with values calculated from the table below:

x	-4	-3	-2	-1	0	1	2	3	4	A
x^3	-64	-27	-8	-1	0	1	8	27	64	B
$+3x$	-12	-9	-6	-3	0	3	6	9	12	C
-20	-20	-20	-20	-20	-20	-20	-20	-20	-20	D
$y=x^3+3x-20$	-96	-56	-34	-24	-20	-16	-6	16	56	$E=B+C+D$

The curve $y = x^3$ is plotted from lines A and B.

The curve $y = x^3 + 3x - 20$ is plotted from lines A and E.

Example from mechanics. A body is projected vertically with a velocity of u ft. per sec. Its height (s ft.) above the point of projection at a time t sec. after projection is given by

$$s = ut - \tfrac{1}{2}gt^2,$$

where g is the acceleration due to gravity, namely, 32 ft. per sec. per sec. Take $u = 40$ ft. per sec., and plot s vertically against t horizontally from $t = 0$ to $t = 2\tfrac{1}{2}$ sec., taking values of t for every quarter second. From the graph find the maximum height of the body, the time taken to reach the maximum height, and the time of flight.

$$\begin{aligned} s &= ut - \tfrac{1}{2}gt^2 \\ &= 40t - \tfrac{1}{2} \times 32t^2 \\ &= 40t - 16t^2. \end{aligned}$$

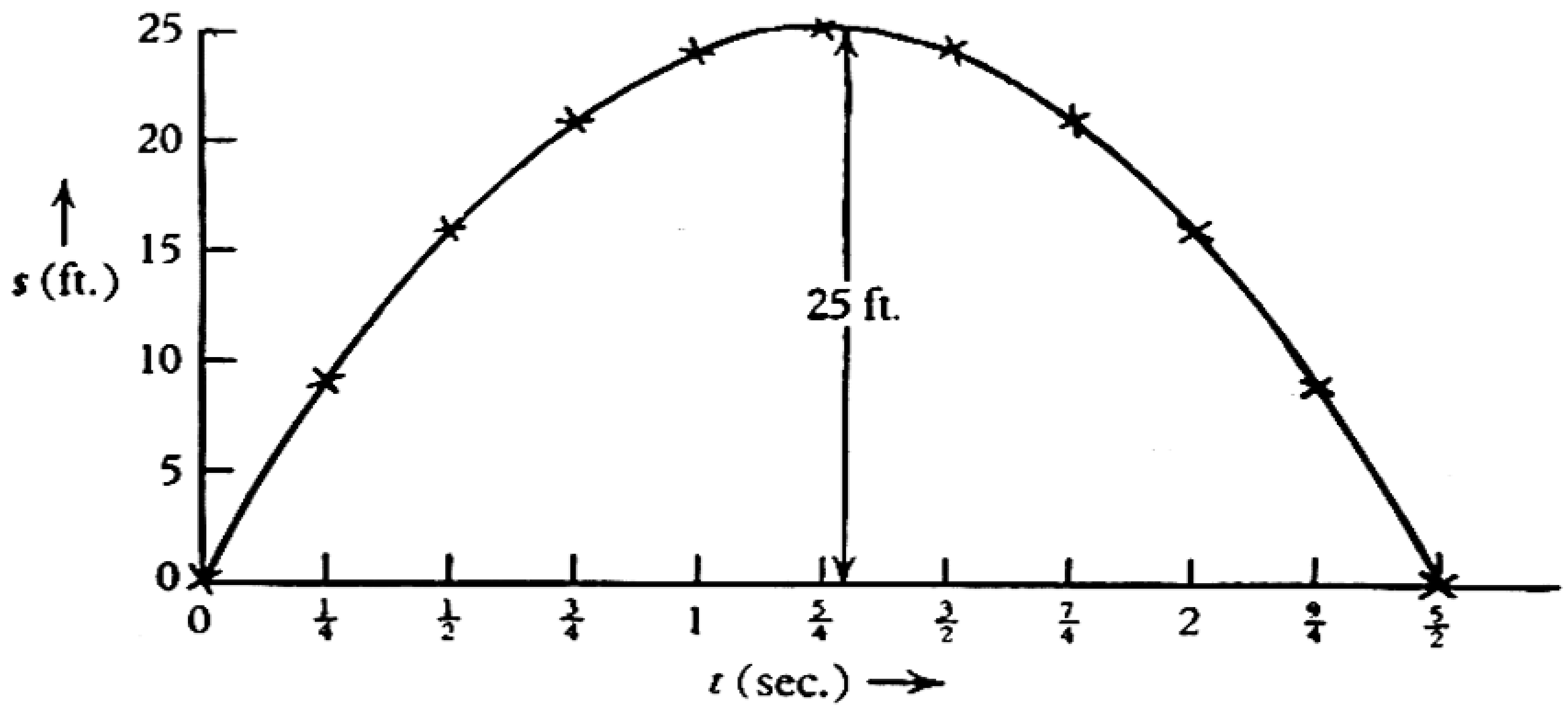


Fig. 116

t (sec.)	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2	$\frac{9}{4}$	$\frac{5}{2}$
t^2	0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{9}{16}$	1	$\frac{25}{16}$	$\frac{9}{4}$	$\frac{49}{16}$	4	$\frac{81}{16}$	$\frac{25}{4}$
$-16t^2$	0	-1	-4	-9	-16	-25	-36	-49	-64	-81	-100
$40t$	0	10	20	30	40	50	60	70	80	90	100
$s = 40t - 16t^2$	0	9	16	21	24	25	24	21	16	9	0

From the graph (fig. 116) the maximum height of the body is 25 ft., the time to reach the maximum height is $1\tfrac{1}{4}$ sec., and the time to reach the ground again is $2\tfrac{1}{2}$ sec. [*Note.* Writing y for s and x for t, the equation is $y = -16x^2 + 40x$, which is recognised as that of a parabola with vertex at the top.]

Examples for Chapter 28

1. Plot on the same axes and to the same scale, the graphs of $y = x^2 - 2x - 3$, and $y = \frac{3}{2} + x - \frac{1}{2}x^2$, taking values of x from -2 to $+4$.

Write the co-ordinates of the vertices of the parabolas, the equation of their common axis, and the values of x at which the curves intersect the axis of x. (*Chap.* 28)

2. Plot $xy = 16$ and show that the two parts of the curve are symmetrical about the line $y = x$.

Write the co-ordinates of the points of intersection of the straight line and the curve.

Plot the line $y = \frac{1}{4}x$ and write the co-ordinates of the points of intersection of this line and the curve. (*Chaps.* 28 *and* 27)

3. Plot $y = 2x^2 - 5x + 2$ for values of x from -3 to $+5$. Read off the values of x for which $y = 0$.

Show that these are answers which satisfy the equation

$$2x^2 - 5x + 2 = 0,$$

by substitution.

Write the minimum value of y and the corresponding value of x. (*Chap.* 28)

4. Plot $y = x^3$ and $y = x - x^3$ on the same axes from $x = -2$ to $x = +2$, using half-units for x.

Write as accurately as possible the abscissae of their points of intersection.

Write the values of x for the points at which $y = x - x^3$ intersects the axis of x, and, by substitution, show that they satisfy the equation $x - x^3 = 0$. (*Chap.* 28)

5. When a body is projected upwards at an angle of 45° the curve it describes is given by $y = x - \frac{32x^2}{V^2}$, where V is the velocity of projection in feet per second. Plot the curve for a body projected with a velocity of 16 ft. per sec., starting from $x = 0$, and find (a) how far away from the point of projection it strikes the ground again, (b) its greatest height. (*Chap.* 28)

6. The work done by an electric motor per second is given by $W = VI - I^2R$, where I is the current in amperes, and V and R are constants. Plot W against I for the case when $V = 40$, $R = 20$, starting from $I = 0$, and using quarter units for I. Find from the graph (*a*) the maximum work done per second, and the current used when this is being done, (*b*) two values of I for which the work done is zero. (*Chap.* 28)

7. The ordinates taken at equal distances on an indicator diagram (including the end ordinates) are 1·90, 1·90, 1·75, 1·50, 1·30, 1·10, 0·90, 0·75, 0·6, 0·5, 0·4 in. If the scale of pressure is 40 lb. per sq.in. to 1 in., find the average pressure on the piston.

Find the area of the diagram if its length is 3·6 in. (*Chap.* 25)

8. Draw a triangle ABC in which $AB = 6$ cm., $AC = 3$ cm., and $BC = 4{\cdot}5$ cm. Let the bisector of angle A meet BC in D. Measure BD and CD and compare the ratios AB/AC and BD/DC.

Produce BA to any point E, and bisect the angle CAE. Let this bisector meet BC produced in F. Compare the ratios AB/AC and BF/CF.

Thus verify the statement: 'The bisector of an angle of a triangle (internal or external) divides the opposite side into two parts (internally or externally) in the ratio of the lengths of the sides containing the angle bisected.' (*Chap.* 23)

9. Draw a circle on graph paper, so that its centre is at the origin of co-ordinates, and its radius is 5 in. Select any four points which lie on the circle, and measure their abscissae and ordinates as accurately as possible. Deduce that every point on the circumference obeys the relation $x^2 + y^2 = 25$. (*Chap.* 27)

10. The ports of a throttle regulator consist of four sectors of an annulus, each making an angle of 20° at the centre, and having outer and inner radii $2\frac{1}{2}$ in. and $1\frac{3}{4}$ in. Find the total area of the ports, taking $\pi = \frac{22}{7}$. (*Chap.* 24)

Answers to Examples for Chapter 28

1. $(1, -4)$ and $(1, 2)$ vertices. Axis $x = 1$. Points of intersection of axis of x $(-1, 0)$ and $(3, 0)$.

2. $(4, 4)$ and $(-4, -4)$; $(8, 2)$ and $(-8, -2)$.

3. $(\frac{1}{2}, 0)$, $(2, 0)$, i.e. $x = \frac{1}{2}$ and $x = 2$. Minimum value of $y = -1\frac{1}{8}$. Corresponding value of $x = 1\frac{1}{4}$.

4. $x = -0{\cdot}71$, $x = 0{\cdot}71$, -1, 0, $+1$.

5. (*a*) Distance travelled horizontally = 8 ft. (*b*) Greatest height = 2 ft.

6. (*a*) Maximum work = 20, corresponding current = 1. (*b*) $I = 0$ and $I = 2$.

7. 45·8 lb. per sq.in.; 4·122 sq.in.

10. $2\frac{19}{84}$ sq.in.

CHAPTER 29

VOLUMES AND SURFACE AREAS. PRISMS AND PYRAMIDS

The space occupied by a cube of edge 1 ft. is called 1 cu.ft. Similarly, that occupied by a cube of edge 1 cm. is called 1 cu.cm. (1 c.c.). It is not necessary, however, that a cubic foot of any material should be shaped as a cube, it is only necessary that it should occupy the same space as 1 ft. cube. The volume of a rectangular solid is obtained by imagining it to be divided into unit cubes. Fig. 117 illustrates the principle. The number of inch cubes in the solid is $8 \times 5 \times 6$, hence the volume is 240 cu.in. By similar reasoning it is seen that 1728 cu.in. are equal to 1 cu.ft., and $100 \times 100 \times 100$ (10^6) c.c. are equal to 1 cu.m.

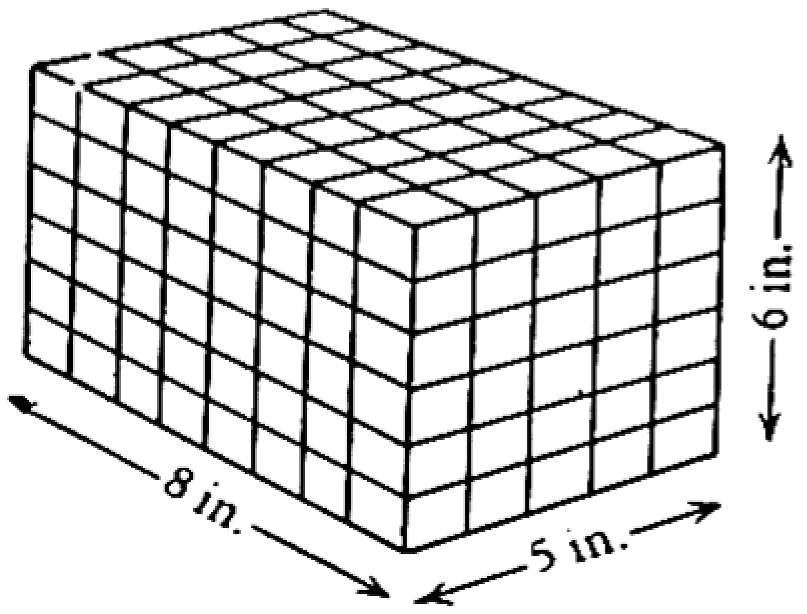
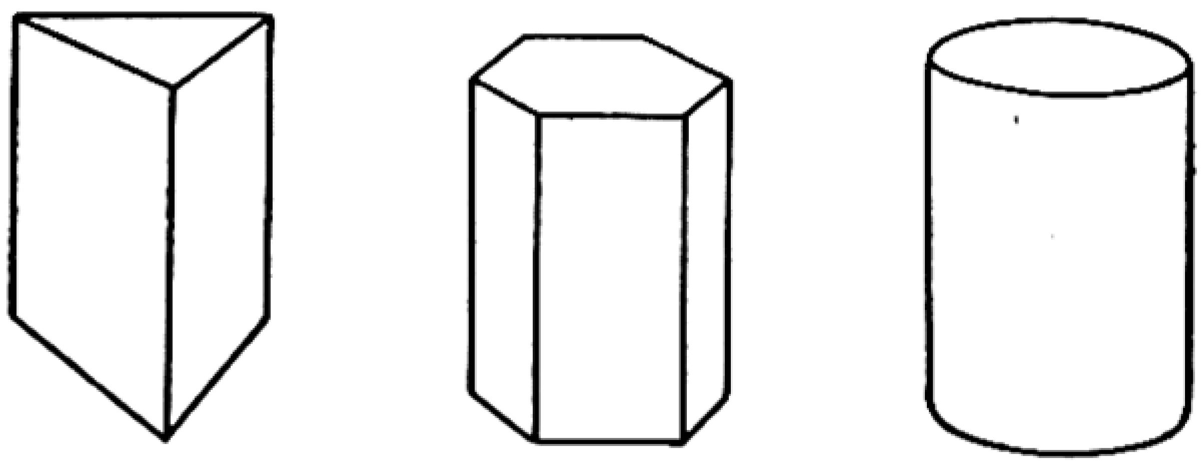


Fig. 117

Prisms

A solid with two plane ends the same shape and size joined by straight sides is called a prism. The shape of the end is used to describe the prism. Fig. 118 shows three prisms including the circular prism which is called a cylinder. If the ends are perpendicular to the sides, the prism is called a right prism, but if the word 'right' is omitted, it is to be understood that a right



Triangular prism **Hexagonal prism** **Cylinder (or circular prism)**

Fig. 118

prism is meant. We have seen from fig. 117 that the volume of a rectangular solid is given by

$$\text{Volume} = \text{length} \times \text{breadth} \times \text{thickness},$$

i.e. $$\text{volume} = \text{area of one face} \times \text{thickness}.$$

Similarly, the volume of any prism may be given by

$$\text{Volume} = \text{area of plane end} \times \text{thickness},$$

or $$\text{volume} = \text{area of plane end} \times \text{perpendicular distance between ends}.$$

In the case of the cylinder, the same formula becomes

$$\text{Volume} = \pi r^2 h$$

(where r is the radius of the plane end, and h is the perpendicular distance between the ends).

Surface area of prism. In many cases the surface area consists of the sum of the areas of the ends and the areas of a number of rectangles which are the sides. The case of the cylinder differs in that its curved surface consists of a rectangle, whose length is the circumference of the cylinder, and whose breadth is the height of the cylinder (see fig. 119).

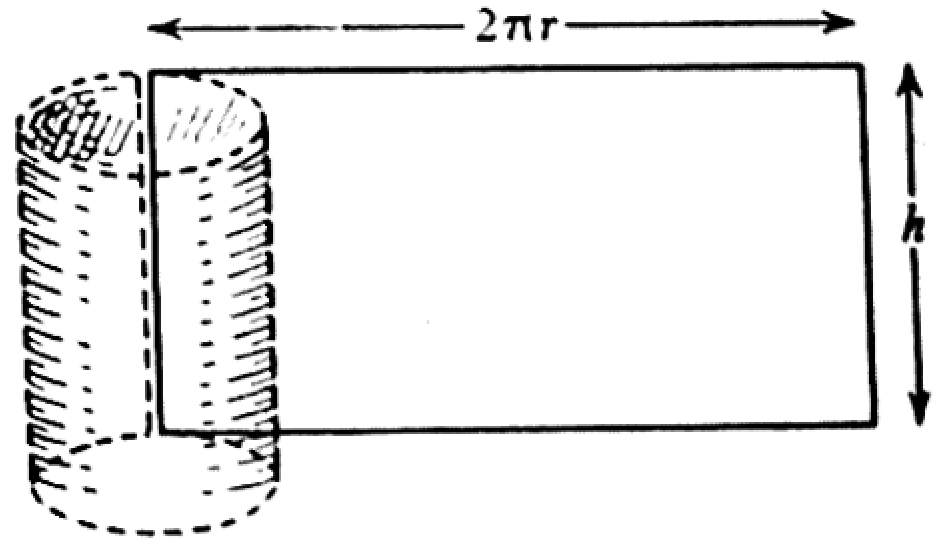


Fig. 119

Area of curved surface of a cylinder $= 2\pi rh$ (r and h as above).

Total surface of solid cylinder $= 2\pi rh + 2\pi r^2$.

The student should note that many common volumes are prisms, although they cannot be classified easily into triangular, rectangular, or simply described types. The volume of earth removed from a railway cutting which has uniform cross-section is a prism whose ends are trapezia. The volume of water flowing past a given point on the bank of a stream in a given time is the volume of a prism whose end is the area of cross-section of the stream at the point considered, and whose length is the distance travelled by the water in the given time. Again, the volume of metal in a plate is that of a prism whose end has the area of the face of the plate, and whose height is the thickness of the metal plate.

Pyramids

A solid with a plane end and straight sides meeting in a point is called a pyramid. As in the case of prisms, the shape of the plane end is used to describe the pyramid. A circular pyramid is called a cone. The axis of a pyramid is a line joining the point to the centre of the base. When the axis is perpendicular to the base, the solid is a right pyramid. Unless otherwise stated, pyramids are assumed to be of this type (see fig. 120).

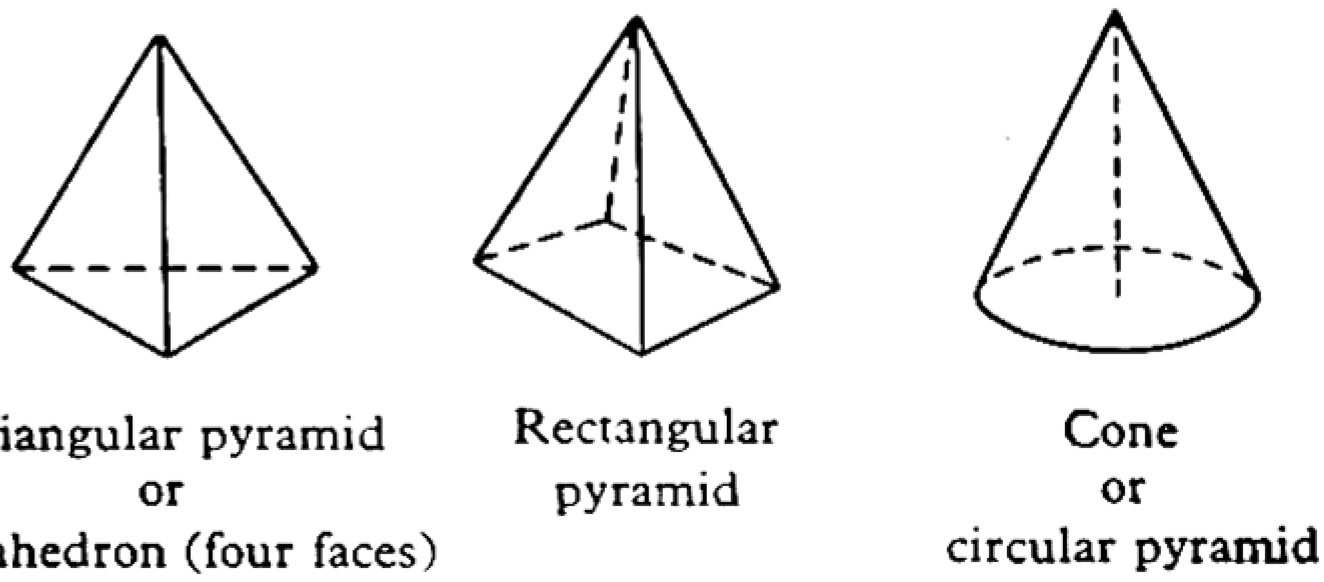


Fig. 120

It is impossible to prove the formula for the volume of a pyramid at this stage, but a particular case may be illustrated. Fig. 121 (*a*) represents a foot cube. It is then sawn into three pyramids of equal volume as shown in fig. 121 (*b*). The bases of these three pyramids have all the same area, namely, 1 sq.ft., and their heights are all equal to 1 ft. Thus each pyramid has the same base and the same height as the original prism (although they are not right pyramids), and their volumes are each one-third of the volume of the original prism.

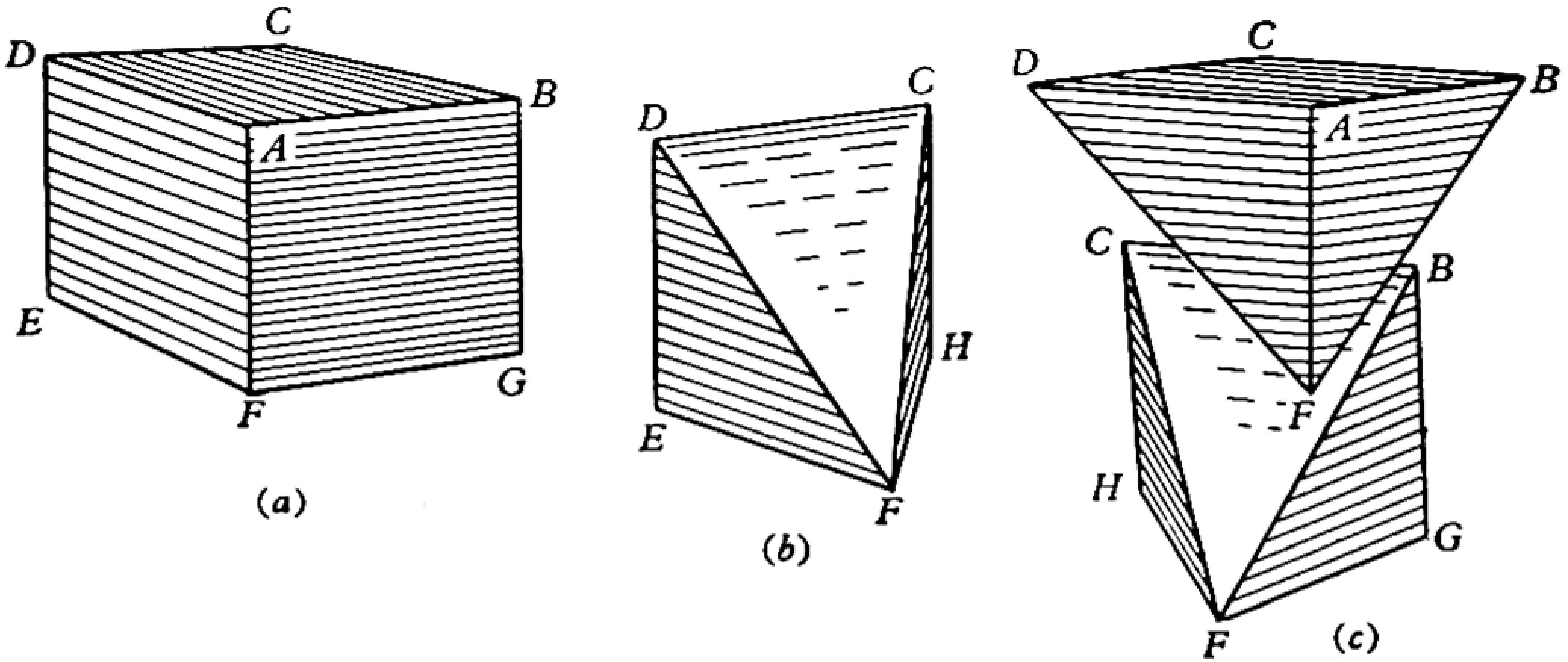


Fig 121

It is true in all cases that the volume of a pyramid is one-third of the volume of a prism which has the same base area and the same perpendicular height.

Volume of pyramid $= \frac{1}{3} \times$ area of plane base $\times$ perpendicular height.

The volume of a cone is therefore $V = \frac{1}{3}\pi r^2 h$ (where r is the radius of the plane base, and h is the height).

Area of curved surface of cone

The curved surface of a right cone is seen from fig. 122 to be a sector of a circle of radius s, where s is the slant height of the cone. The arc of the sector is of length $2\pi r$ (i.e. the circumference of the base). Now the area of the complete circle of which the sector is a part is πs^2, so the area of the sector is πs^2 multiplied by the fraction $\dfrac{2\pi r}{2\pi s}$.

$$\text{Area of curved surface} = \pi s^2 \times \frac{2\pi r}{2\pi s} = \pi r s$$

$$= \pi \times \text{radius of base} \times \text{slant height.}$$

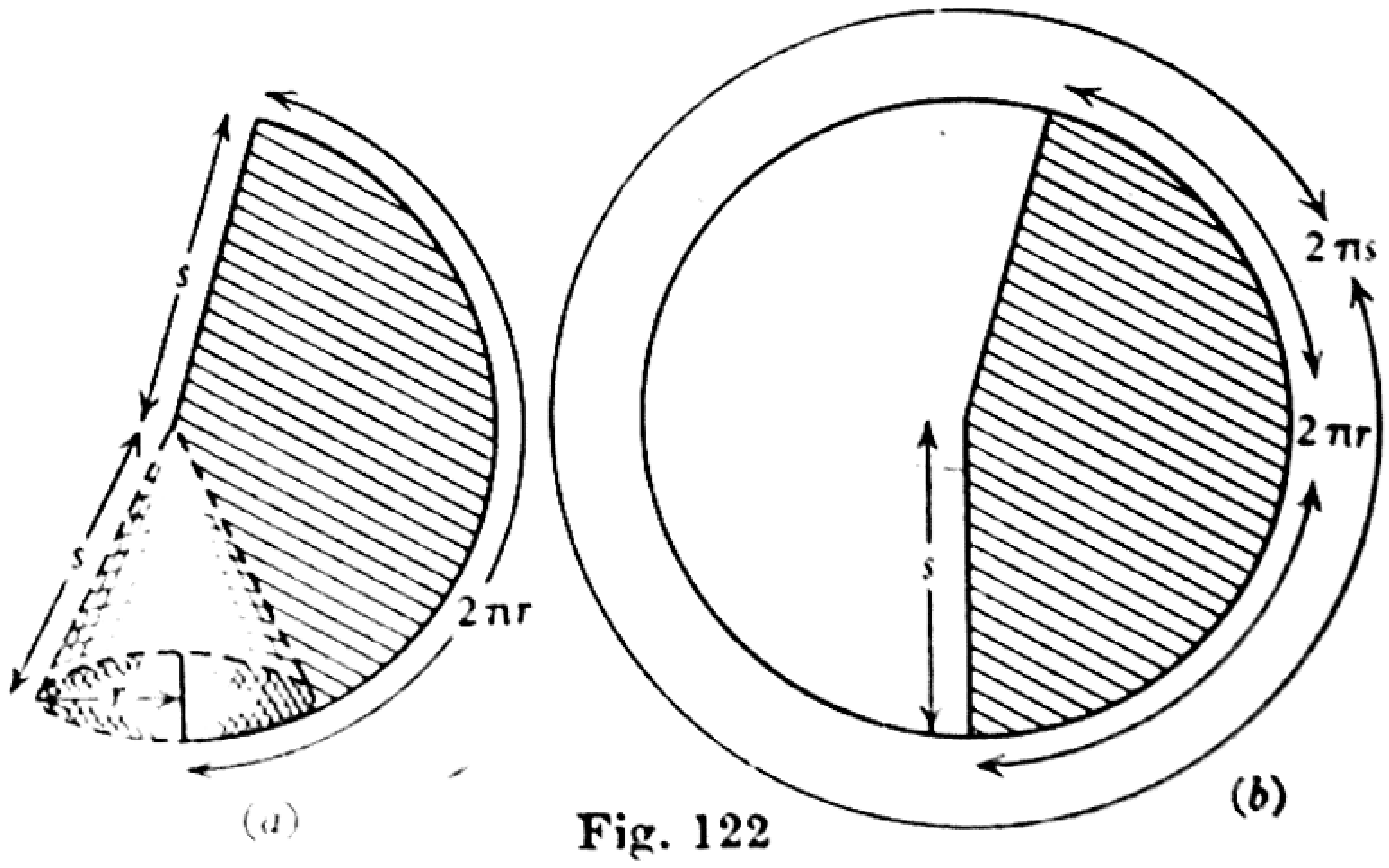


Fig. 122

Example. Find the volume of 20 ft. of girder whose section is shown in fig. 123.

The solid is a prism of length 20 ft., and area of base as shown

Areas of three rectangles which comprise the base are

$$7 \times 1 \text{ sq.in.} + 7 \times 1 \text{ sq.in.} + 5 \times 1 \text{ sq.in.}$$

$$= 19 \text{ sq.in.}$$

$$= \tfrac{19}{1728} \text{ sq.ft.}$$

$$\text{Volume of 20 ft. of girder} = \tfrac{19}{1728} \times 20 \text{ cu.ft.}$$
$$= \tfrac{380}{1728} \text{ cu.ft.}$$
$$= 0{\cdot}22 \text{ cu.ft. (approx.).}$$

Example. A stone post consists of a cylinder surmounted by a cone. The total height is 4 ft., while the height of the cylindrical portion is 3 ft. The radius of the base of the cone, and of the cylinder, is 5 in. Sketch the vertical section of the conical part to scale and so determine its slant height. Hence calculate the total area of curved surface, and the volume of the post.

Fig. 124 shows the slant height to be 13 in.

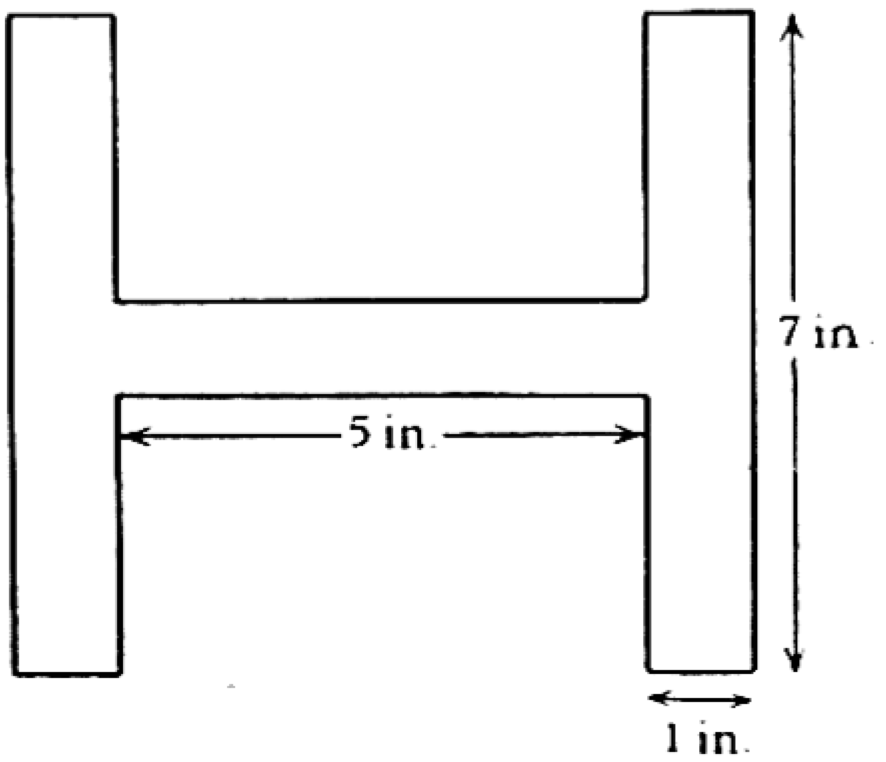


Fig. 123

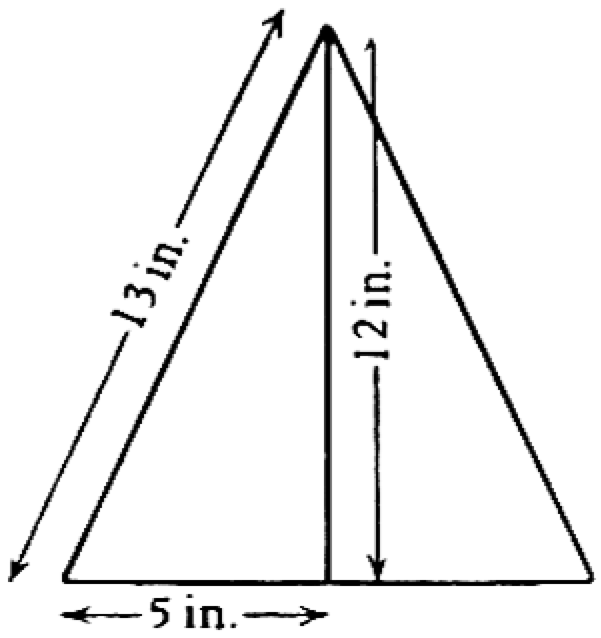


Fig. 124

$$\text{Area of curved surface of cylinder} = 2\pi \times \text{radius} \times \text{height}$$
$$= 2\pi \times 5 \times 36 \text{ sq.in.}$$

$$\text{Area of curved surface of cone} = \pi \times 5 \times 13 \text{ sq.in.}$$

$$\text{Total area} = 360\pi + 65\pi \text{ sq.in.}$$
$$= 425\pi \text{ sq.in.}$$
$$= \frac{425\pi}{144} \text{ sq.ft.}$$
$$= \underline{9{\cdot}272 \text{ sq.ft.}}$$

$$\text{Volume of cylinder} = \pi \times r^2 \times h$$
$$= \pi \times 25 \times 36 \text{ cu.in.}$$

$$\text{Volume of cone} = \frac{\pi \times r^2 \times 12}{3}$$
$$= \frac{\pi \times 25 \times 12}{3}$$
$$= 100\pi \text{ cu.in.}$$

Numbers	Logs
425	2·6284
3·142	0·4972
	3·1256
144	2·1584
9·272	0·9672
3142	3·4972
1728	3·2375
1·819	0·2597

$$\begin{aligned}\text{Total volume} &= 900\pi + 100\pi \text{ cu.in.}\\ &= 1000\pi \text{ cu.in.}\\ &= 3142 \text{ cu.in.}\\ &= \tfrac{3142}{1728} \text{ cu.ft.}\\ &= \underline{1{\cdot}819 \text{ cu.ft.}}\end{aligned}$$

Volumes by the mid-ordinate rule

The interior of a sunken cement water tank has **a** shape something like fig. 125, the inside walls being irregular. The areas of cross-section are measured for a number of equidistant levels $A, B, C, D, \ldots, J$. It is required to find the volume of the tank approximately.

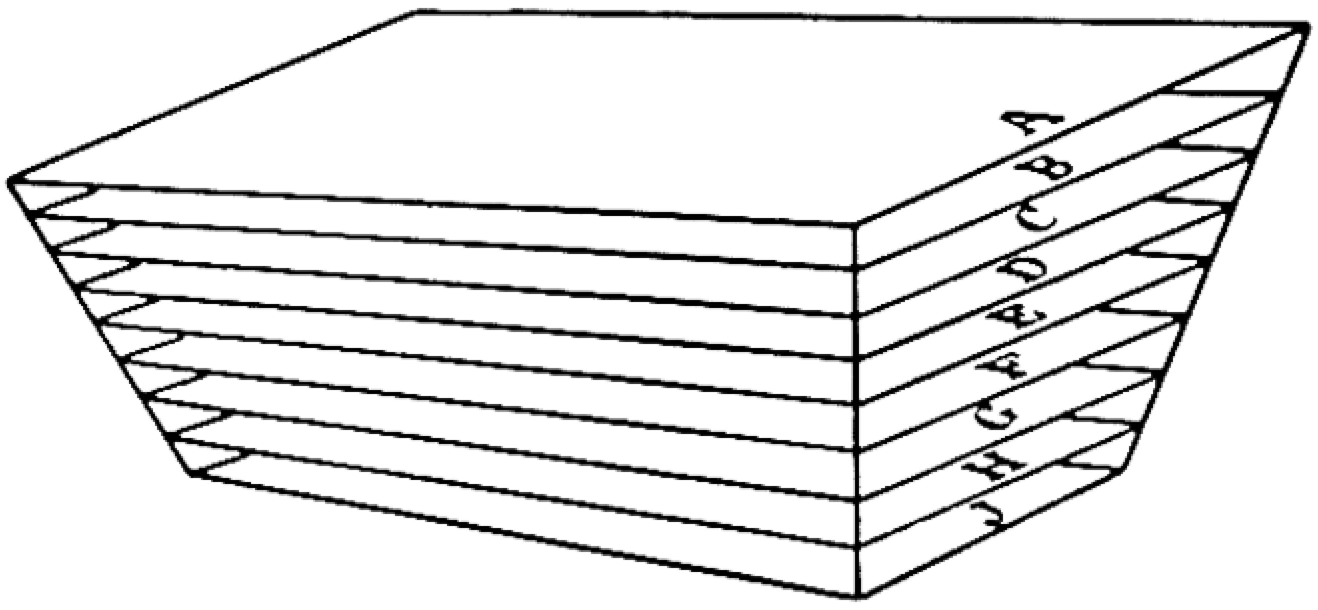


Fig. 125

Now the volume of water between levels J and H is approximately equal to that of a prism whose base is the area half-way between J and H, and whose height is the vertical distance between J and H. If we call the area half-way between J and H the mid-area of J and H, and proceed to find the mid-areas between all the pairs of sections given, we may add the volumes of the small prisms together to obtain the approximate volume of the tank.

The mid-ordinate rule may therefore be used to determine volumes when the ordinates represent areas.

Volume = average of mid-ordinates (area) × distance between end ordinates.

For this problem the areas are as follows:

$A = 135$ sq.ft., $B = 122$ sq.ft., $C = 109$ sq.ft.,
$D = 98$ sq.ft., $E = 84$ sq.ft., $F = 70$ sq.ft.,
$G = 55$ sq.ft., $H = 44$ sq.ft., $J = 30$ sq.ft.

The total depth is 12 ft.

The mid-ordinates are thus: $128\frac{1}{2}$, $115\frac{1}{2}$, $103\frac{1}{2}$, 91, 77, $62\frac{1}{2}$, $49\frac{1}{2}$, 37 sq.ft.

Total of mid-ordinates $= 664\frac{1}{2}$ sq.ft.

Average of mid-ordinates $= 664\frac{1}{2} \div 8 = 83\frac{1}{16}$ sq.ft.

Volume = average of mid-ordinates × depth

$= 83\frac{1}{16} \times 12$ cu.ft.

$= 996\frac{3}{4}$ cu.ft.

Examples for Chapter 29

1. A rivet consists of a cylindrical head, of diameter $\frac{1}{4}$ in., and depth $\frac{1}{8}$ in., and a bolt $\frac{1}{8}$ in. diameter and length $\frac{1}{2}$ in. Find the volume of metal in 1 gross of such rivets.

Compare the total volume with that of a cylindrical rod $\frac{1}{4}$ in. radius and 9 in. long. (*Chap.* 29)

2. (*a*) An inkwell consists of a cylinder $1\frac{1}{4}$ in. radius and 4 in. high, with a conical depression in the top, of the same radius as the cylinder, and $2\frac{1}{2}$ in. deep. Find the air space when ink is poured in to the level of the vertex of the cone.

(*b*) Express the volume of ink when filled to this level as a ratio to the volume of the air space. (*Chap.* 29)

3. Find the volume of earth removed in making a railway cutting 50 yd. long, width at the bottom 12 ft., and width at the top 36 ft., its depth being 10 ft. In another part of the same line a rock cutting 50 yd. long is made with the same base width, but the top width is only 26 ft. wide, the depth being the same as before. How much less earth is excavated in this case? (*Chap.* 29)

4. A hydrometer made of tin-plate is in the shape of a cylinder with conical ends. The depth of the cylinder is 6 cm., while the lengths of the conical ends are 3 cm. The diameter is $2\frac{1}{2}$ cm. Find the volume of the hydrometer. Draw the elevation of one cone to scale and measure its slant height. Hence find the total surface area of the instrument. (Take $\pi = \frac{22}{7}$.) (*Chap.* 29)

5. The hold of a ship is 10 ft. deep. Areas of cross-section at equal depths 1 ft. apart are 240, 240, 238, 236, 228, 220, 212, 202, 190, 176 and 160 sq.ft. Find the volume of the hold approximately. *(Chap.* 29)

6. A pond is 10 ft. deep in the deepest part. Areas of water surface were measured for every foot depth as follows: 125, 195, 260, 290, 325, 340, 360, 370, 380, 390, 400 sq.yd. Find the volume of water in the pond when full.

During a drought the water level fell 3 ft. Find the volume of water evaporated. *(Chap.* 29)

7. Plot (1) $y = x^2$, (2) $y = -x^2$ from $x = -3$ to $x = +3$ using half-units for x, on the same axes.

Also on the same axes and with the same scales plot (3) $y^2 = x$ by taking values of y from -6 to $+6$.

Join the points of intersection of (1) and (3) and of (2) and (3) and write the equations of the two straight lines drawn. *(Chaps.* 28 *and* 27)

8. Find the area of the face of the plate shown in fig. 126. The centre of the arc radius 2⅛ in. is at the corner of the 3 in. square, and the smaller arcs are semicircles. *(Chap.* 24)

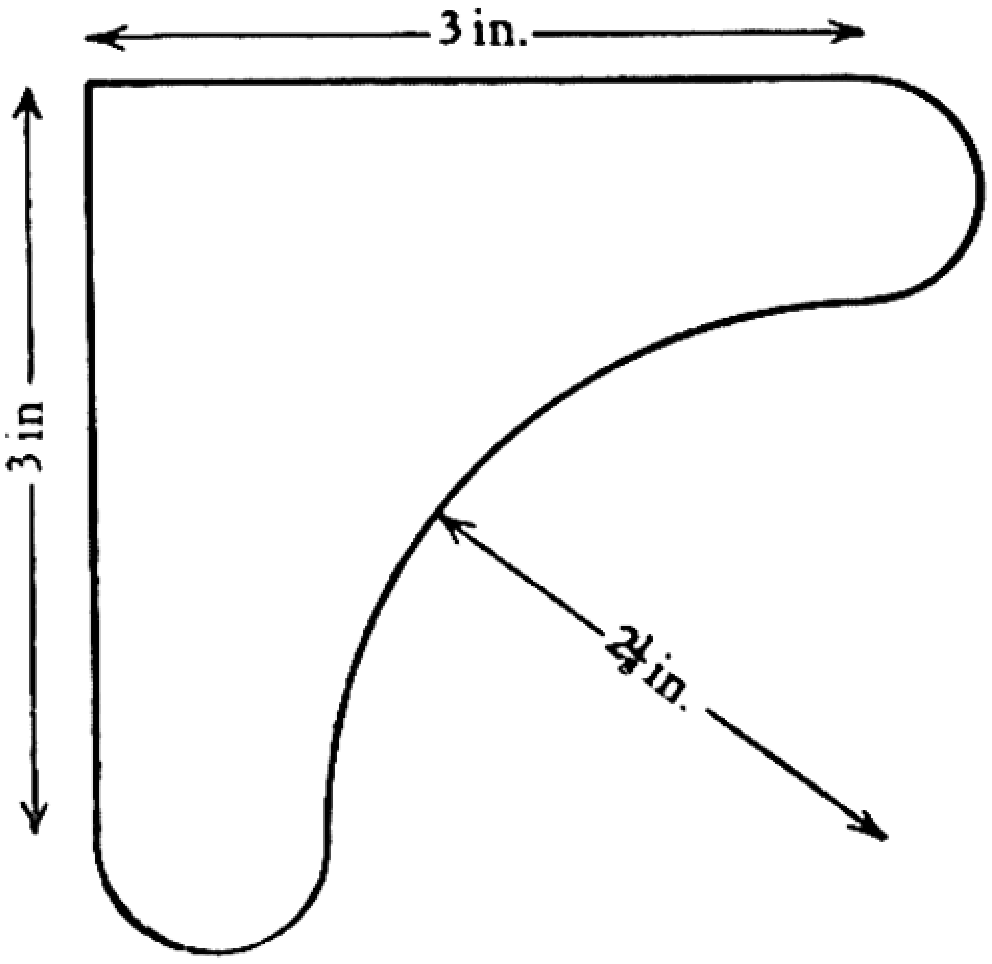


Fig. 126

9. A metal rod has a triangular section. The triangle is isosceles, with two angles 75° and the shortest side 1⅛ in. It is required to find the diameter of the smallest cylinder into which the rod will fit. Draw the triangle to scale, and draw the

perpendicular bisectors of two of the sides of the triangle. Their point of intersection gives the centre of the circle required. Measure the diameter of the circle. Show that a rod whose section is a regular hexagon of side $1\frac{1}{2}$ in. would fit inside the same cylinder.
(*Chap.* 23)

10. (*a*) Find the equation of the straight line which passes through the points $(1, -2)$ and $(-1, 2)$.

(*b*) The equation connecting the effort (E) and load (W) for a machine is of the form $E = aW + b$, where a and b are constants. When $W = 10$, $E = 7{\cdot}5$, and when $W = 20$, $E = 12{\cdot}5$. Plot these points on a graph, with E vertically and W horizontally, and hence determine the equation of the machine.
(*Chap.* 27)

Answers to Examples for Chapter 29

1. Volume $= 1\frac{43}{56}$ cu.in. in both cases.

2. (*a*) 8·183 cu.in. (*b*) $\frac{9}{10}$.

3. $1333\frac{1}{3}$ cu.yd.; $227\frac{7}{9}$ cu.yd.

4. $72\frac{19}{28}$ sq.cm.

5. 2142 cu.ft.

6. 1057·5 cu.yd.; 385 cu.yd.

7. $y = x$; $y = -x$.

8. 6·055 sq.in.

10. (*a*) $y = -2x$. (*b*) $E = \frac{1}{2}W + 2\frac{1}{2}$.

CHAPTER 30

THEOREM OF PYTHAGORAS. APPLICATIONS

In order to find the slant height of a cone when given the vertical height and the radius of the base, it has been necessary, in the previous chapter, to make a scale drawing of a right-angled triangle and to measure the longest side. The purpose of this chapter is to show the truth of a theorem which will enable us to calculate the longest side without drawing, and to apply the theorem to other problems in geometry and mensuration.

The longest side of a right-angled triangle is the side opposite the right angle, and is called the hypotenuse.

The theorem of Pythagoras

This theorem states that the square on the hypotenuse is equal to the sum of the squares on the other two sides.

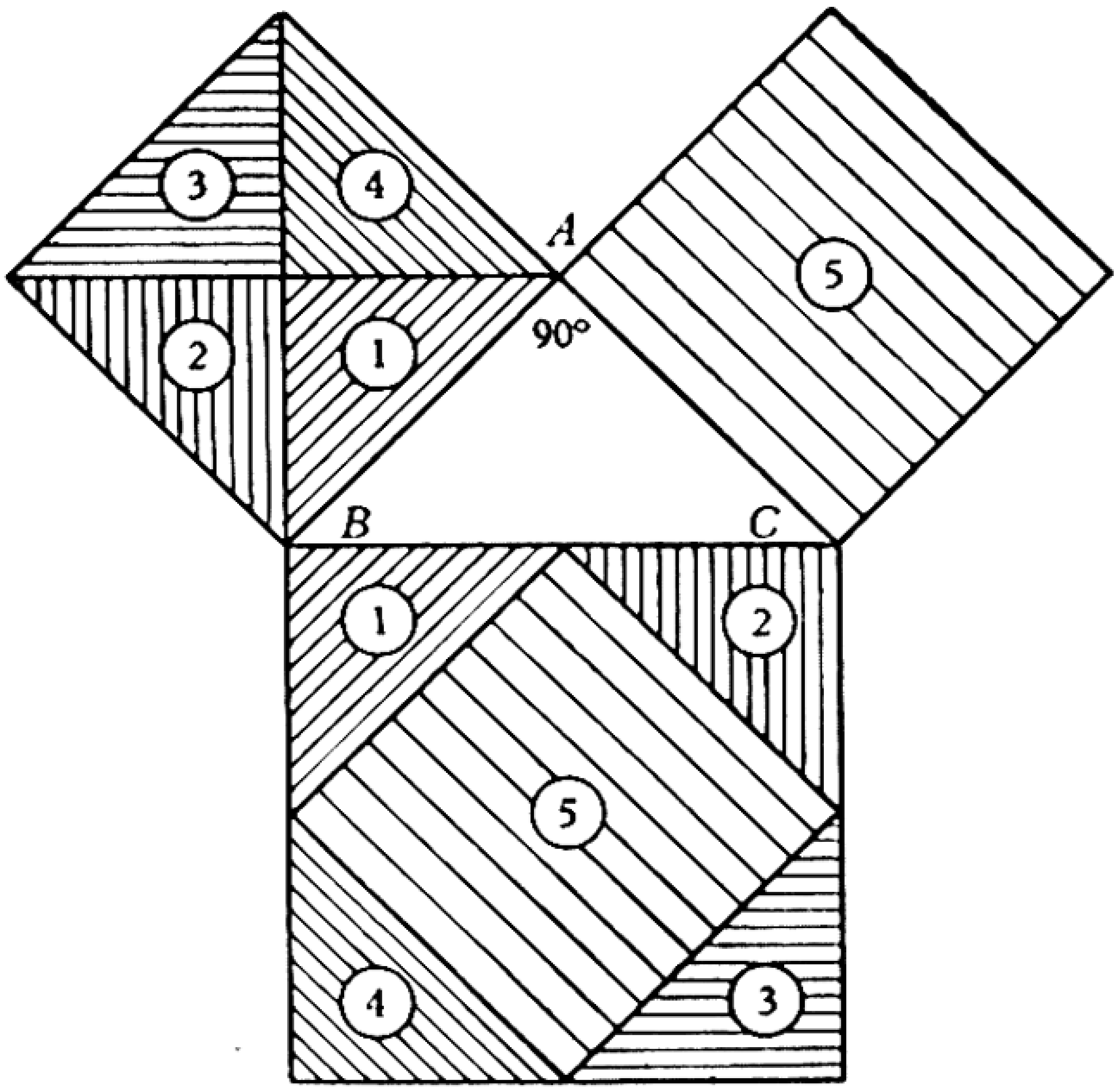


Fig. 127

Fig. 127 shows the simplest case. *ABC* is a right-angled triangle which is also isosceles. Squares are drawn on *AB*, *AC* and *BC*, the last being the hypotenuse. Now figures 1, 2, 3, 4 and 5 will fit together exactly to make the square on *BC*, showing that the sum of the squares on *AB* and *AC* is equal to the square on *BC*.

Fig. 128 shows another right-angled triangle *ABC*. Again, figures 1, 2, 3, 4 and 5 fit exactly to make the square on the hypotenuse. Whatever lengths the sides may have, provided the triangle is right-angled, a similar figure to this one may be drawn, proving diagrammatically that the theorem is true. The converse of the theorem is also true, that is, if it is found that the sum of the squares on two sides of a triangle is equal to the square on the third side, it follows that the triangle is right-angled.

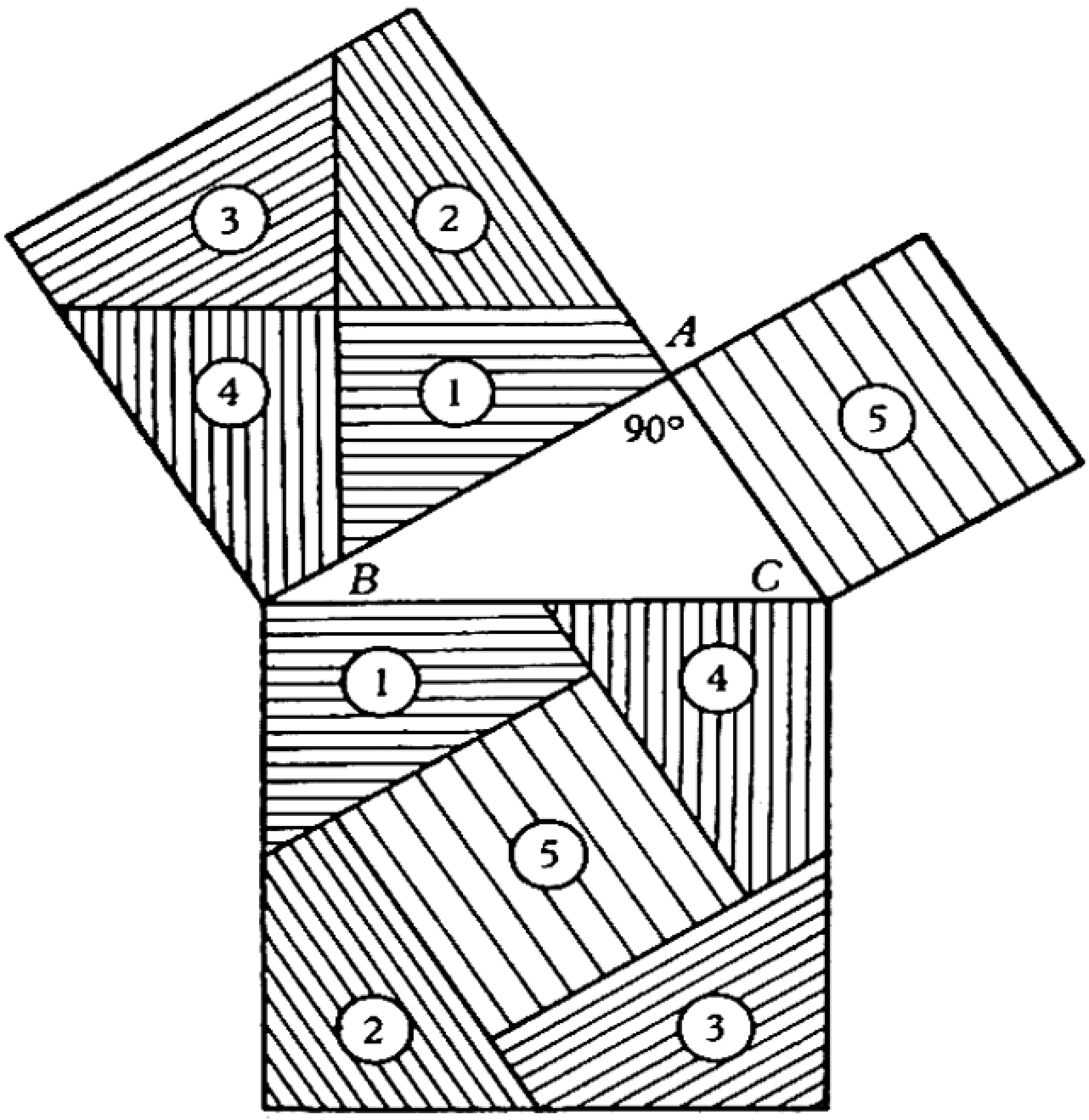


Fig. 128

A triangle whose sides are in the ratio 3:4:5 is right-angled because

$$3^2+4^2 = 9+16 = 25 = 5^2.$$

Similarly, a triangle whose sides are in the ratio 5:12:13 is right-angled because

$$5^2+12^2 = 25+144 = 169 = 13^2,$$

and also for the ratio 7:24:25, for

$$7^2+24^2 = 49+576 = 625 = 25^2.$$

These are the simplest cases in which the three sides of a right-angled triangle may be expressed as rational quantities. Frequently it is found that one side is irrational (see Chapter 4).

Consider a right-angled isosceles triangle as in fig. 127. The sides AB and AC are equal in length. Let these be of length a. Then

$$AB^2+AC^2 = a^2+a^2 = 2a^2,$$

so the square on BC has an area of $2a^2$ and $BC = \sqrt{(2a^2)} = a\sqrt{2}$. The sides are in the ratio $a:a:a\sqrt{2}$, i.e. $1:1:\sqrt{2}$. If $AB=AC=6$ in., then $BC = 6\sqrt{2}$ in.

Again, consider the triangle whose angles are 30°, 60° and 90°. Fig. 129 shows two such triangles placed side by side so as to form an equilateral triangle. As the sides are equal in length, choose that CA shall be a units in length, then BC must be $2a$ units.

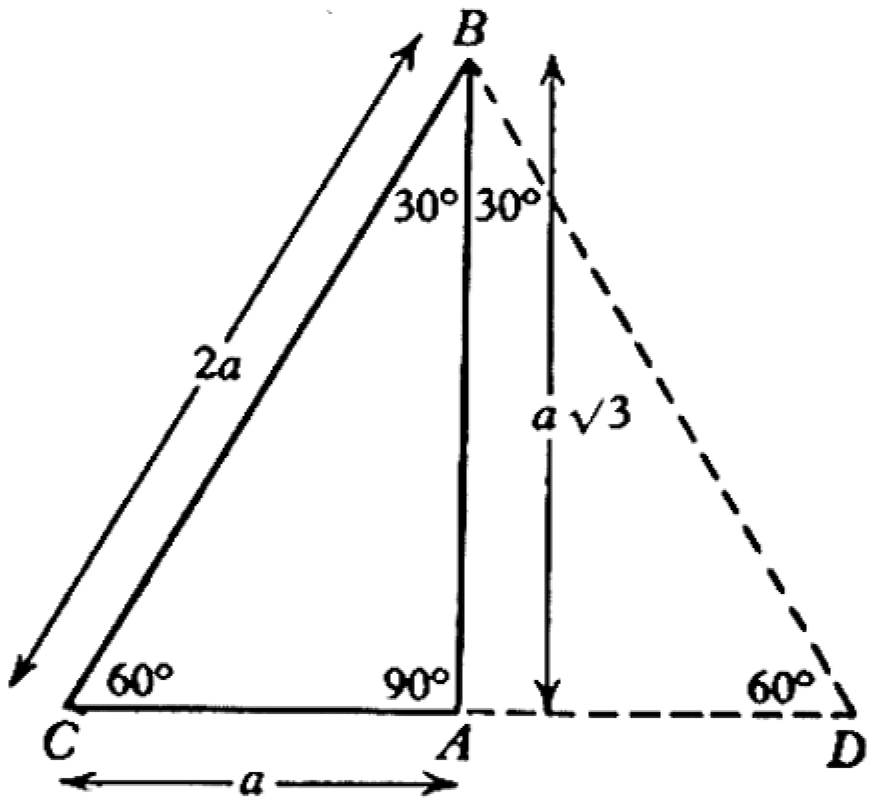


Fig. 129

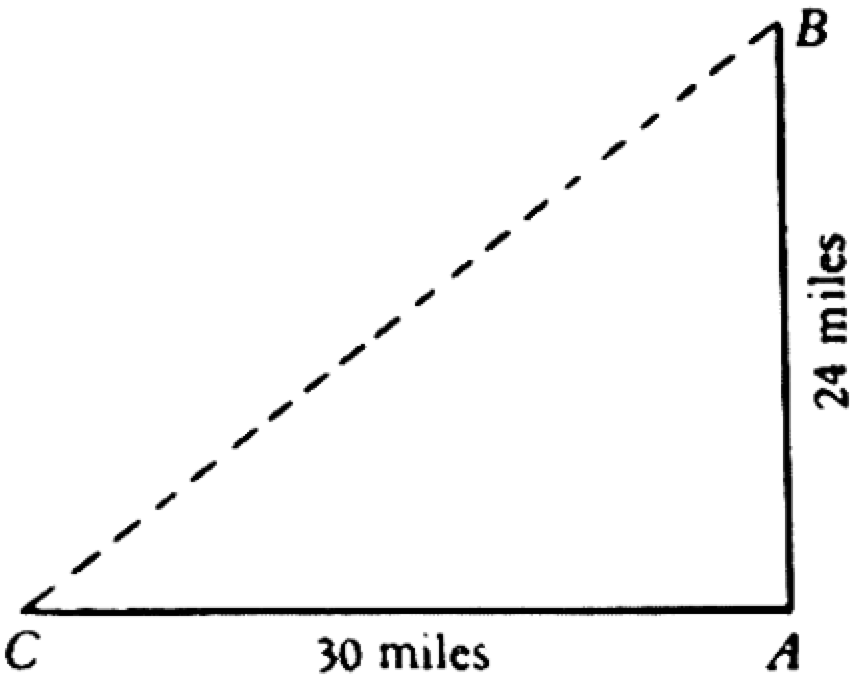


Fig. 130

Now $BC^2 = AC^2+AB^2$, i.e.

$$(2a)^2 = a^2+AB^2,$$

i.e.
$$4a^2 = a^2+AB^2,$$

whence
$$AB^2 = 3a^2.$$

$$\therefore\ AB = \sqrt{(3a^2)} = a\sqrt{3}.$$

Hence a 60°, 30°, 90° triangle has sides in the ratio a; $a\sqrt{3}$; $2a$ or $1:\sqrt{3}:2$.

Applications

Example. Two ships leave port together. One travels due north at 12 m.p.h. and the other travels due west at 15 m.p.h. Find their distance apart after 2 hr.

The distances travelled in 2 hr. are 24 miles and 30 miles respectively, and their distance apart is the line BC (fig. 130).

Now $BC^2 = AC^2 + AB^2$ (by the theorem of Pythagoras)

$= 30^2 + 24^2$

$= 900 + 576$

$= 1476.$

Numbers	Logs
1476	3·1690
38·41	1·5845

$\therefore\ BC = \sqrt{1476} = 38{\cdot}41$ miles.

Distance apart after 2 hr. is 38·41 miles.

Example. Find the distance between two opposite corners of a rectangular solid 8 in. by 6 in. by 4 in.

Fig. 131 shows the solid; the length required is DG. [Note that angle $GHE = 90°$, and angle $DEG = 90°$.]

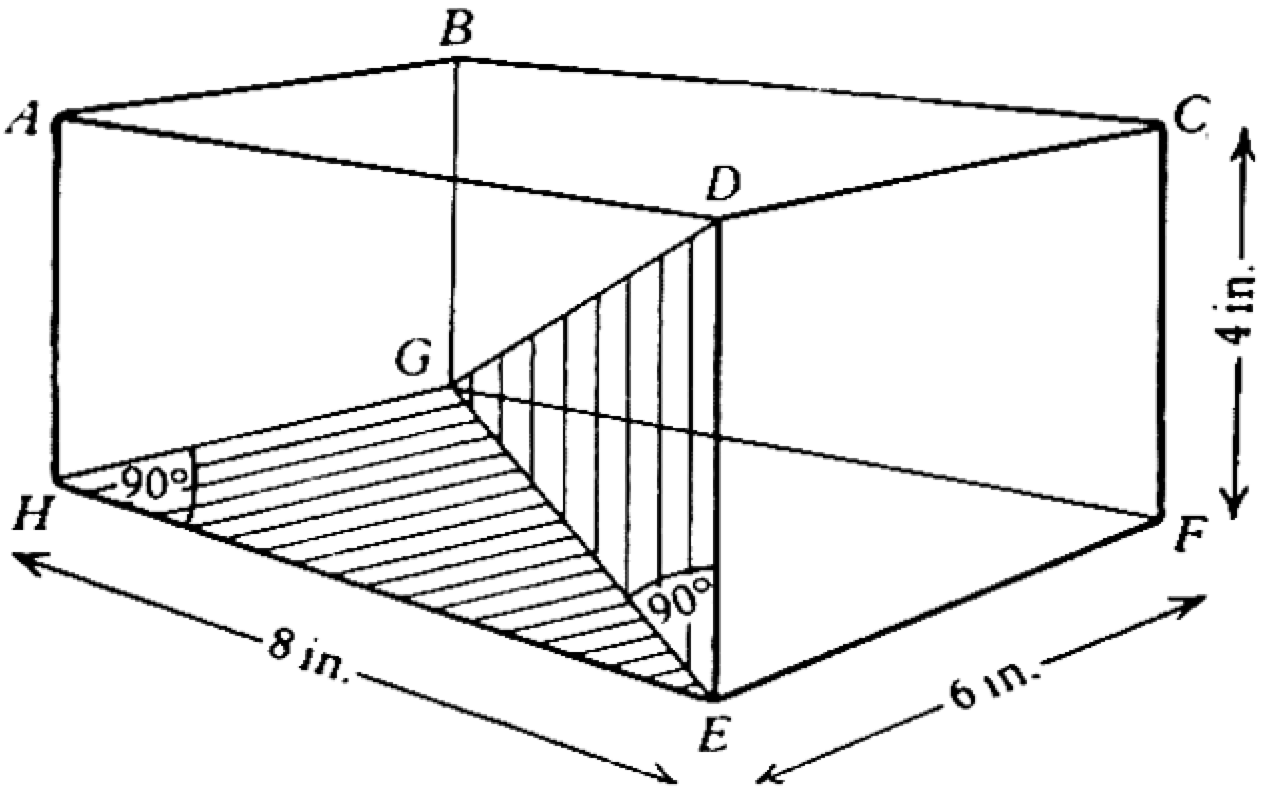


Fig. 131

The line GE must be found first.

To find GE.

In the right-angled triangle EHG, $GE^2 = HE^2 + HG^2$

$= 8^2 + 6^2$

$= 64 + 36$

$= 100.$

$\therefore\ GE = 10$ in.

To find GD.

In the right-angled triangle GED, $DG^2 = DE^2 + GE^2$.

$DG^2 = 4^2 + 10^2$

$= 116.$

$\therefore\ DG = \sqrt{116}$

$= 10{\cdot}78$ in.

Numbers	Logs
116	2·0645
10·78	1·0328

The length of the diagonal required is 10·78 in.

Example. Find the area of a hexagon of side 1·4 in. [The simplest way to draw a regular hexagon is to describe a circle whose radius is equal to the side of the hexagon, then mark off distances on the circumference representing chords of length equal to the radius, as in fig. 132.]

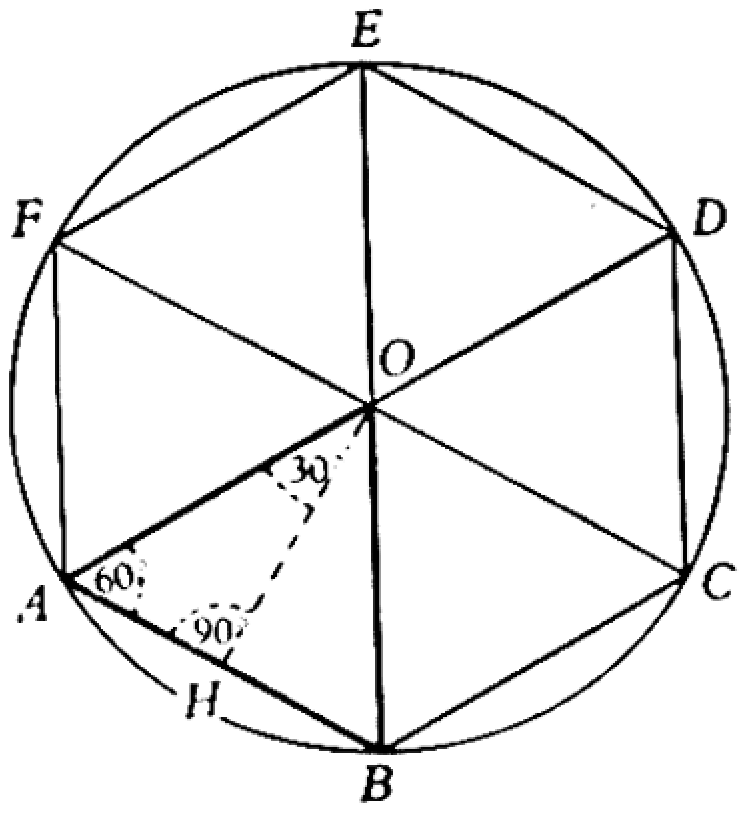


Fig. 132

Since a hexagon has an interior angle of 120° it will divide into six equilateral triangles, each of which contains one-sixth of the area of the hexagon.

Consider the triangle ABO.

$$\text{Area of triangle } ABO = \tfrac{1}{2}\text{ base} \times \text{altitude}$$
$$= \tfrac{1}{2}AB \times OH$$
$$= \tfrac{1}{2} \times 1{\cdot}4 \times 0{\cdot}7\sqrt{3}$$

(Note that AOH has sides in the ratio $1:\sqrt{3}:2$)

$$= 0{\cdot}7 \times 0{\cdot}7\sqrt{3}.$$

Thus, area of hexagon

$$= 6 \times 0{\cdot}7 \times 0{\cdot}7\sqrt{3} \qquad \text{(A)}$$
$$= 6 \times 0{\cdot}7 \times 0{\cdot}7 \times 1{\cdot}732$$
$$= \underline{5{\cdot}093 \text{ sq.in.}}$$

Numbers	Logs
6	0·7782
0·7	$\bar{1}$·8451
0·7	$\bar{1}$·8451
1·732	0·2385
5·093	0·7069

To obtain a formula for the area of a hexagon of given side

Let a be the length of the given side, then from equation (A) above, we may write $\frac{1}{2}a$ for 0·7, because 0·7 represented half the length of a side of the hexagon.

$$\therefore \text{ Area of hexagon} = 6 \times \tfrac{1}{2}a \times \tfrac{1}{2}a \times 1{\cdot}732$$
$$= \frac{3a^2}{2} \times 1{\cdot}732$$
$$= 2{\cdot}598a^2$$
$$\doteqdot 2{\cdot}6a^2.$$

Thus, area of a hexagon is equal to 2·6 × the square of one side (approx.).

Example. Find the perimeter of a trapezium whose parallel sides are 10 in. and 7 in. long, the perpendicular distance between these sides is 8 in., and the non-parallel sides are equal in length.

Let $ABCD$ (fig. 133) be the trapezium.

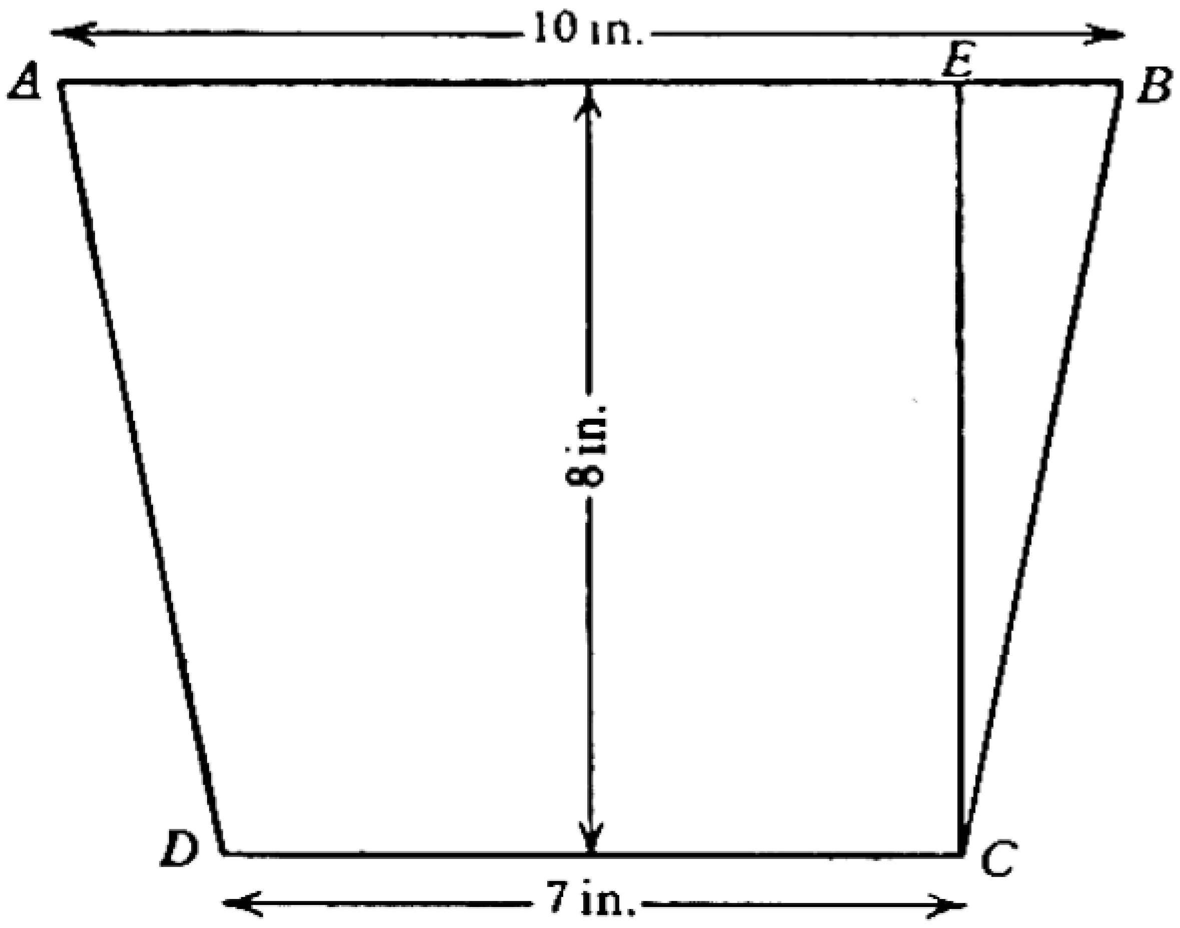


Fig. 133

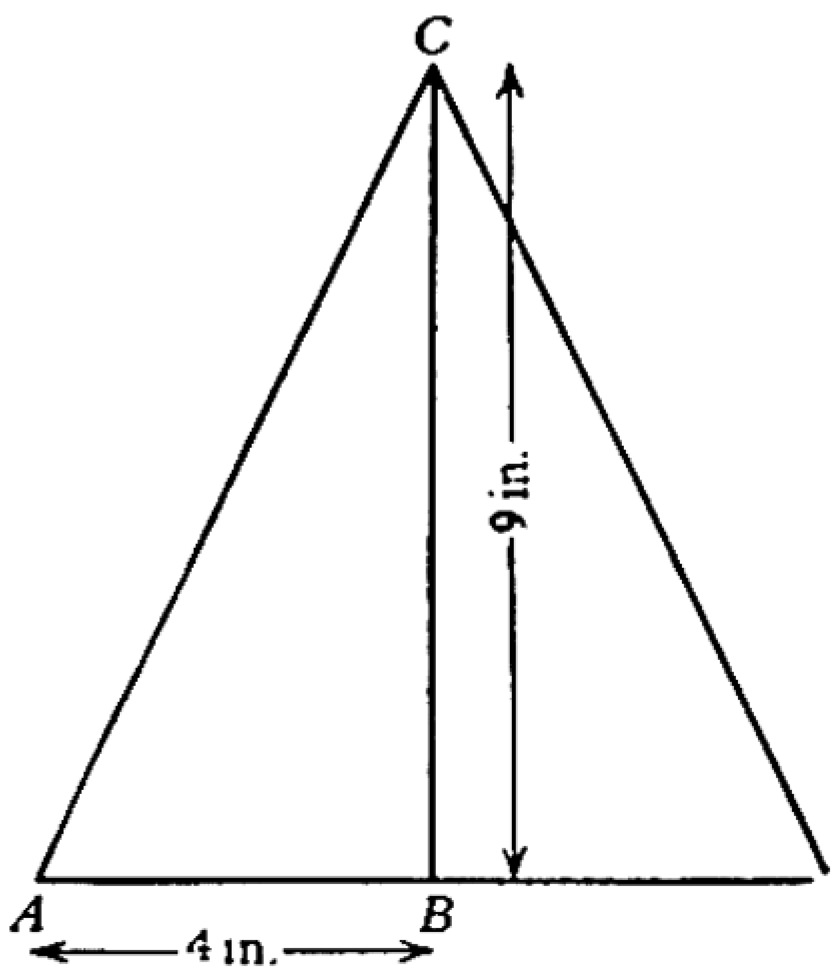


Fig. 134

To find the length of CB (or AD), draw EC perpendicular to AB.

$$\begin{aligned} \text{Now } CB^2 &= EB^2 + EC^2 \\ &= (1\tfrac{1}{2})^2 + 8^2 \\ &= 2\tfrac{1}{4} + 64 \\ &= 66{\cdot}25. \end{aligned}$$

Numbers	Logs
66·25	1·8212
8·139	0·9106

$$\therefore\ CB = \sqrt{(66{\cdot}25)} = 8{\cdot}139.$$

$$\text{Hence, perimeter} = 10 + 7 + 8{\cdot}139 + 8{\cdot}139 \text{ in.}$$

$$= \underline{33{\cdot}278 \text{ in.}}$$

Example. Find the area of the curved surface of a cone whose vertical height is 9 in., and the radius of whose base is 5 in.

The slant height is AC (fig. 134).

$$\begin{aligned} \text{Now } AC^2 &= AB^2 + BC^2 \\ &= 16 + 81 \\ &= 97. \end{aligned}$$

$$\therefore\ AC = \sqrt{97} = 9{\cdot}849 \text{ in.}$$

Numbers	Logs
3·142 4 9·849	0·4972 0·6021 0·9934
123·8	2·0927

$$\begin{aligned} \text{Area of curved surface} &= \pi r s \\ &= \pi \times 4 \times 9{\cdot}849 \\ &= \underline{123{\cdot}8 \text{ sq.in.}} \end{aligned}$$

Examples for Chapter 30

1. A number of triangles ABC are drawn, in all of which the angle BAC is 90°. Find the length of the third side in each case.

(*a*) $AC = 6,\ AB = 8.$ (*b*) $AC = 10,\ BC = 26.$

(*c*) $AC = 3\frac{1}{2},\ BC = 12\frac{1}{2}.$ (*d*) $AB = 8,\ AC = 4.$

(*e*) $AC = 2,\ BC = \sqrt{(13)}.$ (*f*) $AC = 5,\ BC = \sqrt{(74)}.$

(*Chap.* 30)

2. (*a*) Calculate the area of an isosceles triangle whose sides are 2·4 in., 2·4 in. and 2·6 in.

(*b*) Find the area of a plate in the shape of a rhombus of side 2 in., containing two angles of 60°. (*Chap.* 30)

3. A buoy is formed of two cones with their equal plane faces in contact. Their heights are 3 ft. and 4 ft. 6 in. The maximum diameter is 3 ft. Find (*a*) the total surface of the buoy, (*b*) its volume. (*Chaps.* 30 *and* 29)

4. Find (*a*) a diagonal of the floor, (*b*) the distance between two opposite corners of a room 15 ft. long, 12 ft. wide and 11 ft. high. (*Chap.* 30)

5. In order to calculate the distance between two points expressed in Cartesian co-ordinates, the difference in their ordinates is squared, the difference in their abscissae is squared, and the square root of the sum of these two squares gives the required distance. Find the distances between the following pairs of points:

(*a*) $(0, 0)$ and $(3, 3)$. (*b*) $(-3, -3)$ and $(2, 9)$.

(*c*) $(-1, 5)$ and $(2, -1)$. (*d*) $(-6, 5)$ and $(-3, -3)$.

(*Chap.* 30)

6. A railway cutting is 28 ft. wide at rail level, and 52 ft. wide at the top, and its vertical depth is 15 ft. Find the area of one side of the cutting for a length of 70 yd. Find also the volume of earth removed in this length. (*Chaps.* 30 *and* 29)

7. The tension (T lb. wt.) in a revolving rod was measured for different speeds of rotation (v ft. per sec.):

v (ft. per sec.)	10	20	30	40	50
T (lb. wt.)	4	16	36	64	100

Plot T against V, and from the graph determine

(*a*) the tension when the speed is 25 ft. per sec.,

(*b*) the speed when the tension is 70 lb. wt. (*Chap.* 26)

8. A drill for plaster consists of a cylinder 6·3 in. long, diameter 0·6 in., surmounted by a cone of the same diameter and length 0·6 in. Find the volume of metal in 1 gross of these drills. (*Chap.* 29)

9. Plot on the same axes, (*a*) $y = x^2 - x - 6$, (*b*) $y = -4$, (*c*) $y = 3x - 9$, from $x = -4$ to $x = +4$.

Write the co-ordinates of the points of intersection of (*a*) and (*b*), and of (*a*) and (*c*). (*Chap.* 28)

10. ABC is an isosceles triangle in which $AB = AC$. D, E and F are the mid-points of AB, AC and BC respectively. Prove that triangles ADE and ABC are similar, and that the diagram contains four congruent triangles. (*Chap.* 23)

Answers to Examples for Chapter 30

1. (*a*) $BC = 10$. (*b*) $AB = 24$. (*c*) $AB = 12$.
   (*d*) $BC = 8{\cdot}944$. (*e*) $AB = 3$. (*f*) $AB = 7$.

2. (*a*) 2·622 sq.in. (*b*) 3·464 sq.in.

3. (*a*) 38·16 sq.ft. (*b*) 17·69 cu.ft.

4. (*a*) 19·21 ft. (*b*) 22·14 ft.

5. (*a*) 4·242. (*b*) 13. (*c*) 6·708. (*d*) 8·544.

6. 4034 sq.ft.; 126,000 cu.ft.

7. (*a*) 25 lb. wt. (*b*) 41·8 ft. per sec.

8. 2646 cu.in.

9. (2, −4) and (−1, −4); (3, 0) and (1, −6).

Revision Examples on Chapters 21–30

1. In order to find the height of a pylon, two men stand one on either side of it, on a line which passes through the foot of the pylon. The angles between the line of sight to the top of the pylon and the horizontal are 38° and 42°, and the observers are 100 yd. apart. Find, by a scale drawing, the height of the pylon. If the pylon had been 50 yd. from each observer, what would have been the angles between the line of sight of the top and the horizontal? (*Chap.* 21)

2. Calculate the area of the piece of land represented by the field-book entry:

	Yards to B	
	250	
To E, 40	220	
	165	120 to D
To F, 45	120	
	85	110 to C
To G, 60	40	
	From A	

(*Chap.* 24)

3. Calculate the area of the face of the plate shown in fig. 135. (*Chap.* 24)

4. A circular sector, radius 5 in., containing an angle of 150°, is folded to form a cone. Find the radius of the circular base. (*Chap.* 29)

5. Find the volume of metal in a cylindrical pipe 10 ft. long, external diameter 8 in., and internal diameter 6 in. (Take $\pi = \frac{22}{7}$.)

Find the ratio of this volume to the volume of metal in a pipe 20 ft. long, internal diameter 6 in., and external diameter 7 in., and express the difference in weight as a percentage of the weight of the heavier pipe, assuming both pipes to be made of the same metal. (*Chaps.* 29 *and* 24)

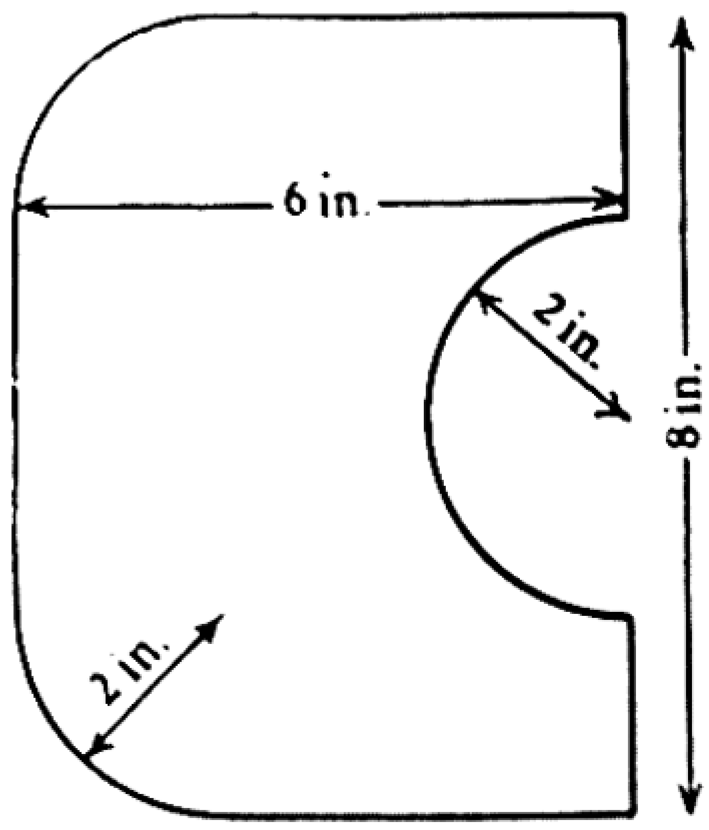


Fig. 135

6. ABC is an isosceles triangle with equal sides AB and AC, and angle $A = 36°$. The bisector of angle A meets BC in D, and the bisector of angle B meets AC in E, and intersects AD in F.

Prove that triangles ABC and BEC are similar. If the angle BAD is written as x, write the following angles as multiples of x: angle ABF, angle DBF, angle BFD, angle AFE, angle ACD, angle BEC, angle ADC, angle BDA, angle AEB, and angle EFD. (*Chap.* 23)

7. AB, BC, CD, DA are rods freely jointed at A, B, C and D, and of lengths 7 in., 20 in., 15 in. and 24 in. respectively. If the angle at A is 90°, calculate the length of the diagonal DB, and find the angle C. Show also that it is impossible to have a right angle at D. (*Chap.* 30)

8. Plot $xy = 25$ and $xy = -25$ on the same axes.

Show that no part of either of these graphs lies within the region bounded by the lines $y = 10-x$, $y = x+10$, $y = -10-x$, $y = x-10$. Find the area bounded by the four lines.

(*Chaps.* 28 *and* 27)

9. The table gives areas of cross-section of a water tank at various depths. Find the weight of water in the tank when full, to the nearest half ton, if 1 cu.ft. of water weighs $62\frac{1}{2}$ lb.

Areas (sq.ft.): 30, 69, 100, 130, 145, 150, 140, 120, 96, 66, 30.

Total depth of tank, 5 ft. (*Chap.* 29)

10. (*a*) Three similar triangles have altitudes h, $2h$ and $3h$. What is the ratio of their areas?

If these triangles represent the vertical sections of three similar pyramids, find the ratio of their volumes.

(*Chaps.* 23 *and* 29)

(*b*) A cylindrical rod 4·5 in. long and diameter 1 in. is filed down to make the largest hexagonal prism; find its volume.

(*Chap.* 30)

11. The distance (s ft.) above its point of projection at a time t sec. after projection, for a body thrown vertically upwards, is given by $s = 64t - 16t^2$. Plot s vertically against t horizontally from $t = 0$ to $t = 4$ using half seconds. Find (*a*) the highest point reached, (*b*) the time taken to reach the highest point, (*c*) the time taken to return to the point of projection, (*d*) the height after 3 sec. (*Chap.* 28)

12. (*a*) Calculate the total length of girders in the framework shown in fig. 136. Find also the height of BD above AE.

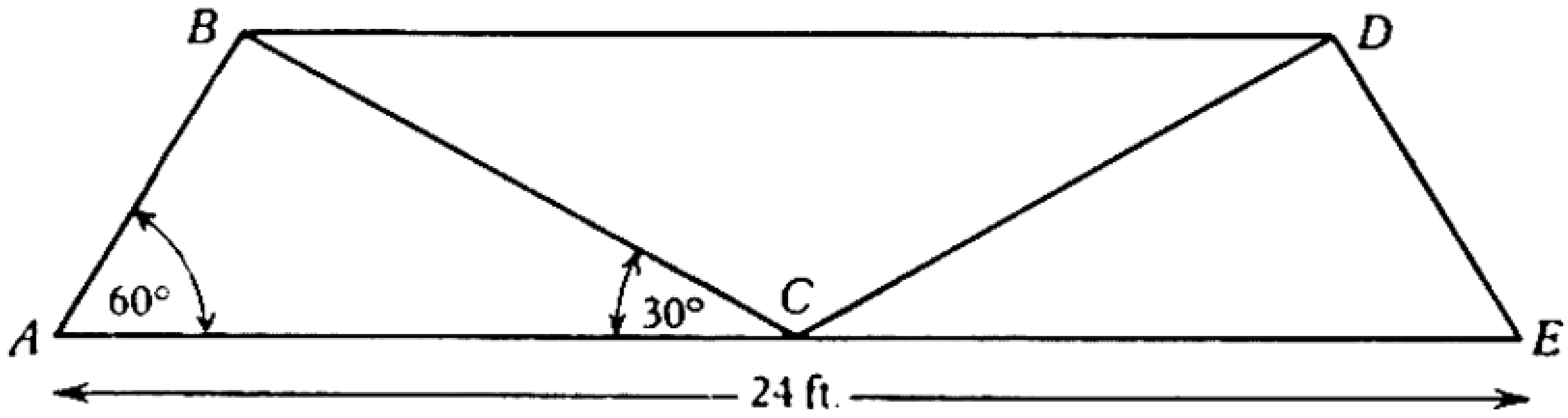


Fig. 136

(*b*) How much greater would be the total length if the structure consisted of three equilateral triangles, the base being still 24 ft. (*Chap.* 30)

Answers to Revision Examples, Chapters 21–30

1. 125·5 ft.; 40°.

2. 29,225 sq.yd.

3. 40 sq.in.

4. $2\frac{1}{12}$ in.

5. $1\frac{19}{36}$ cu.ft.; $\frac{14}{13}$; $7\frac{1}{7}$ %.

6. $2x$, $2x$, $3x$, $3x$, $4x$, $4x$, $5x$, $5x$, $6x$, $7x$.

7. $DB = 25$ in.; angle $C = 90°$; $AB + BC$ not as great as $\sqrt{(24^2 + 15^2)}$.

8. 200 sq. units.

9. $14\frac{1}{2}$ tons.

10. (*a*) 1:4:9; 1:8:27. (*b*) 2·925 cu.in.

11. (*a*) 64 ft. (*b*) 2 sec. (*c*) 4 sec. (*d*) 48 ft.

12. (*a*) 74·78 ft.; 5·2 ft. (*b*) 9·22 ft.

CHAPTER 31

MENSURATION OF FRUSTA AND SPHERES. DENSITY AND SPECIFIC GRAVITY

The sphere

The surface area of a sphere is given by area $= 4\pi r^2$, where r is the radius. (The proof is beyond the scope of this book.) Writing d (diameter) for $2r$, this becomes, area $= \pi d^2$.

Consider a cylinder, open at both ends, into which the sphere will just fit, as in fig. 137. The circumference of the cylinder will be the same as the greatest circumference of the sphere, and the length of the cylinder will be equal to d, the diameter of the sphere. Now the area of curved surface of the cylinder will be circumference × length, i.e. $\pi d \times d = \pi d^2$, which is the same as the area of the surface of the sphere. Thus the area of the surface of a sphere is the same as the area of curved surface of its *circumscribing cylinder.*

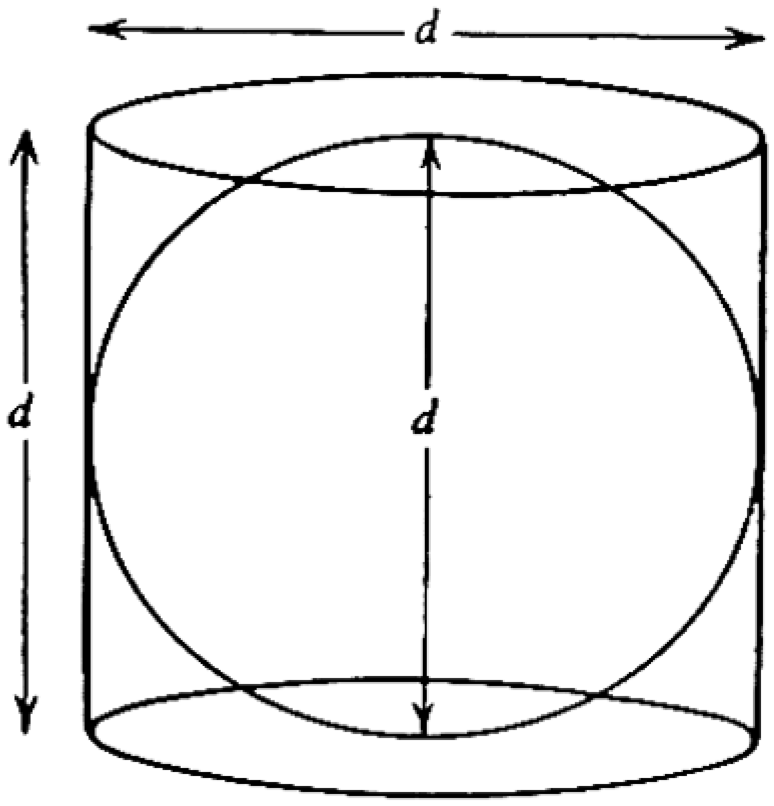


Fig. 137

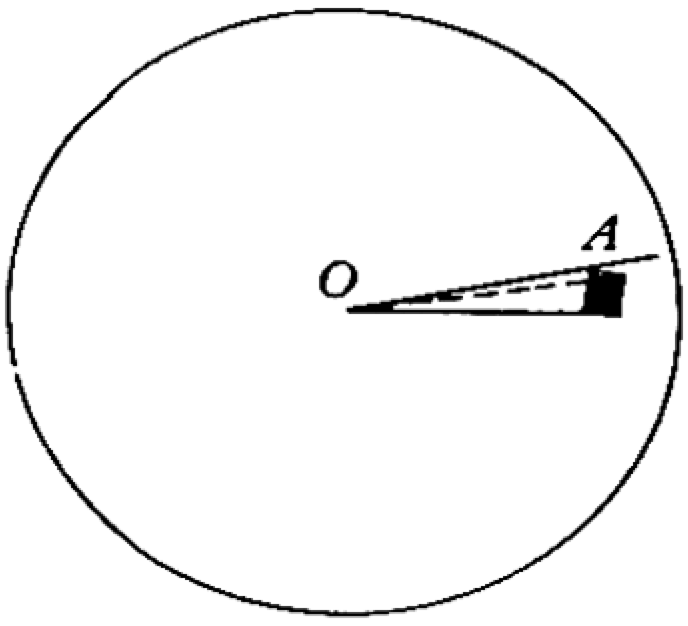


Fig. 138

Volume of a sphere. The volume of a sphere is given by

$$V = \frac{4\pi r^3}{3}.$$

This is illustrated by fig. 138. Consider a pyramid whose base is 1 sq. unit in area, and whose vertex is at the centre of the sphere.

Now, volume of pyramid

$$= \frac{\text{area of base} \times \text{height}}{3} = \frac{1 \times r}{3}.$$

Also, the number of such pyramids necessary to fill the volume of the whole sphere is the number of times the area of the base of one of them will divide into the total surface area of the sphere; that is, $\frac{4\pi r^2}{1}$ or $4\pi r^2$.

Volume of sphere

$$\begin{aligned} &= \text{volume of one pyramid} \times \text{number of pyramids} \\ &= \frac{1 \times r}{3} \times 4\pi r^2 \\ &= \frac{4\pi r^3}{3}. \end{aligned}$$

Frusta

A frustum of a pyramid is the part intercepted between two parallel planes. Thus, if a pyramid or cone is cut across by a plane parallel to the base, the volume between the base and the plane of intersection is called a frustum. It is advisable to deal with each frustum as the difference between the volumes of two pyramids by applying the geometry of similar triangles.

Example. Find the volume and total surface area of a frustum of a cone, radii 5 in. and 3 in., and height 4 in.

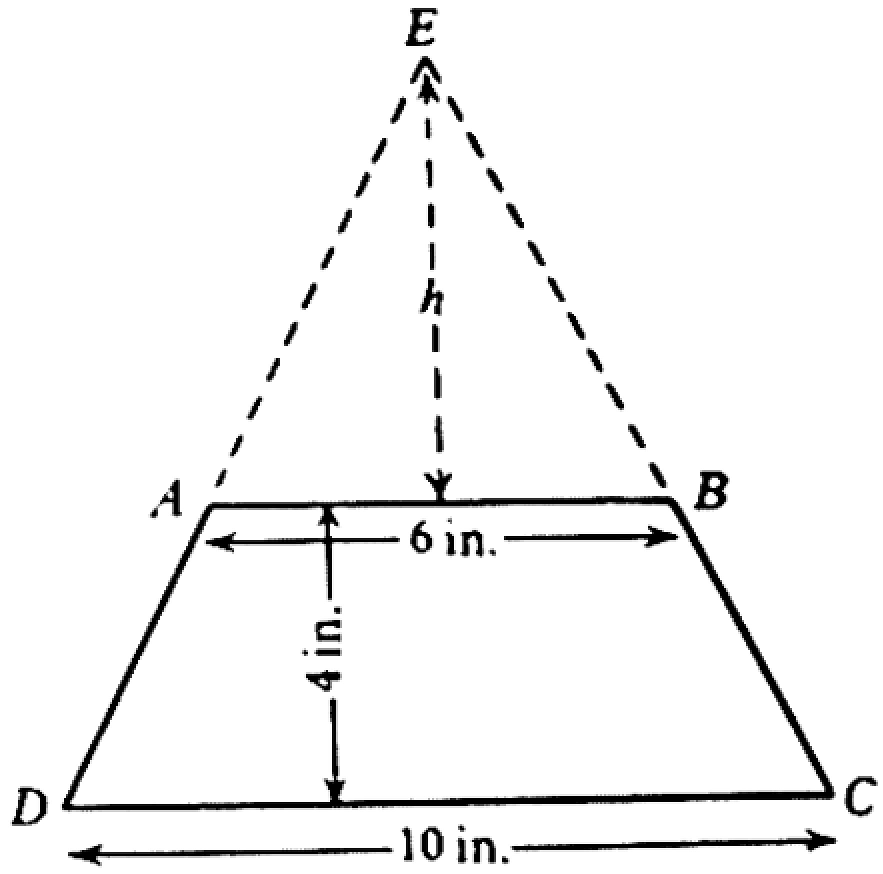


Fig. 139

Let $ABCD$ be a vertical section through the frustum (fig. 139), and let ECD be a vertical section through the complete cone from which the frustum was cut.

Now EAB and EDC are similar triangles; therefore

$$\frac{h}{6} = \frac{h+4}{10} \quad \text{(Chapter 23)},$$

i.e. $\frac{1}{6}$ of $h = \frac{1}{10}$ of $(h+4)$,

$$\therefore\ h = \tfrac{6}{10} \text{ of } (h+4) = \tfrac{6}{10} \times h + \tfrac{6}{10} \times 4,$$

$$\therefore\ h = \frac{3h}{5} + \frac{12}{5}, \qquad \therefore\ \tfrac{2}{5}h = \tfrac{12}{5}, \qquad \therefore\ h = 6.$$

Thus the height of the complete cone was $6 + 4 = 10$ in.

Volume of frustum

= volume of cone radius 5, height 10 minus volume of cone radius 3, height 6

$$= \frac{\pi \times 25 \times 10}{3} - \frac{\pi \times 9 \times 6}{3}$$

$$= \frac{250\pi}{3} - 18\pi = \frac{196\pi}{3}\,\text{cu.in.}$$

$= 65·33 \times 3·142$ cu.in.

= 205·2 cu.in.

To find the area of surface:

$$DE^2 = (\text{height})^2 + (\text{radius of base})^2$$

$$= 100 + 25.$$

$$\therefore\ DE = \sqrt{(125)} = 11·18\,\text{in.}$$

Also $AE^2 = 6^2 + 3^2$

$$= 36 + 9$$

$$= 45.$$

$$\therefore\ AE = \sqrt{(45)} = 6·708.$$

Numbers	Logs
65·33	1·8151
3·142	0·4972
205·2	2·3123
125	2·0969
11·18	1·0485
45	1·6532
6·708	0·8266

Curved surface of frustum

= curved surface of cone radius 5, height 10 minus curved surface of cone radius 3, height 6

$$= \pi \times 5 \times 11·18 - \pi \times 3 \times 6·708$$

$$= \pi(55·90 - 20·124)$$

$$= \pi \times 35·776$$

= 112·4 sq.in.

Area of plane surfaces

$$= \pi \times (5^2) + \pi \times (3^2)$$

$$= 25\pi + 9\pi$$

$= 34\pi$ sq.in.

= 106·9 sq.in.

3·142	0·4972
35·78	1·5537
112·4	2·0509
3·142	0·4972
34	1·5315
106·9	2·0287

Thus, total surface area of frustum = 112·4 + 106·9 sq.in.

= 219·3 sq.in.

Density and specific gravity

The density of a substance is the mass per unit volume of the substance. Mass means the quantity of matter in a body, and is numerically equal to its weight. Thus if a piece of iron weighs 5 lb., its mass is 5 lb. In order to give the density of a substance it is necessary to give its weight (or mass) and to give also the volume which has this mass. The density of water is usually taken to be 62·3 lb. per cu.ft., and the density of copper is 8·93 g. per c.c. or 555 lb. per cu.ft.

A solid will float in a liquid whose density is greater than its own, but will sink in a liquid whose density is less than its own. So oak of density 50 lb. per cu.ft. (or 0·8 g. per c.c.) will float in water whose density is 62·3 lb. per cu.ft. (or 1 g. per c.c.), but ebony, whose density is 75 lb. per cu.ft. (or 1·2 g. per c.c.) will sink in water. The explanation of the flotation of hollow steel ships is similar, but this will be studied with the principle of Archimedes in mechanics.

$$\text{Density} = \frac{\text{Mass (or weight)}}{\text{Volume}}.$$

Therefore, weight of a body = volume × density.

Note that the units for volume and density must obviously correspond, e.g. if the volume is in cubic feet, the density must be expressed in lb. per cubic foot; while a volume in c.c. is multiplied by a density in grams per c.c. to give a total weight in grams.

The *specific gravity* of a substance is given by

$$\frac{\text{Weight of given volume of the substance}}{\text{Weight of same volume of water}}.$$

Thus the specific gravity of a substance is the ratio between its density and that of water. (In the case of gases, hydrogen is taken as the standard substance instead of water.)

One c.c. of copper weighs 8·93 g., and, as 1 c.c. of water weighs 1 g., the specific gravity of copper is $\frac{8{\cdot}93}{1} = 8{\cdot}93$. (Specific gravities are merely figures, for, as ratios, they have no dimensions.)

The specific gravity of sea water is 1·03, so 1 c.c. of sea water weighs 1·03 g.

Suppose we require the weight of 50 cu.ft. of iron whose specific gravity is 7·2.

We know that iron is 7·2 times as heavy as water, therefore

$$\text{weight of 1 cu.ft. of iron} = 7{\cdot}2 \times 62{\cdot}3\text{ lb.},$$

so

$$\text{weight of 50 cu.ft. of iron} = 50 \times 7{\cdot}2 \times 62{\cdot}3\text{ lb.}$$
$$= \underline{22{,}430\text{ lb.}}$$

Numbers	Logs
50	1·6990
7·2	0·8573
62·3	1·7945
22,430	4·3508

$$\begin{aligned}\text{Weight of a body} &= \text{volume} \times \text{density}\\ &= \text{volume} \times \text{specific gravity} \times \text{weight of unit volume of water.}\end{aligned}$$

[For students who do not require volumes of spheres and frusta, but do require density, problems 7–10 below will be adequate.]

Examples for Chapter 31

1. Find the total surface area of a cylindrical tank with hemispherical ends if the diameter of the tank is 4 ft. and the total length 12 ft. Find also its volume. (*Chap.* 31)

2. (*a*) Find the volume of a copper rivet which consists of a hemispherical head, radius 0·2 in., and a cylinder, length 0·6 in., and radius 0·1 in.

(*b*) Calculate the weight of one gross of such rivets if the density of copper is 555 lb. per cu.ft. (*Chap.* 31)

3. A buoy consists of a frustum of a cone whose ends have radii 2 ft. and 3 ft. and whose height is 4 ft., to which is attached a hemisphere of radius 3 ft. at the larger end. Find the volume and surface area of the buoy. (*Chap.* 31)

4. Find the weight of a hollow brass sphere, inner and outer radii 10 in. and 11 in. respectively, if the specific gravity of brass is 8·6, and 1 cu.ft. of water weighs 62·3 lb. (*Chap.* 31)

5. Find the area of parchment in a lamp-shade in the shape of a frustum of a cone (without ends), vertical height 10 in., and radii of ends 5 in. and 3 in. (*Chap.* 31)

6. A bucket shaped as the frustum of a cone is 10 in. deep, and the diameters of top and bottom are respectively 1 ft. and 7 in. Find the weight of water in the bucket when full, taking the density of water as 62·3 lb. per cu.ft. (*Chap.* 31)

7. Find the weight of a bracket made of cast iron, 18 in. long, whose section is shown in fig. 140. The specific gravity of cast iron is 7·2, and the density of water 62·3 lb. per cu.ft. (*Chap.* 31)

8. A water tank is of rectangular section, 15 ft. long and 9 ft. wide. Water is poured in until the depth is 28 in. Compare the thrust on the base with the thrust on one end.

[The thrust on the base is a force equal to the weight of water in the tank, while the thrust on one end is equal to the area of the wetted surface of the end multiplied by half the depth of water, multiplied by the weight of 1 cu.ft. of water, provided all measurements are in feet.] (*Chap.* 31)

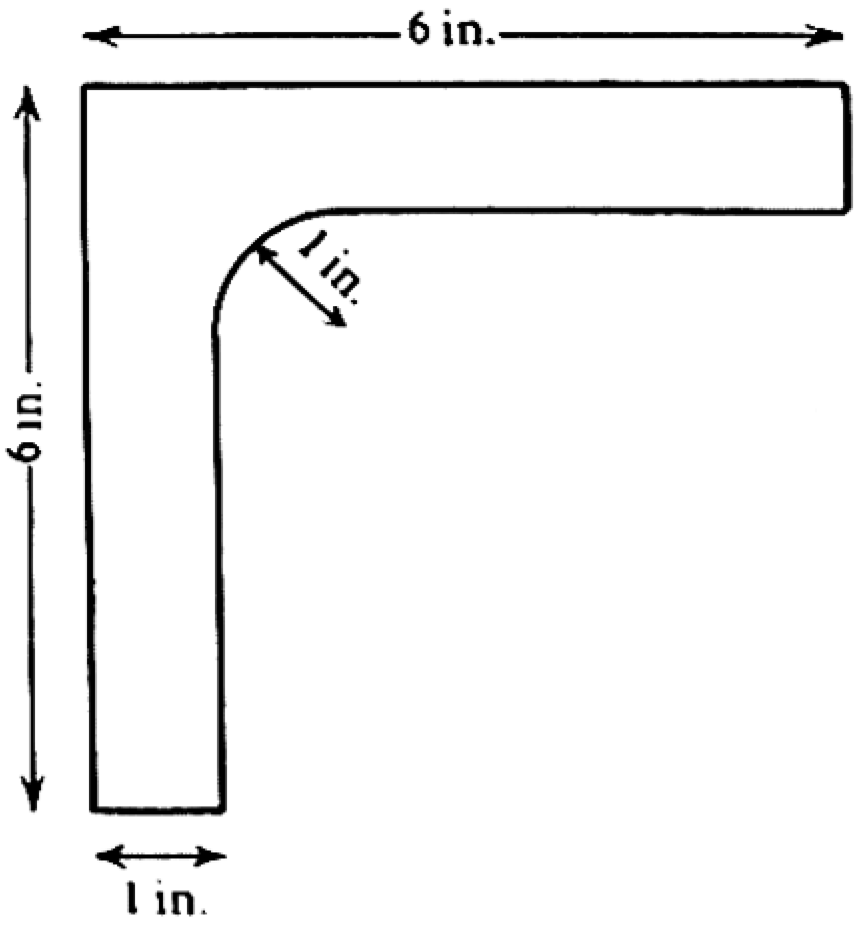


Fig. 140

9. A cylinder of metal 4 cm. long and 2 cm. in diameter is drilled to a depth of 2·5 cm. with a 1 cm. drill. (Assume the hole to be cylindrical.) The solid weighs 75·28 g. Find the specific gravity of the metal. (*Chap.* 31)

10. A hexagonal plate of copper whose side is 3 in. and whose thickness is ¼ in. is drilled through in six places by a ¼ in. drill. Find the weight of the plate if 1 cu.ft. of copper weighs 555 lb. (*Chap.* 31)

Answers to Examples for Chapter 31

1. 150·8 sq.ft.; 134·1 cu.ft.

2. (*a*) 0·0356 cu.in. (*b*) 1·65 lb.

3. 136 cu.ft.; 133·9 sq.ft.

4. 430 lb. **5.** 256·4 sq.in.

6. 26·14 lb. **7.** 52·38 lb.

8. 19,630 lb. wt.; 1526 lb. wt.

9. 7·1. **10.** 1·86 lb.

CHAPTER 32

TRIGONOMETRIC RATIOS

This chapter and five following chapters deal with that branch of mathematics which enables heights and distances to be calculated through a knowledge of the ratios of the lengths of sides of right-angled triangles. The subject is called Trigonometry, the prefix 'tri' indicating that the fundamental ratios are three in number. We hope to be able to accomplish by calculation results which up to now have been possible only by scale drawing.

Consider a right-angled triangle (fig. 141), and denote one angle by θ (*theta*). This is the angle we wish to study, but since the definitions of the ratios we are about to define will apply to any angle between 0° and 90°, the symbol θ will stand for such angles in general.

The sides of the triangle are given special names.

The longest side, that is, the side opposite the right angle, is called the *hypotenuse*.

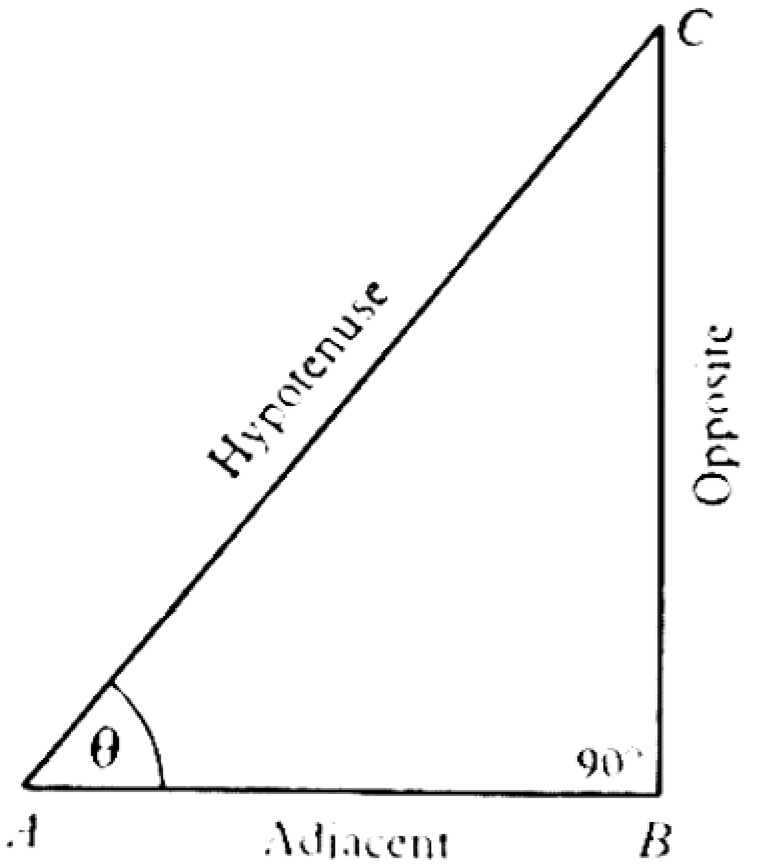


Fig. 141

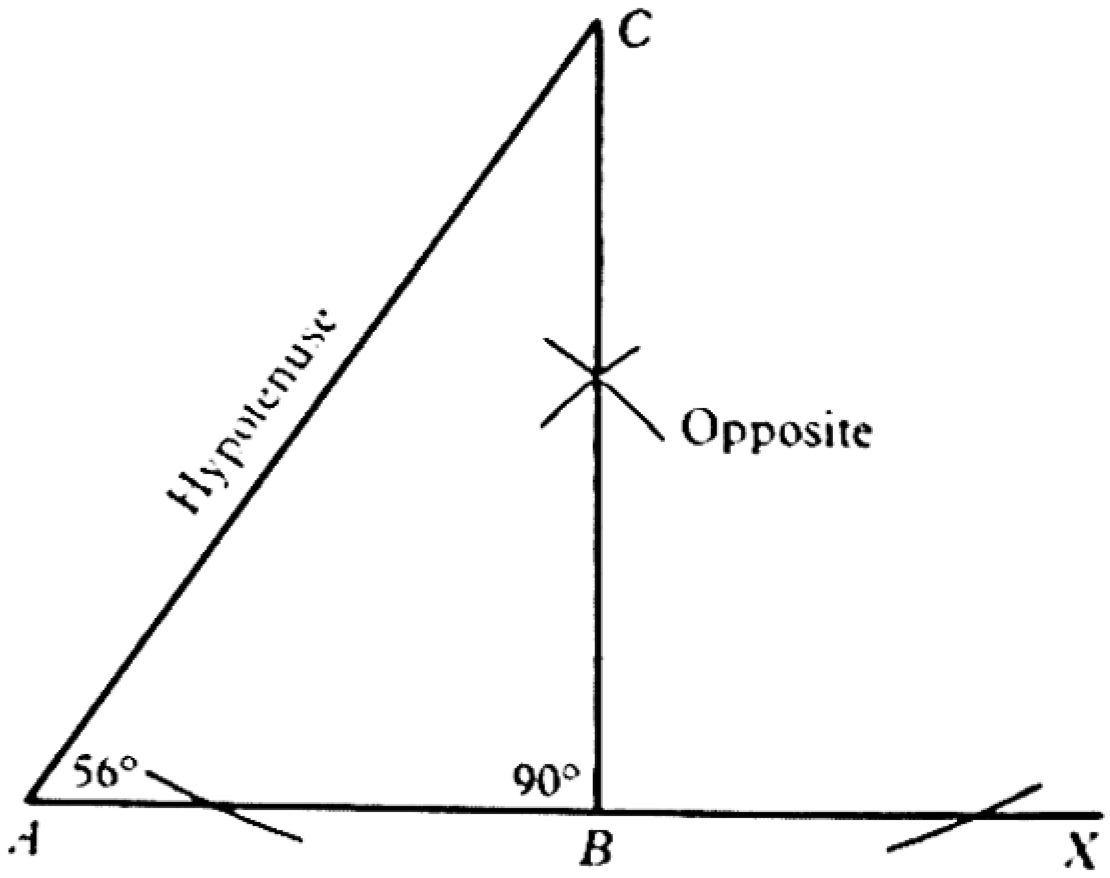


Fig. 142

The side opposite the marked angle θ, is called the *opposite* side.

The third side, which must be one of the arms of the angle θ, is called the *adjacent* side.

The ratio $\dfrac{\text{Length of opposite side}}{\text{Length of hypotenuse}}$ is called the *sine* of the angle θ (written $\sin\theta$).

The ratio $\dfrac{\text{Length of adjacent side}}{\text{Length of hypotenuse}}$ is called the *cosine* of the angle θ. (written $\cos\theta$)

The ratio $\dfrac{\text{Length of opposite side}}{\text{Length of adjacent side}}$ is called the *tangent* of the angle θ. (written $\tan\theta$)

These are the three fundamental trigonometric ratios of an angle.

Note. They are merely ratios, that is, a length divided by a length, so they have no dimensions. Thus sin 30° is always equal to $\frac{1}{2}$, for all right-angled triangles which contain an angle of 30° are similar, and the ratio of corresponding sides is the same for them all.

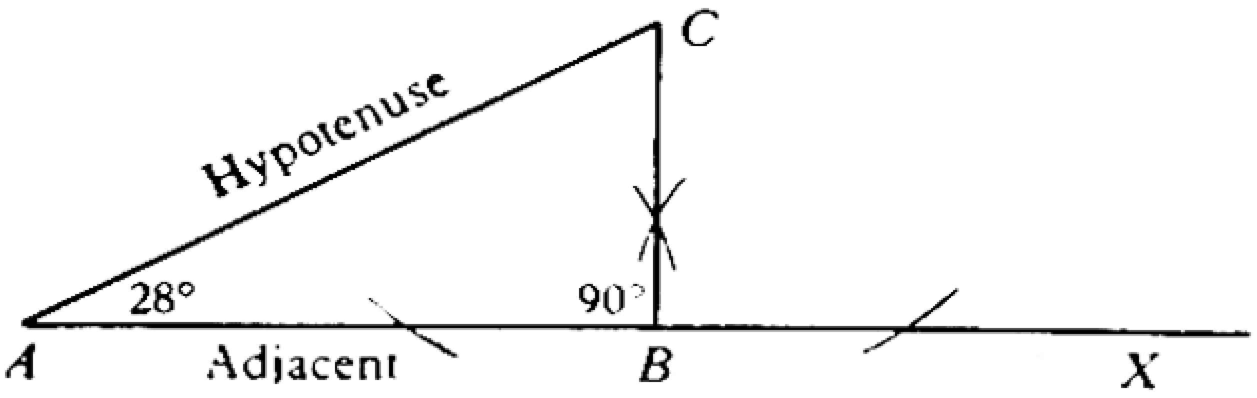


Fig. 143

To find the sine, cosine or tangent of an angle by scale drawing

Sin 56°. The ratio *sine* is $\dfrac{\text{Length of opposite side}}{\text{Length of hypotenuse}}$, and the calculation will be easier if we arrange that the denominator is some simple number like unity or 10. Let the hypotenuse be measured as 10 units. We have, therefore, to construct a triangle containing an angle of 90°, an angle of 56°, and a hypotenuse 10 units in length (see fig. 142). Draw a horizontal line AX. At A measure an angle of 56°, and draw AC 10 units long. From C draw a perpendicular to intersect AX in B. Then ABC is a triangle from which sin 56° may be measured.

$$\text{Sin } 56° = \frac{BC}{AC} = \frac{8{\cdot}3}{10} \text{ (by measurement)} = 0{\cdot}83.$$

Cos 28°. The ratio *cosine* is $\dfrac{\text{Length of adjacent side}}{\text{Length of hypotenuse}}$. Again it will be convenient if the denominator, i.e. the hypotenuse, is 10 units in length. Draw AX, and at A measure an angle of 28°. Draw AC 10 units long and from C draw CB perpendicular to AX (fig. 143).

Thus $\cos 28° = \dfrac{AB}{AC} = \dfrac{8{\cdot}8}{10}$ (by measurement) $= 0{\cdot}88$.

Tan 48°. The ratio *tangent* is $\dfrac{\text{Length of opposite side}}{\text{Length of adjacent side}}$. We shall make the adjacent side 10 units long. Measure AB 10 units long. At A make an angle of 48° and draw AX in this direction (fig. 144). Draw BC perpendicular to AB, to intersect AX at C. Thus

$$\tan 48° = \frac{CB}{AB} = \frac{11{\cdot}2}{10} = 1{\cdot}12.$$

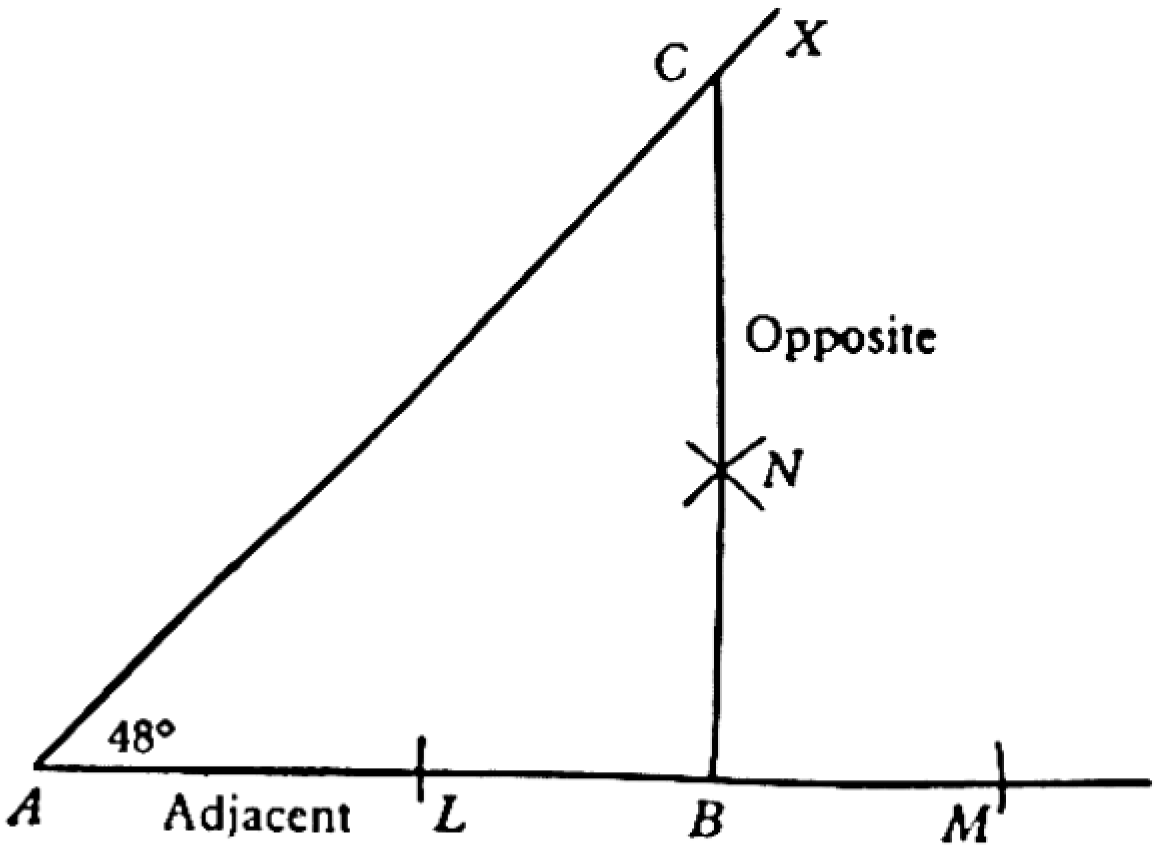


Fig. 144

To find the angle which has a given sine, cosine or tangent by drawing

The angle whose sine is 0·75. Let θ be the angle, then

$$\sin\theta = \frac{0{\cdot}75}{1} = \frac{7{\cdot}5}{10} = \frac{\text{Opposite}}{\text{Hypotenuse}}.$$

Draw a base-line AX, of any length (fig. 145). Centre A, radius 10 units, and describe an arc above AX. Slide the set-square along AX, with a ruler beside it, until a point on the arc 7·5 units from AX is found. This is the point C. Draw CB perpendicular to AX, and join CA. Then angle CAB is the angle whose sine is 0·75.

By measurement, $\theta = 48\frac{1}{2}°$.

The angle whose cosine is 0·48. Let θ be the angle, then

$$\cos\theta = \frac{0{\cdot}48}{1} = \frac{4{\cdot}8}{10} = \frac{\text{Adjacent}}{\text{Hypotenuse}}.$$

Draw a base-line AB 4·8 units in length (fig. 146). From B erect a perpendicular BX. Centre A, radius 10 units, and draw an arc to cut BX at C. Join AC. Then angle CAB is the required angle θ.

By measurement $\theta = 61\frac{1}{2}°$.

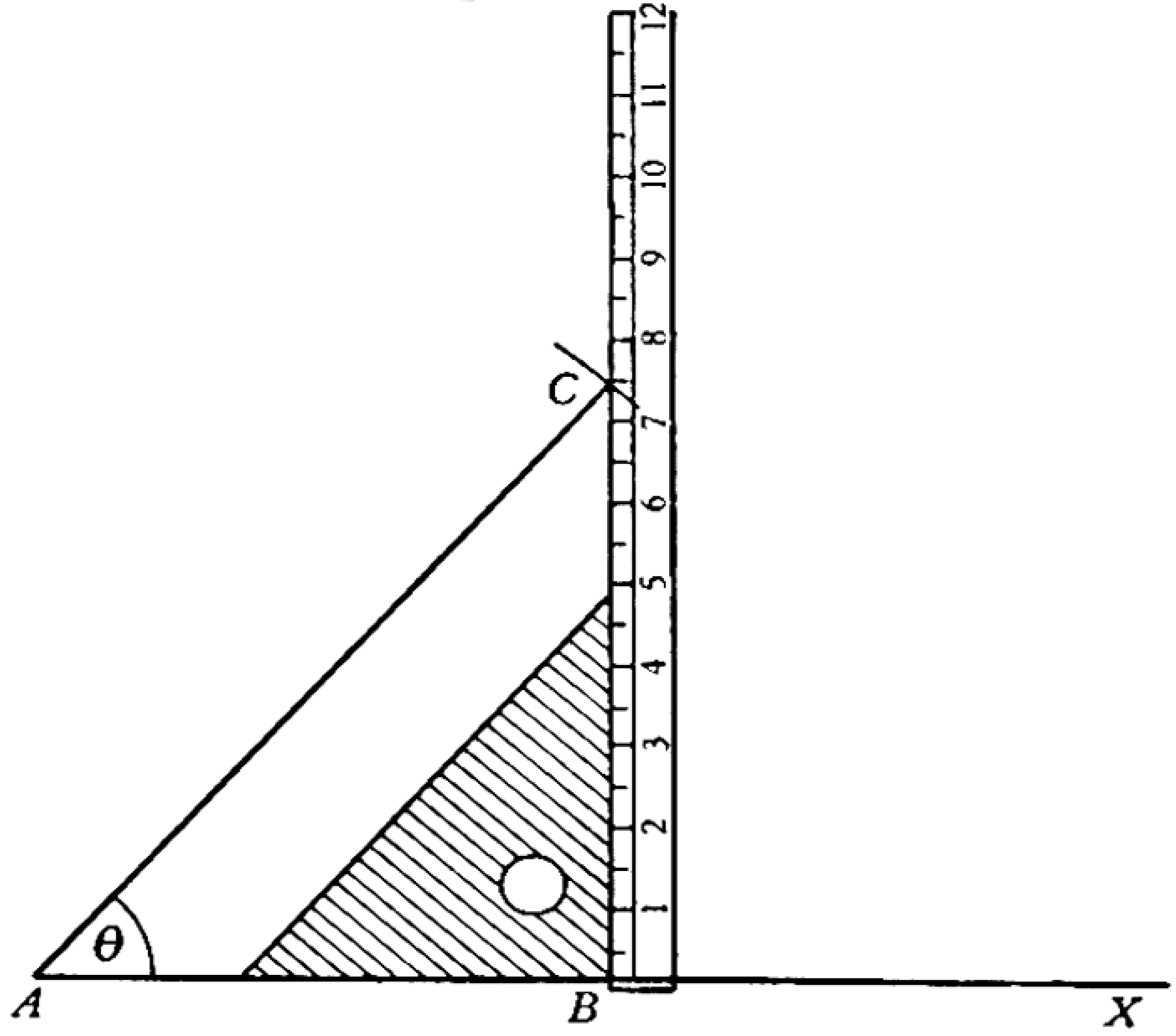


Fig. 145

The angle whose tangent is 0·94. Let θ be the angle, then

$$\tan\theta = \frac{0{\cdot}94}{1} = \frac{9{\cdot}4}{10} = \frac{\text{Opposite}}{\text{Adjacent}}.$$

Draw a base-line AB, 10 units long (fig. 147). From B erect a perpendicular 9·4 units in length to C. Join AC, then angle CAB is the angle θ.

By measurement $\theta = 43\frac{1}{2}°$.

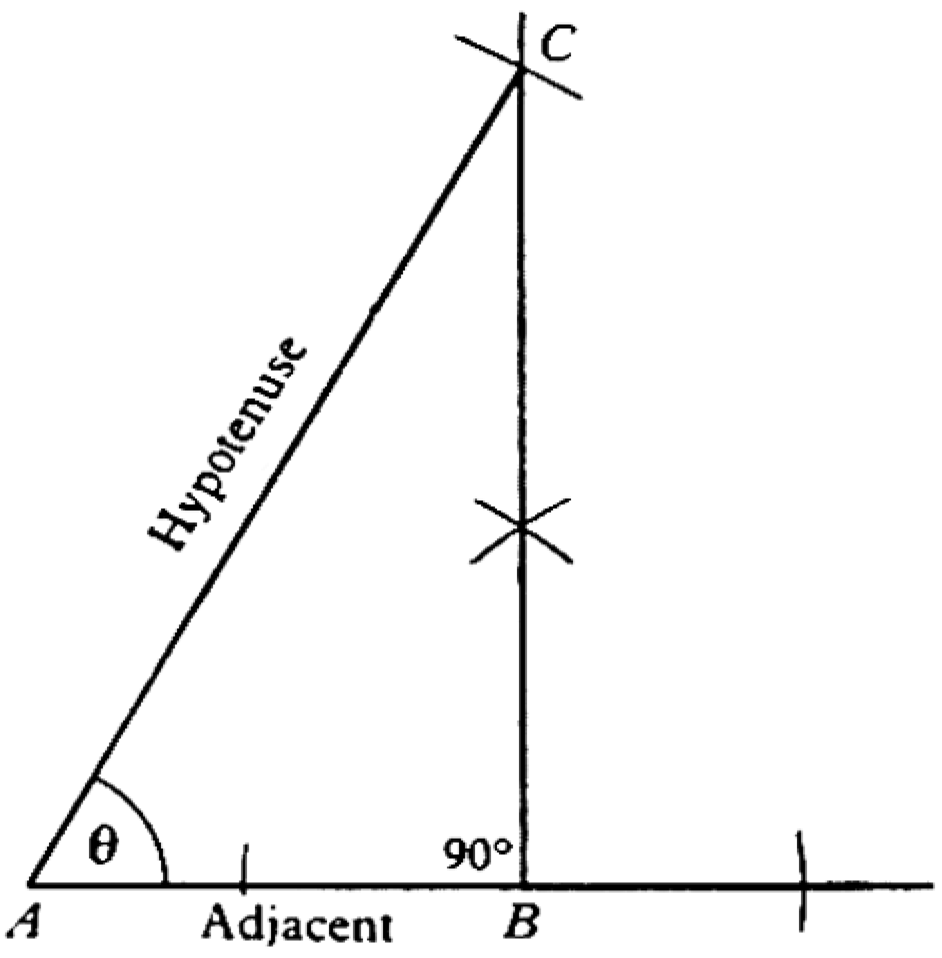


Fig. 146

Use of tables

Tables of sines, cosines and tangents are always available, so the student is not required to find the ratios by scale drawing except when specially asked to do so for an exercise.

(1) *Table of sines*. Angles in degrees are given in the left-hand column, while the main columns of the page subdivide the degrees for intervals of 6 min. The difference columns on the right give the corrections for angles between these 6 min. intervals.

Required the sine of 21° 32′. Look in the main column headed 30′ opposite the figure 21° on the left. This column gives 0·3665, and is the sine of 21° 30′. We require the sine of an angle 2′ greater, and we see opposite 0·3665, in the difference column headed 2′, the figure 5. Add 5 to 3665, giving 3670. Thus

$$\sin 21^\circ 32' = 0{\cdot}3670.$$

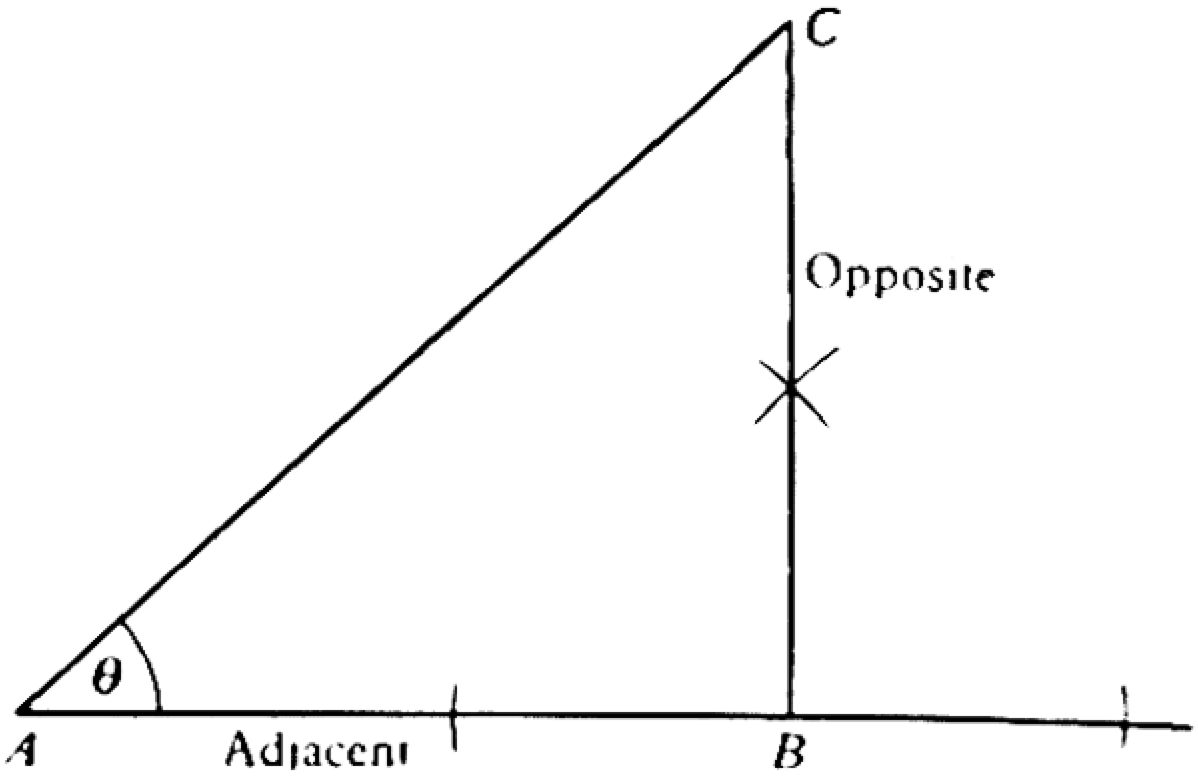


Fig. 147

(2) *Table of cosines.* Before using the table, let us examine fig. 148. Three angles θ_1, θ_2 and θ_3 are increasing in magnitude. All three hypotenuses are the same length. It is obvious that the value of the fraction $\frac{\text{Adjacent}}{\text{Hypotenuse}}$ *decreases* as the angle *increases*. Thus the larger the angle (between 0° and 90°) the smaller the cosine. Incidentally, the same diagram shows that the fraction $\frac{\text{Opposite}}{\text{Hypotenuse}}$ increases as the angle increases, therefore the larger the angle the greater its sine.

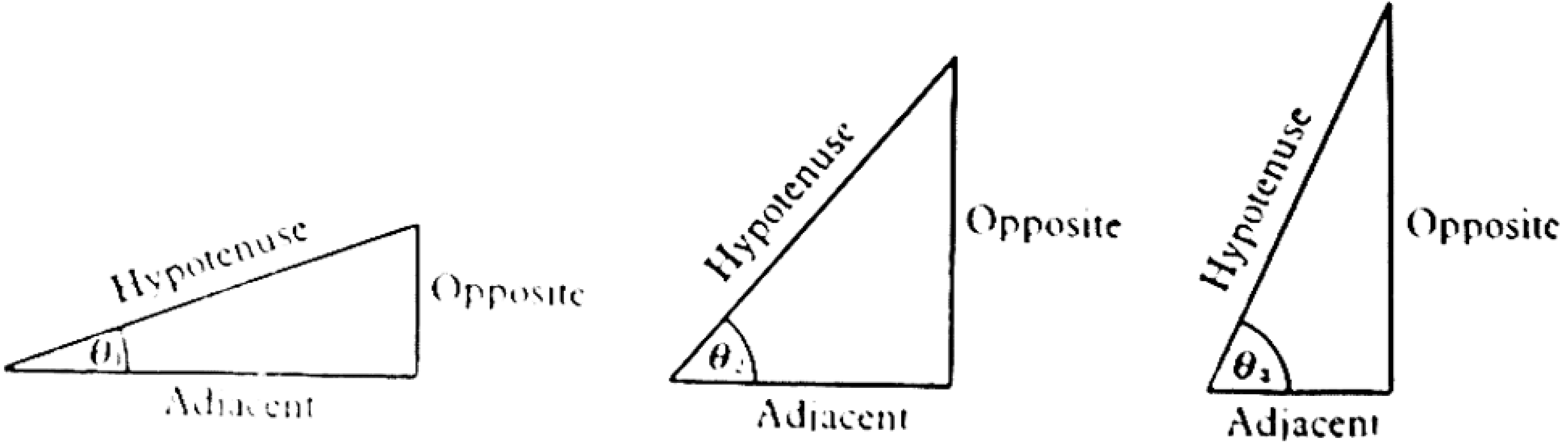


Fig. 148

Required the cosine of 21° 32′. Look in the main column headed 30′ opposite the figure 21° in the left-hand column. This gives 0·9304. Now the angle whose cosine we require is 2′ *greater* than the angle $21^\circ 30'$, so its cosine is *less* than the cosine of $21^\circ 30'$. The mean difference column headed 2, opposite 9304 gives the figure 2, so we *subtract* 2 from 9304, and find

$$\cos 21^\circ 32' = 0{\cdot}9302.$$

(3) *Table of tangents.* Referring again to fig. 148, we see that as the angle increases the ratio $\frac{\text{Opposite}}{\text{Adjacent}}$ increases, so the tangent increases with the angle as in the case of sines.

The table of tangents is used in exactly the same way as the table of sines.

Thus: When using the tables of sines or tangents, *add* mean differences. When using the table of cosines *subtract* mean differences.

Reversed use of tables. (*Given a ratio, to find the corresponding angle*)

(1) *Table of sines.* Required the angle whose sine is 0·5683.

The nearest number less than 5683 in the main columns is 0·5678 which occurs in the column headed 36′ opposite the angle 34°. This is 5 short of 5683. We find 5 in the opposite mean difference column headed 2.

Thus the angle whose sine is 0·5683 is $34^\circ 36' + 2'$, i.e. $34^\circ 38'$.

(2) *Table of cosines.* Required the angle whose cosine is 0·7587.

The safest policy is to look in the main columns for the nearest number *either side* of the given number. We find that 7581 occurs in the column headed 42′ opposite 40°, and that 7593 occurs in the column headed 36′ opposite the same number of degrees. Therefore the angle we require lies *between* $40^\circ 36'$ and $40^\circ 42'$. Now 7593 differs by 6 from 7587, and the opposite difference column containing a figure 6 is the column headed 3′. Our answer, then, differs from $40^\circ 36'$ by 3′, and since we know that it lies between $40^\circ 36'$ and $40^\circ 42'$, it must be $40^\circ 39'$. The angle whose cosine is 0·7587 is $40^\circ 39'$.

(3) *Table of tangents.* The tangent table is used in the same way as the sine table.

The student may check the following:

Required the angle whose tangent is 1·0398. Now 1·0392 corresponds to $46^\circ 6'$, and the mean difference column containing 6 is headed 1. Thus the angle whose tangent is 1·0398 is $46^\circ 7'$.

Complementary angles. It has been explained in Chapter 21 that two acute angles are said to be complementary if their total is 90°. Now it is obvious that if a right-angled triangle contains an angle of 40°, the third angle is 50° in order that the three angles may total 180°. Thus the two acute angles of any right-angled triangle are complementary. If one of these angles is referred to as θ, the other must be $90° - \theta$. Fig. 149 shows two triangles exactly the same. In the first triangle the angle whose ratios are required is θ, and the sides are labelled appropriately. In the second triangle the angle whose ratios are required is $(90° - \theta)$, and the sides are again labelled appropriately, but while the hypotenuse must be the same in both cases, the opposite side and the adjacent side have changed. This is necessary because BC is *opposite* to θ, while AB is opposite to $(90° - \theta)$.

$$\sin\theta = \frac{BC}{AC} \quad \text{and} \quad \cos(90° - \theta) = \frac{BC}{AC},$$

$$\cos\theta = \frac{AB}{AC} \quad \text{and} \quad \sin(90° - \theta) = \frac{AB}{AC}.$$

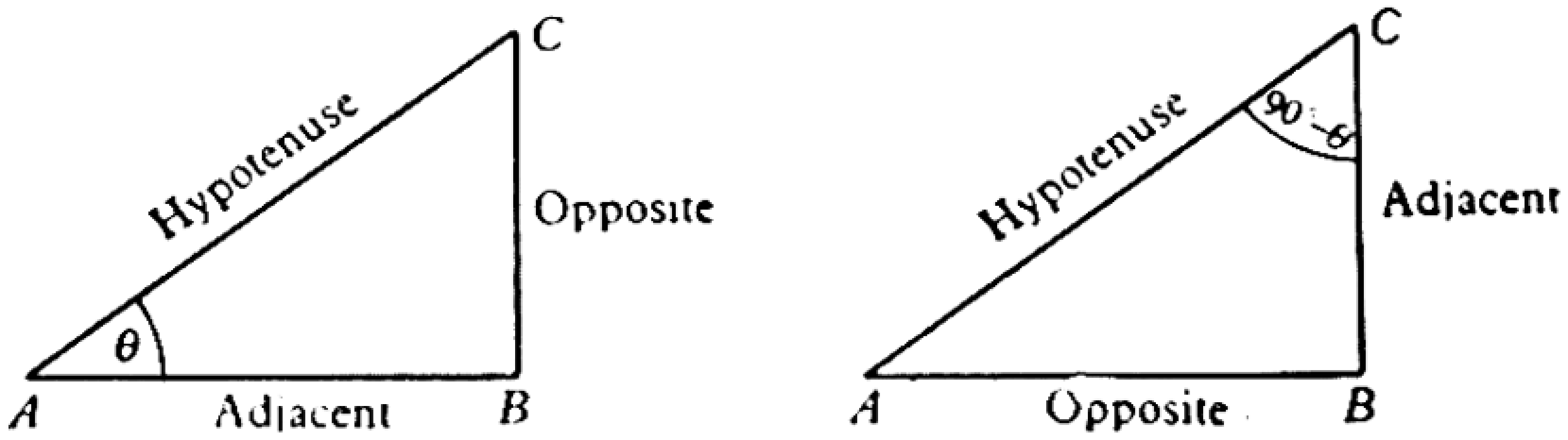


Fig. 149

Therefore the sine of an angle is the same as the cosine of its complement, and the cosine of an angle is the same as the sine of its complement. Hence

$$\sin 30° = \cos 60°, \quad \cos 20° = \sin 70°, \quad \sin 75° = \cos 15°.$$

For this reason some examining bodies do not issue a table of cosines to their candidates. When a cosine is required, the sine of the complement of the angle is looked up in the table of sines.

Examples for Chapter 32

1. Find, by scale drawings, (*a*) sin 42°, (*b*) cos 55°, (*c*) tan 34°, and check your answers from the tables. (*Chap.* 32)

2. Find, by scale drawings, (*a*) the angle whose sine is $\frac{1}{2}$, (*b*) the angle whose cosine is 0·34, (*c*) the angle whose tangent is 1·24, and check your answers from the tables. (*Chap.* 32)

3. A vertical flagstaff 30 ft. high is supported by a tight rope 34·64 ft. long attached to the top of the staff and to the ground. Calculate the angle between the rope and the horizontal. Check by a scale drawing. What would be the angle between the rope and the horizontal if its length were increased to 42·42 ft.? (*Chap.* 32)

4. Draw any triangle, as large as possible. Let A, B and C be the angles, and a, b and c be the lengths of the sides opposite these angles respectively. Measure a, b and c, and A, B and C as accurately as possible, and with the help of tables show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$ (*Chap.* 32)

5. Taking any acute angle you wish, as θ, with the help of tables verify

(i) $[\sin\theta]^2 + [\sin(90-\theta)]^2 = 1,$

(ii) $\dfrac{1}{[\sin(90-\theta)]^2} = 1 + [\tan\theta]^2.$ (*Chap.* 32)

6. By means of tables, find the angles in the following right-angled triangles:

(i) Sides 3 in., 4 in. and 5 in.

(ii) Sides 5 in., 12 in. and 13 in.

(iii) Sides 7 in., 24 in. and 25 in. (*Chaps.* 32 *and* 30)

7. A column of mercury 50·1 mm. long rests horizontally in a capillary tube whose bore is 1 mm. Find the weight of the column, assuming its ends to be hemispheres, if the specific gravity of mercury is 13·60. (*Chap.* 31)

8. Plot $y = x^3 - x^2 - 2x$ from $x = -3$ to $x = +3$ in half-units. On the same axes plot $y = x^3$ for the same range, and write the abscissae of the points of intersection of the two graphs. (*Chap.* 28)

9. A rain gauge consists of a cylindrical can 12 in. long, and diameter 4 in. In the top of this is soldered an inverted cone of the same diameter as the cylinder and 3 in. deep. A small hole is drilled at the vertex of the cone. Calculate the area of sheet metal in the instrument and also find the volume of water necessary to fill the cylinder up to the level of the vertex of the cone. (*Chap.* 29)

10. A and B are two points on shore 880 yd. apart. C and D are two rocks out at sea. Angles ADB and ACD are found to be equal. Angle BAC is 60°, and $AC = 1760$ yd. Draw the perpendicular bisectors of BC and AB and let them meet in E. Centre E, radius EA, and describe a circle. Hence show that D lies on the circle, and find the angle CDA. (*Chap.* 23)

Answers to Examples for Chapter 32

3. 60°, 45°.

6. (i) 36° 52′ and 53° 8′. (ii) 22° 37′ and 67° 23′.
(iii) 73° 44′ and 16° 16′.

7. 0·542 g.

8. $x = 0$ and $x = -2$.

9. 186·1 sq.in.; 113·1 cu.in.

10. Angle $CDA = 90°$.

CHAPTER 33

TRIGONOMETRIC RATIOS FOR SPECIAL ANGLES. VECTORS

Sometimes it is convenient to use three more ratios besides sin, cos and tan. The other three are called cosecant, secant and cotangent, and are usually written cosec, sec and cot.

The *cosecant* of an angle is the reciprocal of the *sine* of the angle.

The *secant* of an angle is the reciprocal of the *cosine* of the angle.

The *cotangent* of an angle is the reciprocal of the *tangent* of the angle.

$$\text{cosecant} = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\text{sine}},$$

$$\text{secant} = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\text{cosine}},$$

$$\text{cotangent} = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\text{tangent}}.$$

It follows that

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta}, \quad \cos\theta = \frac{1}{\sec\theta} \quad \text{and} \quad \tan\theta = \frac{1}{\cot\theta}.$$

Fig. 150 shows two identical triangles. Now $\tan\theta = CB/AB$ and $\cot(90-\theta) = CB/AB$.

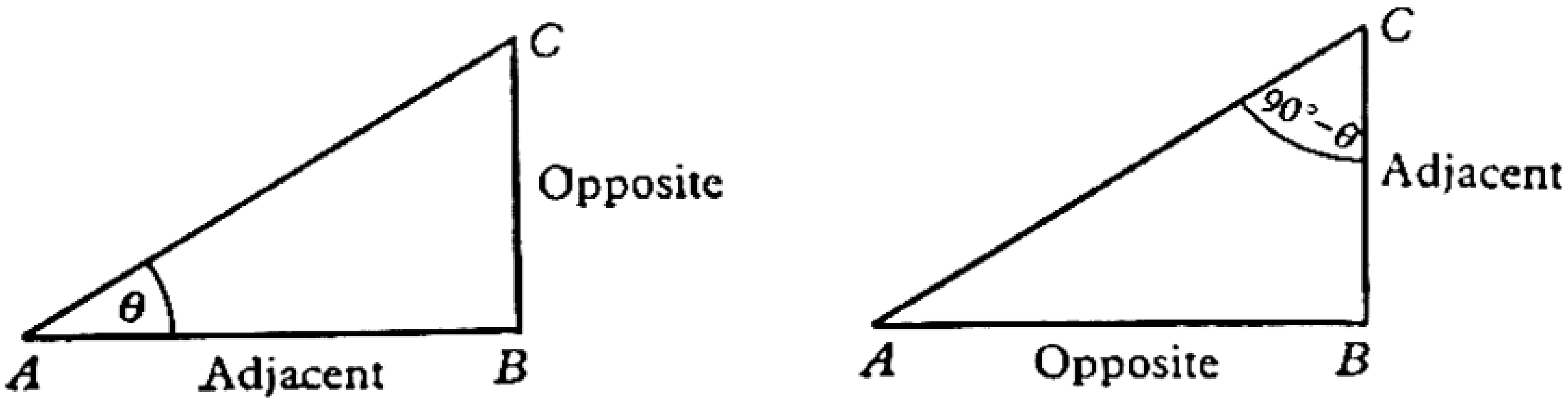


Fig. 150

Thus the tangent of an angle is equal to the cotangent of its complement, and the cotangent of an angle is equal to the tangent of its complement. For this reason, tables of cotangents are not given in most books of tables. If, for instance, the cotangent of 40° is required, the answer is obtained as tan 50° from the table of tangents.

Ratios for special angles

(1) *Angle* 45°. Fig. 151 shows a right-angled triangle containing an angle of 45°. It follows that the third angle is also 45°, and the triangle is isosceles. Let AB be of length a units, then $BC = a$ units. To find the length of the hypotenuse, use the theorem of Pythagoras:

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= a^2 + a^2. \\ \therefore\ AC^2 &= 2a^2. \\ \therefore\ AC &= \sqrt{(2a^2)} = a\sqrt{2}. \end{aligned}$$

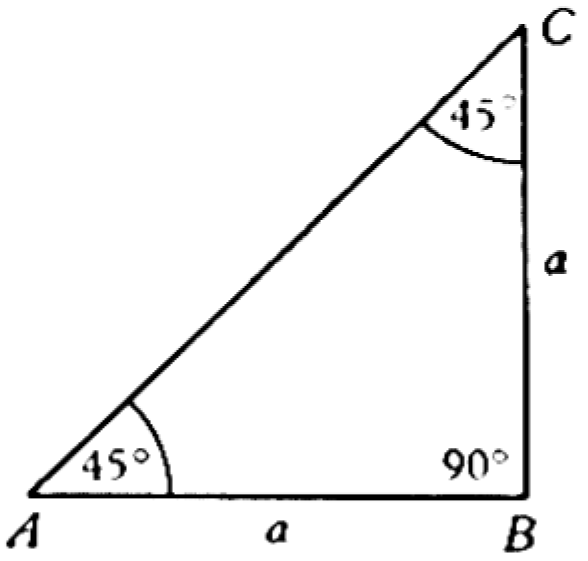


Fig. 151

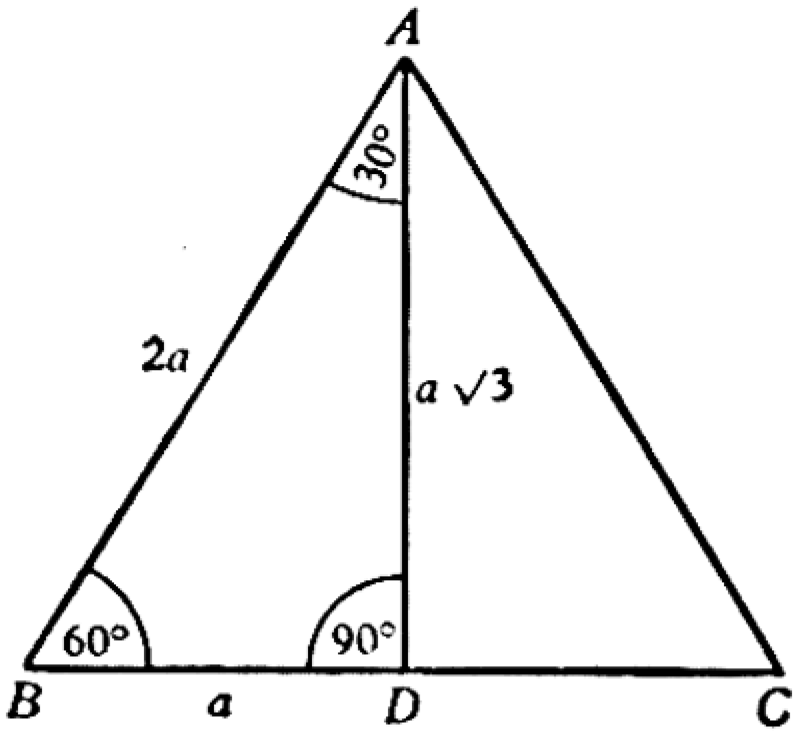


Fig. 152

The sides of a right-angled triangle containing an angle of 45° are thus in the ratio $1:1:\sqrt{2}$.

Thus,

$$\sin 45° = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{1{\cdot}414} = 0{\cdot}7171,$$

$$\cos 45° = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{1{\cdot}414} = 0{\cdot}7171,$$

$$\tan 45° = \frac{a}{a} = 1.$$

From these answers it follows that $\operatorname{cosec} 45° = \sqrt{2}$, $\sec 45° = \sqrt{2}$ and $\cot 45° = 1$.

(2) *Angles* 60° *and* 30°. ABC is an equilateral triangle (fig. 152). AD is drawn perpendicular to BC so that angle $BDA = 90°$, and angle $DAB = 30°$. Let the side of the equilateral triangle be $2a$ units, then $BD = a$ units.

Now

$$\begin{aligned} AB^2 &= BD^2 + AD^2, \\ 4a^2 &= a^2 + AD^2, \end{aligned}$$

whence $$3a^2 = AD^2,$$

$$\therefore\ AD = a\sqrt{3}.$$

It follows that the sides of a 30°, 60°, 90° triangle are in the ratio $1 : \sqrt{3} : 2$.

$$\sin 60^\circ = \frac{AD}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2} = \frac{1{\cdot}732}{2} = 0{\cdot}866.$$

$$\therefore\ \operatorname{cosec} 60^\circ = \frac{1}{0{\cdot}866} = 1{\cdot}155.$$

$$\cos 60^\circ = \frac{BD}{AB} = \frac{a}{2a} = 0{\cdot}5. \qquad \therefore\ \sec 60^\circ = 2.$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{a\sqrt{3}}{a} = \sqrt{3} = 1{\cdot}732.$$

$$\therefore\ \cot 60^\circ = \frac{1}{1{\cdot}732} = 0{\cdot}5774.$$

Also, from the same diagram,

$$\sin 30^\circ = \frac{BD}{AB} = \frac{a}{2a} = 0{\cdot}5. \qquad \therefore\ \operatorname{cosec} 30^\circ = 2.$$

$$\cos 30^\circ = \frac{AD}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2} = 0{\cdot}866. \qquad \therefore\ \sec 30^\circ = 1{\cdot}155.$$

$$\tan 30^\circ = \frac{BD}{AD} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}} = 0{\cdot}5774. \qquad \therefore\ \cot 30^\circ = \sqrt{3} = 1{\cdot}732.$$

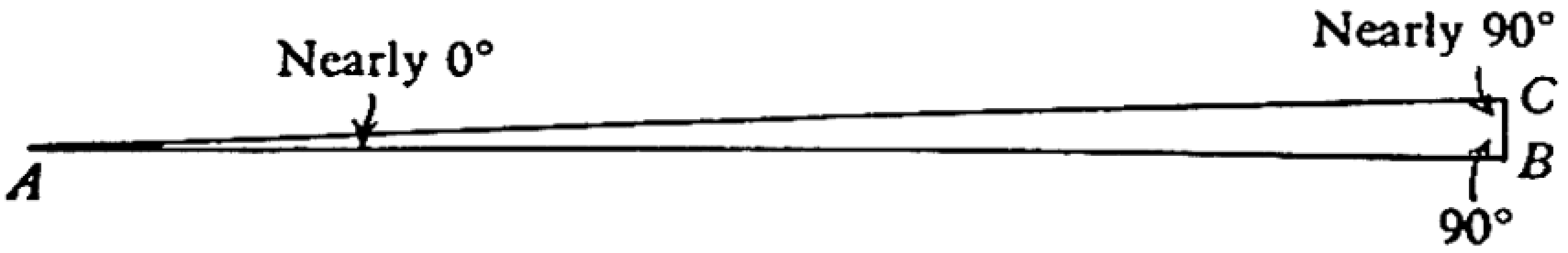


Fig. 153

(3) *Angles* 0° *and* 90°. A right-angled triangle containing another angle of 90° and a third angle 0° becomes merely a straight line. We shall follow the argument better if we draw a right-angled triangle with an angle which is nearly 90°, and the third angle nearly 0°, and consider the values of the ratios as the second and third angles become 90° and 0° respectively. Fig. 153 shows such a triangle. When the angle ACB becomes 90°, the side CB becomes of zero length.

$\sin A = \frac{CB}{AC}$, and when CB becomes zero, $\sin 0° = \frac{O}{AC} = 0$.

$\cos A = \frac{AB}{AC}$, and when CB becomes zero, $AB = AC$, so

$$\cos 0° = \frac{AC}{AC} = 1.$$

$\tan A = \frac{CB}{AB}$, and when CB becomes zero, $\tan 0° = \frac{O}{AB} = 0$.

$\sin C = \frac{AB}{AC}$, and when CB becomes zero, $AB = AC$, so

$$\sin 90° = \frac{AC}{AC} = 1.$$

$\cos C = \frac{CB}{AC}$, and when CB becomes zero, $\cos 90° = \frac{O}{AC} = 0$.

$\tan C = \frac{AB}{CB}$. The ratio is very large, and as CB becomes less in value, the ratio becomes still greater. Suppose $AB = 10$ and $CB = 0{\cdot}1$, then $\frac{AB}{CB} = 100$. When CB decreases to $0{\cdot}01$, then AB/CB becomes 1000. When CB ultimately becomes zero, the ratio AB/CB becomes infinitely great, and $\tan 90°$ is equal to infinity (∞).

It is not advisable to attempt to memorise all these results, but rather the proofs should be fully understood in order that any result may be obtained from first principles when required.

Graphs of sine, cosine and tangent from 0° to 90°

A graph of $y = \sin\theta$ may be plotted by using the values of sines from the tables for ordinates, and angles θ for abscissae. A more interesting method is shown in fig. 154. A quarter circle, radius 1 unit, is divided into nine equal angles of 10°. Radii are drawn through these divisions and perpendiculars drawn to the horizontal from the points where the radii intersect the circumference. Thus the values of $\sin 10°$, $\sin 20°$, $\sin 30°$, etc., are given by the lengths BK, CL, DM, etc., directly (since radius = 1). These lengths are marked as ordinates on the graph corresponding to their appropriate angles. Notice that the sine curve passes through the origin and reaches its peak at 90°, the value here being unity.

The cosine curve is plotted in fig. 155 by a similar method. The radius is again unity, but since the cosines of the angles are the lengths of adjacent sides we arrange to have the adjacent sides vertical by drawing perpendiculars to AO. Hence $OB = \cos 10°$, $OC = \cos 20°$ and so on, so that the lengths OB, OC, …, OJ become lengths of ordinates for the graph. Notice that the cosine curve is the sine curve reversed. Its highest value is unity, at the angle 0°, and it reaches the value zero for the angle 90°.

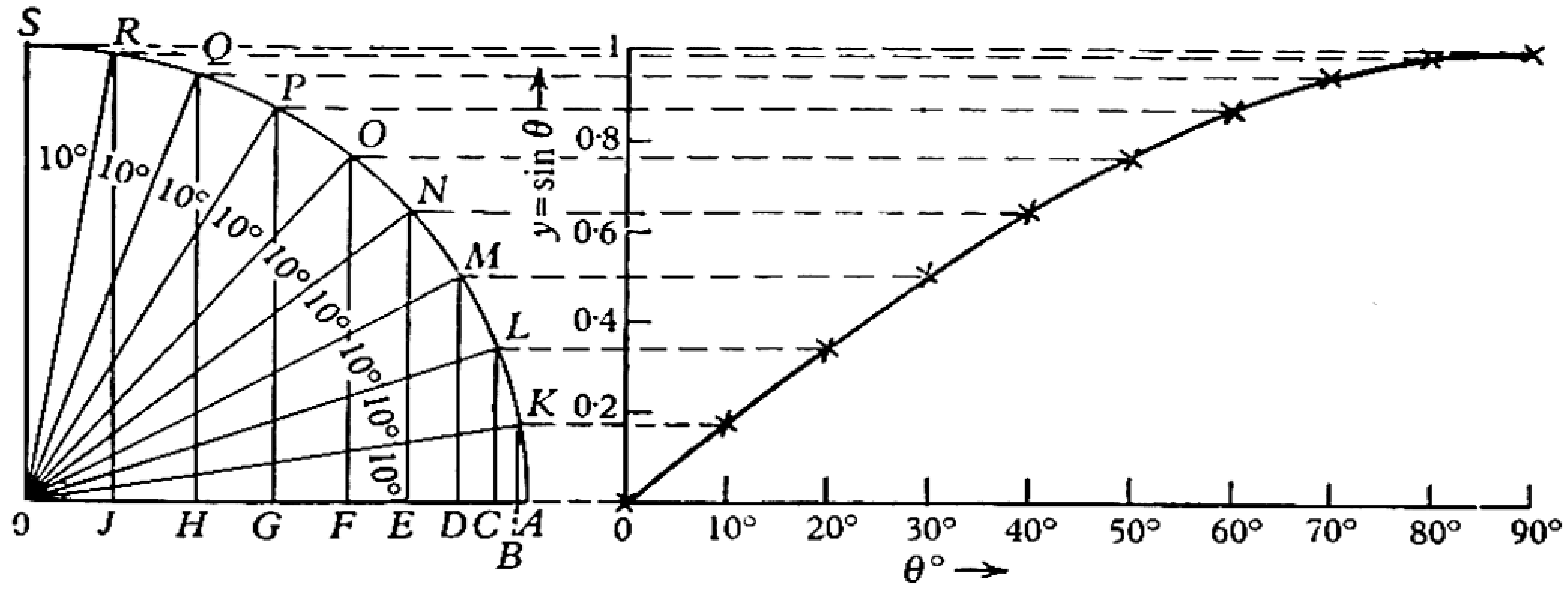


Fig. 154

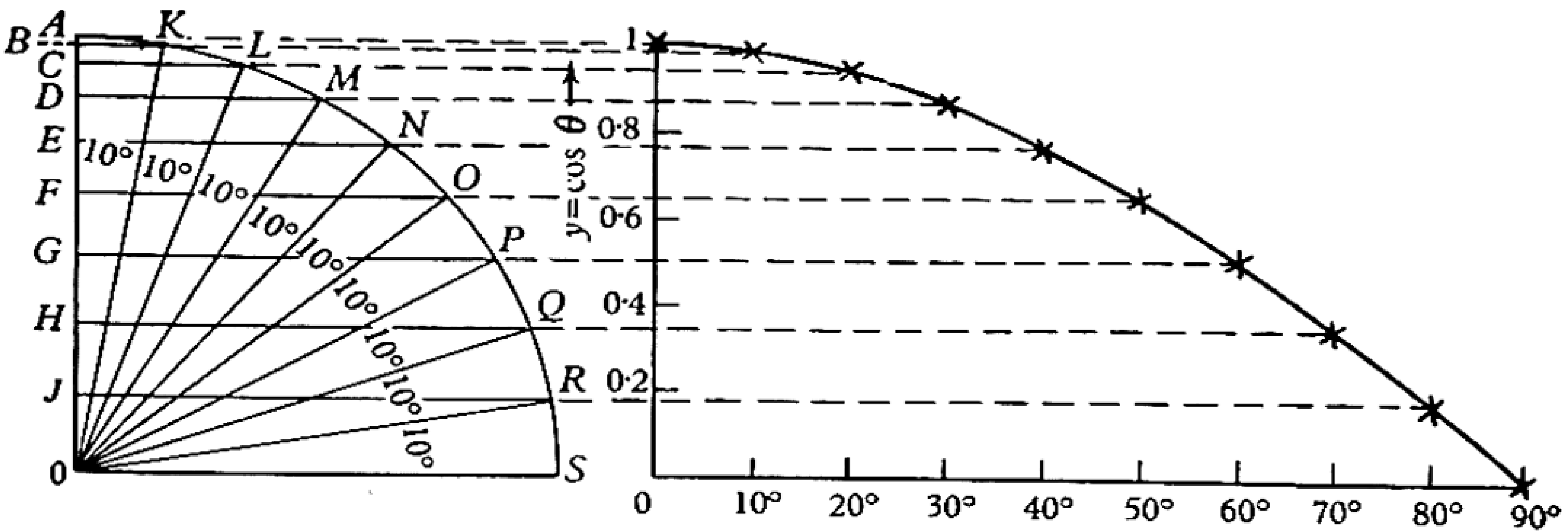


Fig. 155

Imagine the two curves plotted on the same axes. They would obviously intersect at the angle 45°, showing that

$$\sin 45° = \cos 45°.$$

Also, since one curve is the reverse of the other, then

$$\sin 10° = \cos 80°,$$

in fact the sine of an angle is equal to the cosine of its complement. This agrees with what was proved in the previous chapter.

To plot $y = \tan\theta$ by a similar method, we must make the adjacent side unit length (see fig. 156). Since $OA = 1$ unit, the ordinates AB, AC, $AD, \ldots, AG$ represent tan 10°, tan 20°, ..., tan 60° respectively. We cannot complete the graph to 90° because the tangents increase rapidly in value, ultimately reaching the value infinity at 90°. A vertical line through 90° will touch the tangent curve at infinity, so the line $\theta = 90°$ is asymptotic to the curve.

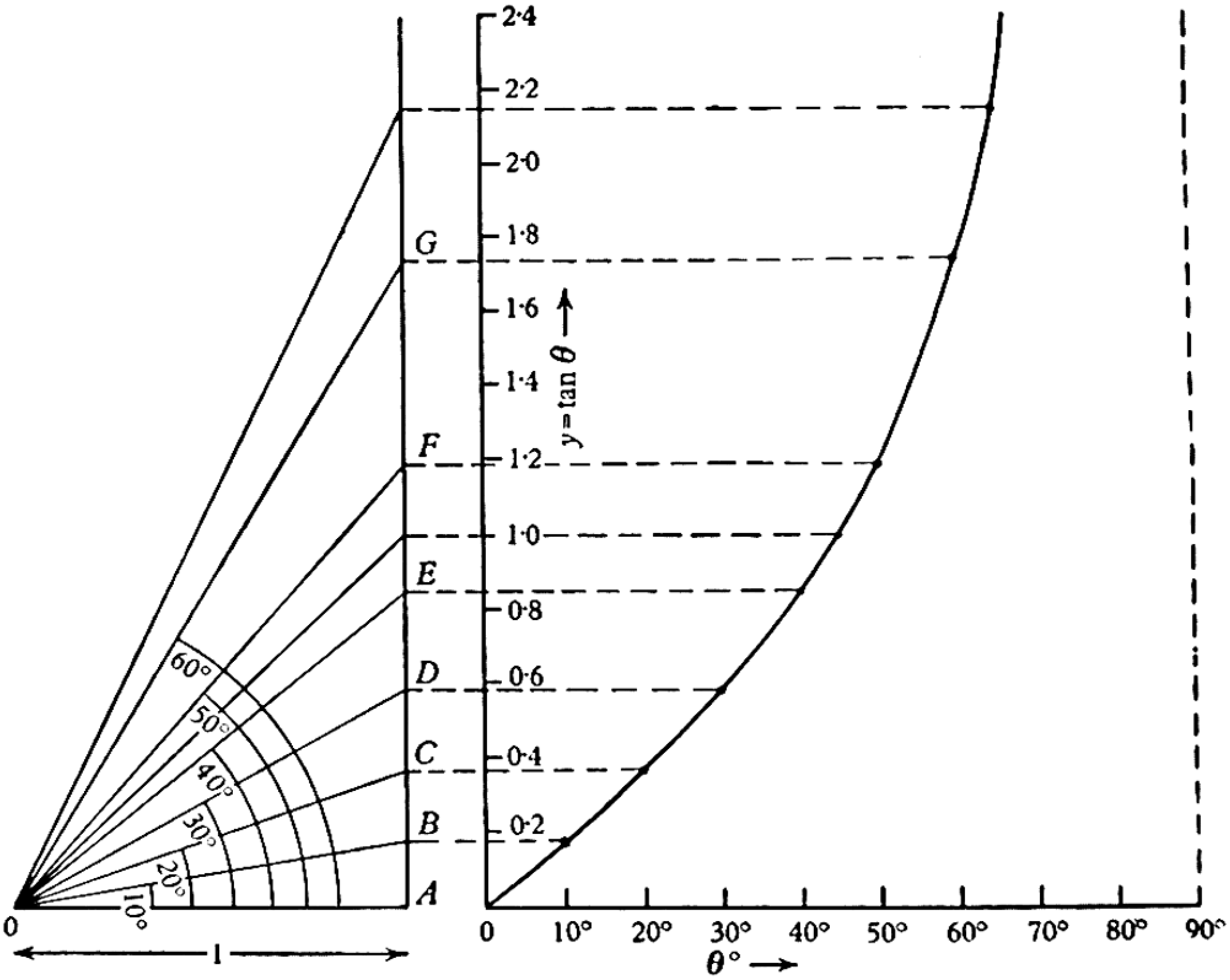


Fig. 156

Vectors

A quantity which has magnitude and direction is called a vector quantity, while a quantity which has magnitude only is called a scalar quantity. Thus 4 lb., 3 ft., 5*s*. are scalar quantities, but a velocity of 3 ft. per sec. in a direction N. 50° E. is a vector quantity, so is a force of 5 lb. wt. acting at 30° to the horizontal. Students will have used vector quantities in mechanics, and will realise that the vectors in common use are velocities, accelerations, forces and momenta. Vectors are added by the parallelogram law, that is, the resultant of two vectors inclined at an angle θ to each other is determined by completing a parallelogram on the two vectors drawn to scale, when the diagonal which passes through their common point represents their resultant in magnitude and direction.

Fig. 157 shows OA, 5 units in length, and OB, 7 units in length, representing forces of 5 lb. wt. and 7 lb. wt. in the directions shown. The two forces have a resultant OC, which is a diagonal of the parallelogram $OBCA$. OC gives the magnitude and direction of a force which could replace OA and OB together. The process of compounding two forces into a single resultant is called *composition* of forces, and the same process may be applied to any other vector quantities. The reverse process, which consists in finding two *components* from a given resultant, is called *resolution* of forces (or vectors generally). Since resolution is usually performed in two directions at right angles to each other, it is a convenient process for us to deal with at the beginning of the study of trigonometry.

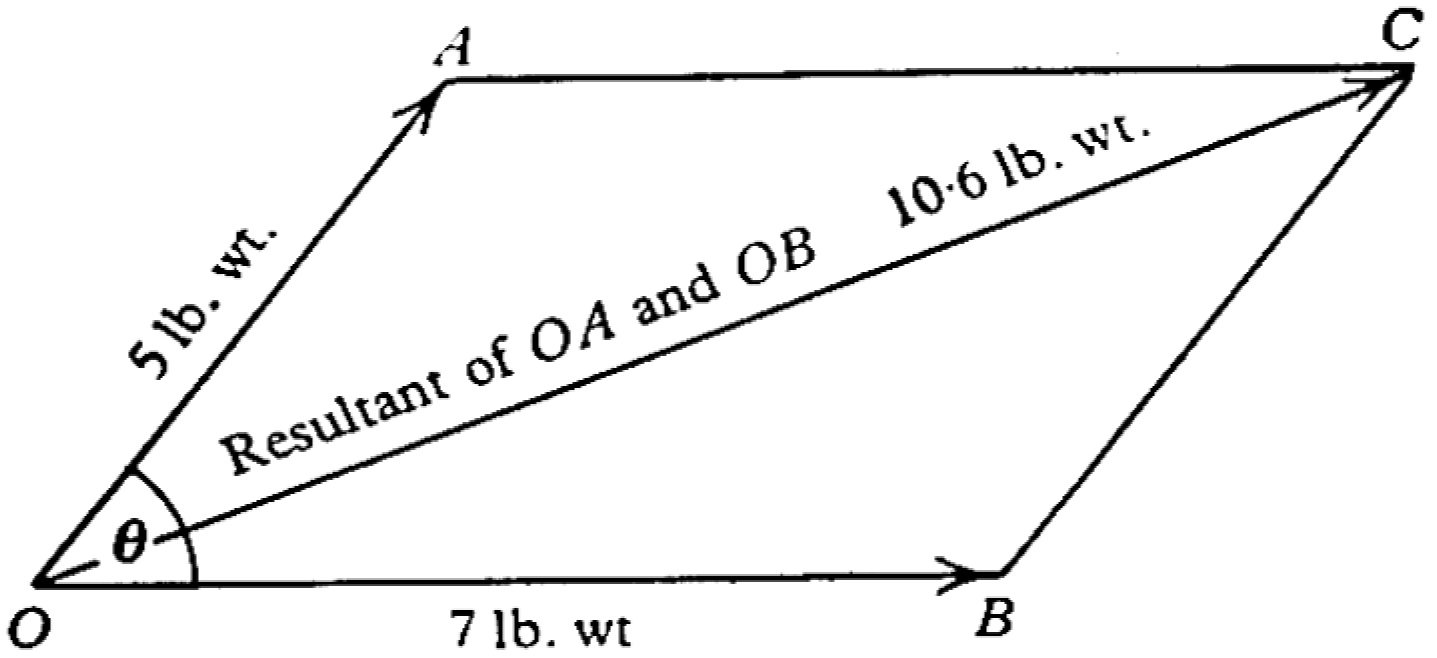


Fig. 157

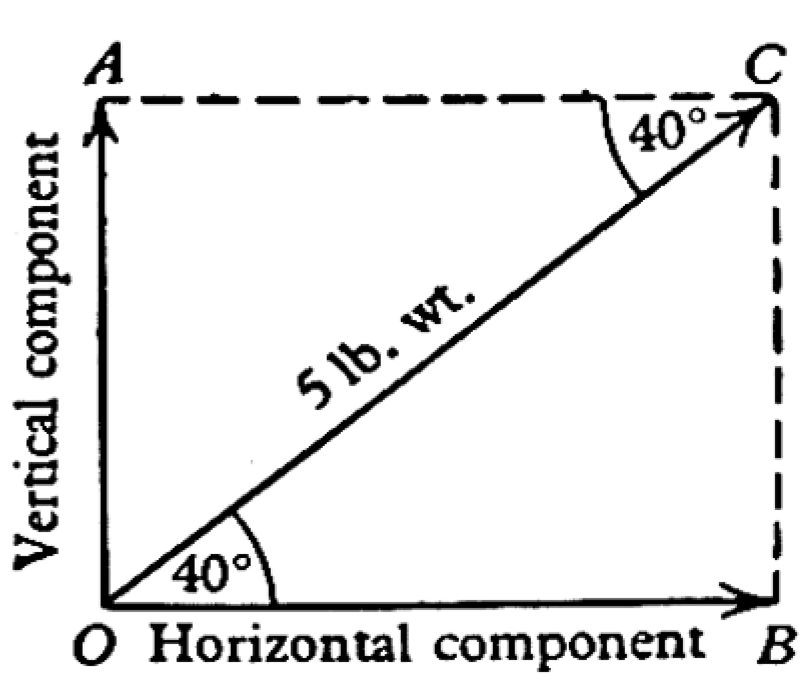


Fig. 158

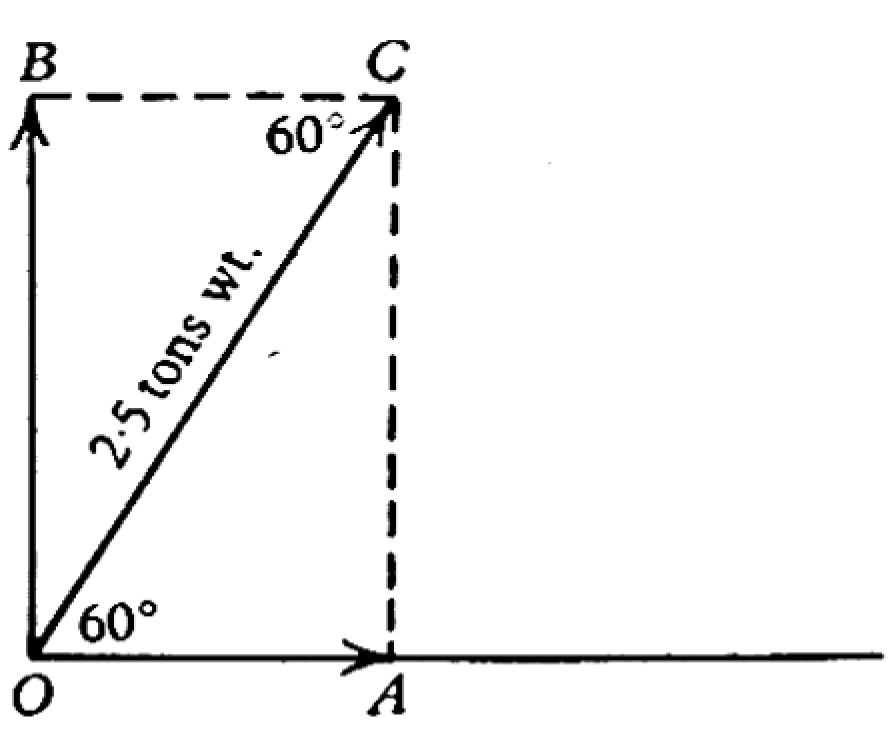


Fig. 159

To find the vertical and horizontal components of a force of 5 lb. wt. inclined at 40° to the horizontal. The horizontal and vertical components OB and OA in fig. 158 are of such lengths that they form two adjacent sides of a parallelogram of which the given force OC is the diagonal. (The parallelogram obviously becomes a rectangle when the components are perpendicular.) A scale drawing is not necessary, for the results required may be obtained from tables.

In the triangle OBC, right-angled at B,

$$\frac{OB}{OC} = \cos 40°, \quad \text{i.e.} \quad \frac{OB}{5} = \cos 40°,$$

therefore

$$OB = 5 \cos 40° = 5 \times 0{\cdot}766 \text{ (from tables)} = 3{\cdot}830 \text{ lb. wt.}$$

In the triangle OAC,

$$\frac{OA}{OC} = \sin 40°, \quad \text{i.e.} \quad \frac{OA}{5} = \sin 40°,$$

therefore $\quad OA = 5 \sin 40° = 5 \times 0{\cdot}6428 = 3{\cdot}2140 \text{ lb. wt.}$

The vertical and horizontal components are thus 3·214 and 3·83 lb. wt.

Example. A force of 2·5 tons wt. acts at 60° to the line of a beam. Find its components along and perpendicular to the beam.

In fig. 159, OA and OB represent these components.

In the triangle OAC,

$$\frac{OA}{OC} = \cos 60° \quad \text{or} \quad \frac{OA}{2{\cdot}5} = \cos 60° = 0{\cdot}5,$$

therefore $\quad OA = 2{\cdot}5 \times 0{\cdot}5 = 1{\cdot}25 \text{ tons wt.}$

In the triangle OBC,

$$\frac{OB}{OC} = \sin 60° \quad \text{or} \quad \frac{OB}{2{\cdot}5} = 0{\cdot}866,$$

therefore $\quad OB = 2{\cdot}5 \times 0{\cdot}866 = 2{\cdot}165 \text{ tons wt.}$

The components along and perpendicular to the beam are 1·25 and 2·165 tons wt.

Examples for Chapter 33

1. A weight of 8 lb. rests on a plank inclined at an angle θ to the horizontal. The weight is considered to be acting vertically downwards. Show that the components of the weight along the plank and perpendicular to the plank are $8 \sin\theta$ and $8 \cos\theta$ respectively. Find these components if $\theta = 30°$.

The frictional force between weight and plank is μR, where R is the component of the weight perpendicular to the plank, and μ is the coefficient of friction for the surfaces. Find the frictional force if $\mu = 0{\cdot}7$. (*Chap.* 33)

2. Two hawsers used for towing a vessel are inclined at angles of 40° to the direction of motion of the ship, one on either side of the line of motion. If the tension in each hawser is 10·5 tons wt., find the resultant force in the direction of motion. What would be the total pull on the vessel if the hawsers were inclined at 45° to the direction of motion? (*Chap.* 33)

3. One side of an equilateral triangle is of length a. Prove that the altitude of the triangle is $\frac{1}{2}a \tan 60°$.

Hence find the area of the triangle, and prove that the area of a hexagon is $\frac{3a^2\sqrt{3}}{2}$, or $2{\cdot}6a^2$ (approx.), where a is the length of one side. (*Chap.* 33)

4. An isosceles triangle contains two sides 2 in. long, and two angles of $67\frac{1}{2}°$. Show that its area is given by

$$2 \sin 22\tfrac{1}{2}° \times 2 \cos 22\tfrac{1}{2}°.$$

Hence calculate the area of a regular octagon inscribed in a circle, radius 2 in. (*Chap.* 33)

5. From the table of sines, work out $\frac{1}{\sin 10°}$, $\frac{1}{\sin 20°}$, up to $\frac{1}{\sin 90°}$.

With these results plot the graph of $y = \operatorname{cosec}\theta$ from 10° to 90°. What would be the value of cosec 0°? What is the least ordinate on the graph? (*Chap.* 33)

6. A crane jib is inclined at an angle θ to the vertical post and is 10 ft. long. The foot of the jib is attached to the post 5 ft. from the bottom of the post. The tie is horizontal and is attached to the top of the jib. Show that the length of the tie is $10 \sin\theta$ ft., and the height of the top of the jib above the level of the foot of the post is $5 + 10 \cos\theta$ ft.

Calculate the length of the tie and the height of the top of the jib when $\theta = 35°$. (*Chap.* 33)

7. Draw a triangle ABC in which angle $C = 90°$, $BC = 10$ units and angle $B = 50°$. Use the diagram to calculate tan 50° and sin 50° as accurately as possible.

Bisect angle A and let the bisector meet BC in D. Show that the value of DC/BD is equal to sin 50°.

Using lengths of sides in the triangles ABC and ADC, show that tan 40° is not the same as twice tan 20°, and that sin 40° is not twice sin 20°. (*Chap.* 32)

8. A solid sphere, radius 3 in., is placed into a rectangular trough containing enough water to immerse the sphere completely. If the base of the trough is a square of 4 in. side, show that the water level will rise $\frac{9\pi}{4}$ in. in the trough.

If the trough had been full of water before the sphere was put in, show that the water which would overflow could be contained in a cone of radius 3 in. and height 12 in. (*Chap.* 31)

9. A hollow conical vessel is formed by folding a sector of a circle of radius 12 in., containing an angle of 120°. Find the area of curved surface and the volume of the cone so formed. (*Chaps.* 29 *and* 30)

10. The bending moment, M, at any point of a uniform beam supported at both ends is $M = Rx - ax^2$, where R is the reaction at one support, a is a constant depending upon the weight of the beam, and x represents the distance from one end. Plot M (lb. ft.) against x ft. for the case where $R = 10$ and $a = 1$, taking x from 1 to 10. Find at what value of x the bending moment is greatest, and find the maximum bending moment. If the actual length of the beam is 10 ft., what is the bending moment at the ends? (*Chap.* 28)

Answers to Examples for Chapter 33

1. Along the plank 4 lb. wt. Perpendicular to the plank 6·928 lb. wt. Friction = 4·85 lb. wt.

2. 16·086 tons wt.; 14·849 tons wt.

4. 11·3136 sq.in.

5. cosec 0° = ∞; cosec 90° = 1 (least ordinate).

6. 5·736 ft.; 13·192 ft.

9. 150·8 sq.in.; 189·5 cu.in.

10. $x = 5$ ft.; maximum bending moment = 25 lb. ft.; bending moment at both ends is zero.

CHAPTER 34

PROBLEMS ON HEIGHTS AND DISTANCES

In order to find the length of any side of a right-angled triangle, it is sufficient to know the value of one angle other than the right angle, and the length of one side. It is always possible to connect the side required and the side known through a trigonometric ratio of the given angle.

Fig. 160 shows triangles (1), (2) and (3), in each of which the right angle is B, and A is the given angle, so in all three cases AC is the hypotenuse, AB is the adjacent side, and BC is the opposite side.

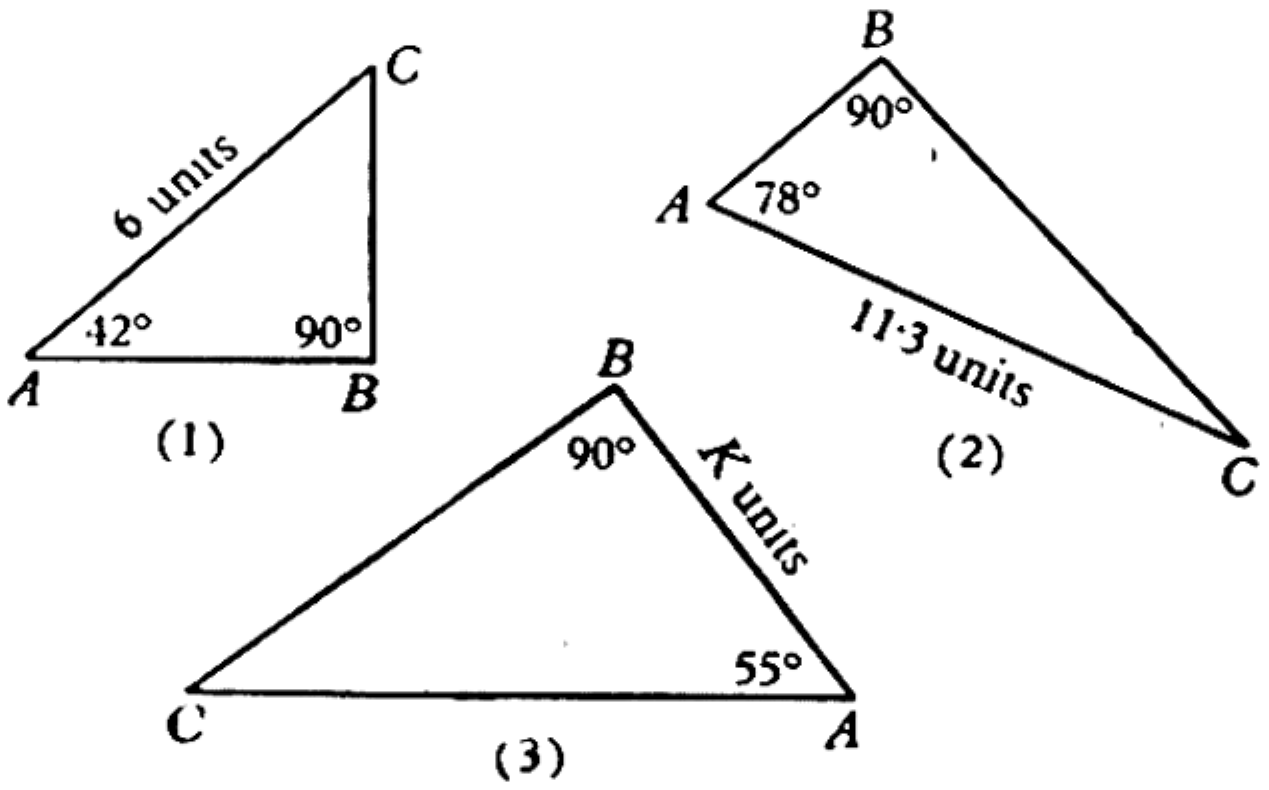


Fig. 160

Consider triangle (1). To find the side AB. Now

$$AB/AC = \cos 42°, \quad \text{or} \quad AB = AC \cos 42°.$$

Since AC is 6, $AB = 6 \cos 42, = 6 \times 0{\cdot}7431 = 4{\cdot}4586$.

Similarly, the side BC is equal to $AC \sin 42° = 6 \sin 42°$, which can be evaluated similarly from the table of sines.

In triangle (2), $AB/AC = \cos 78°$, or $AB = 11{\cdot}3 \cos 78°$.

Also $BC/AC = \sin 78°$, or $BC = 11{\cdot}3 \sin 78°$.

In triangle (3), required the side BC. Now $BC/AB = \tan 55°$, or $BC = K \tan 55°$.

To find the side AC, note that

$$AB/AC = \cos 55°, \quad \text{or} \quad K = AC \cos 55°,$$

therefore

$$AC = \frac{K}{\cos 55^\circ} \quad \text{(dividing both sides by cos 55°)}$$
$$= K \sec 55^\circ.$$

Angles of elevation and depression

When an observer requires to raise his line of sight from the horizontal to view an object, then the angle through which the line of sight is turned from the *horizontal* is called the angle of elevation. When the line of sight is turned downwards through an angle, the angle between the *horizontal* and the line of sight is called the angle of depression. These terms are used frequently and should be clearly understood (see fig. 161).

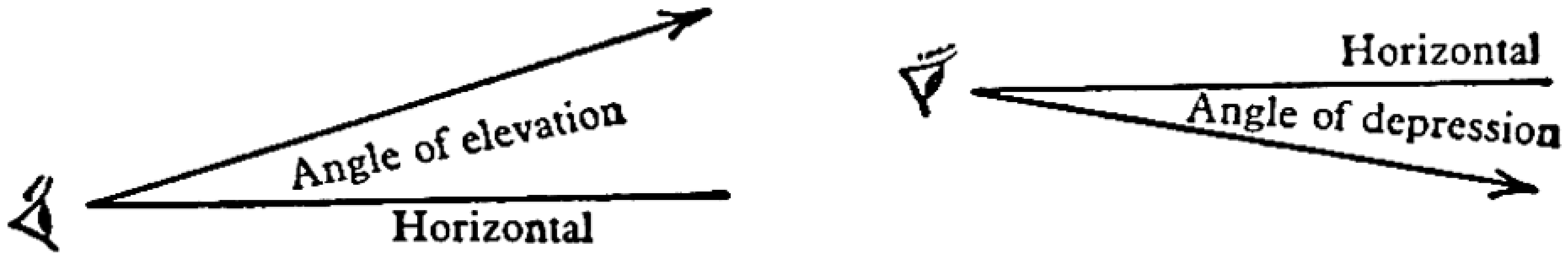


Fig. 161

Example. A pylon stands on level ground. At a point 200 ft. from the foot of the pylon, the elevation of its highest point is 37°. Find the height of the pylon.

Let CB (fig. 162) represent the pylon.

Then
$$\frac{CB}{AB} = \tan 37^\circ,$$

i.e.
$$CB = AB \tan 37^\circ = 200 \tan 37^\circ$$
$$= 200 \times 0{\cdot}7536 \text{ (from tables)}$$
$$= \underline{150{\cdot}72 \text{ ft.}}$$

$$\underline{\text{Height of pylon} = 150{\cdot}7 \text{ ft. (approx.).}}$$

Example. The legs of a tripod camera stand are 4 ft. 8 in. in length, and the angle between any pair of legs is 32°. Find the distance apart of the feet.

Let AB and AC be one pair of legs (fig. 163). Let AD be perpendicular to BC.

Now $\frac{BD}{AB} = \sin 16^\circ$, or $BD = AB \sin 16^\circ = 56 \sin 16^\circ$ (in in.)

$$= 56 \times 0{\cdot}2756 \text{ in.}$$
$$= 15{\cdot}43 \text{ in.}$$

$$
\begin{aligned}
\text{Distance between the legs} &= 2 \times 15{\cdot}43 \text{ in.} \\
&= 30{\cdot}86 \text{ in.} \\
&= \underline{30{\cdot}9 \text{ in. (approx.).}}
\end{aligned}
$$

Problems involving more than one right-angled triangle

Example. From a window 12 yd. above the ground an observer notices that the angle of depression of the foot of a building across the road (which is level) is 27°, and that the elevation of the top of the same building is 31°. Find the height of the building and the width of the road.

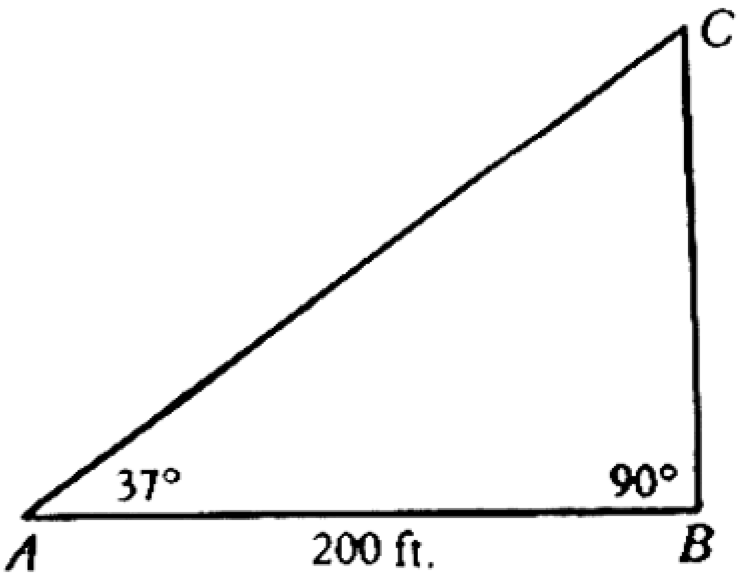


Fig. 162

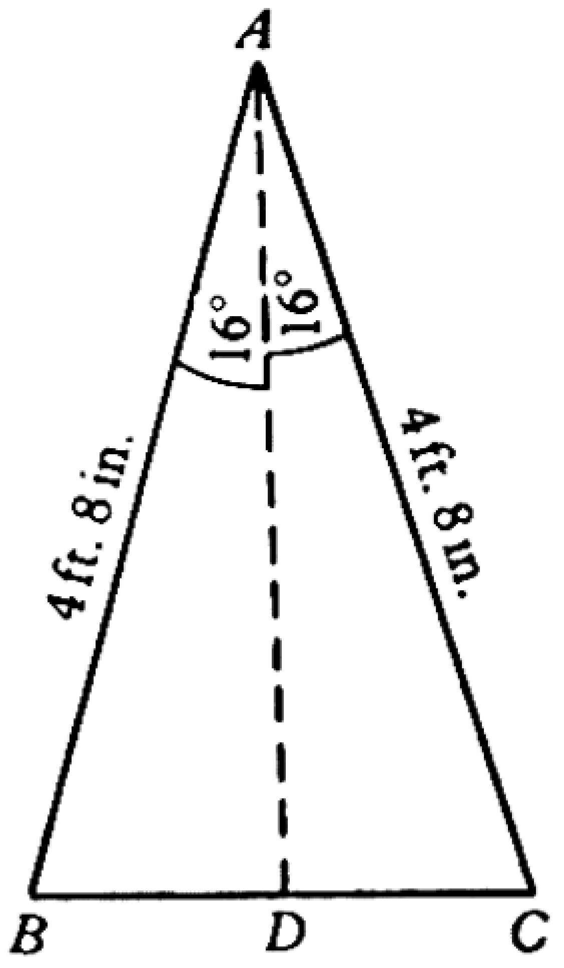


Fig. 163

Let O be the observer, AB the street, and BC the opposite building, in fig. 164. Let OD be a horizontal through O.

It will be convenient to find the width of the road first.

In triangle ABO, $OA/AB = \tan 27°$,

or $\qquad 12 = AB \tan 27°$

$$
\begin{aligned}
\text{i.e.} \qquad AB &= \frac{12}{\tan 27°} \qquad \text{(Dividing both sides by } \tan 27°\text{)} \\
&= 12 \cot 27° \\
&= 12 \tan 63° \qquad \text{(because } \cot 27° = \tan 63°\text{)} \\
&= 12 \times 1{\cdot}9626 \text{ yd.} \\
&= 23{\cdot}5512 \text{ yd.} \\
&\doteqdot 23{\cdot}55 \text{ yd.}
\end{aligned}
$$

Now $AB = OD$, so in triangle ODC,

$$\frac{CD}{OD} = \tan 31°,$$

i.e. $$CD = 23{\cdot}55 \tan 31°$$

$$= 23{\cdot}55 \times 0{\cdot}6009$$

$$= 14{\cdot}16 \text{ yd.}$$

Numbers	Logs
23·55	1·3720
0·6009	$\bar{1}{\cdot}7788$
14·16	1·1508

The height of the building is the sum of CD and BD, i.e. $12 + 14{\cdot}16$, or 26·16 yd. Thus, height of the building = 26·16 yd., and the width of the road = 23·55 yd.

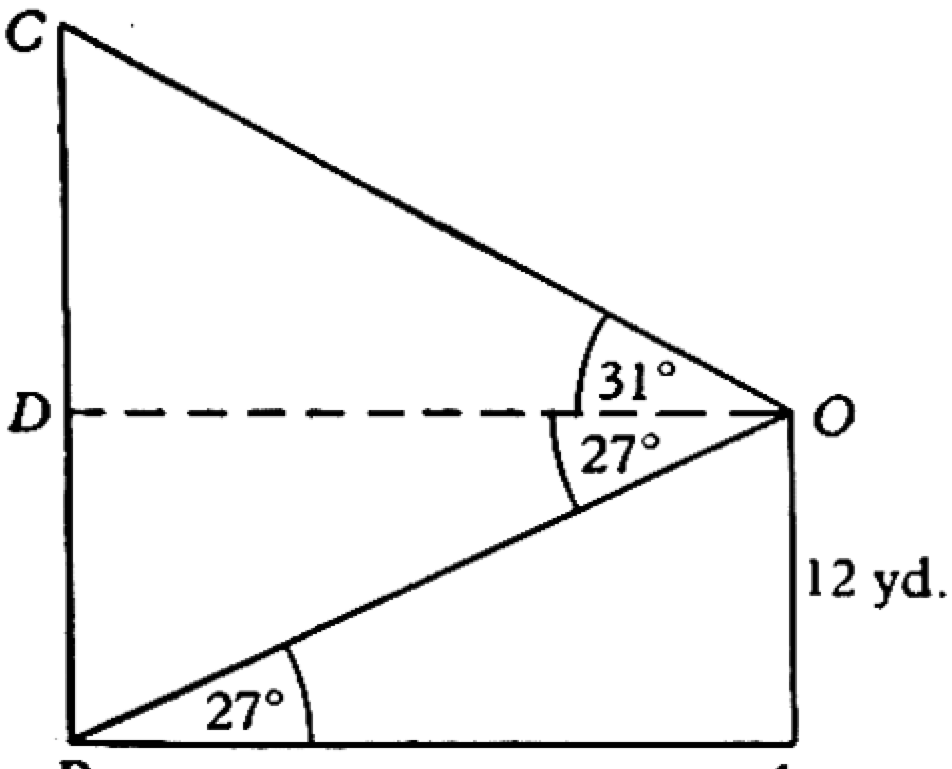


Fig. 164

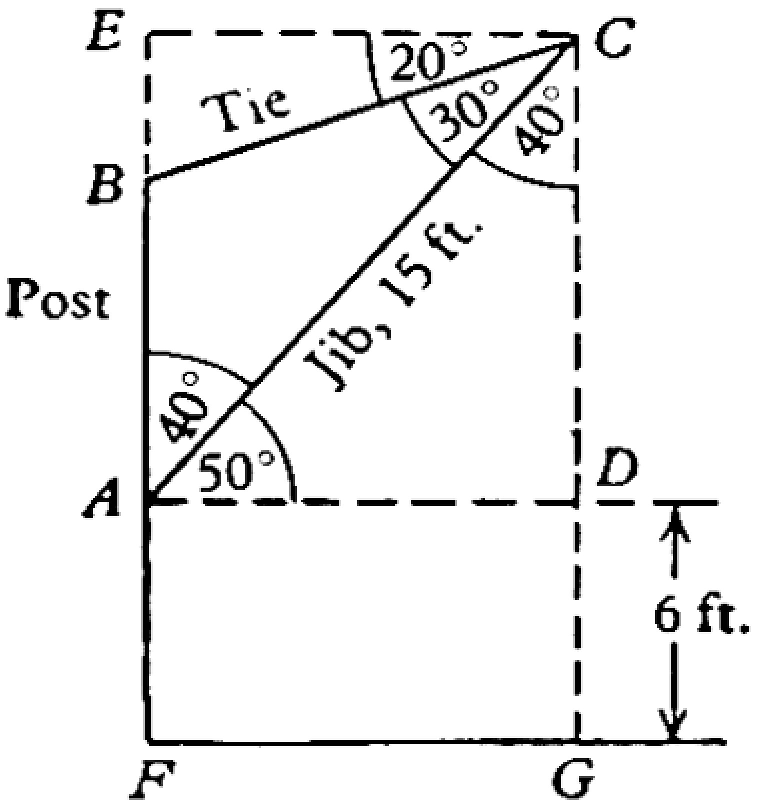


Fig. 165

Example. The jib of a crane, 15 ft. long, makes an angle of 40° with the vertical post to which it is attached at a point 6 ft. above ground level. The tie makes an angle of 30° with the jib. Find the height of the top of the jib above ground level, and the length of the tie.

To find the height of the jib top above ground level requires the length CD (fig. 165).

In the triangle ACD, $\dfrac{CD}{AC} = \sin 50°$,

i.e. $$CD = 15 \times 0{\cdot}766$$

$$= 11{\cdot}49 \text{ ft.}$$

Therefore, height of top of jib above ground level

$$= 11{\cdot}49 + 6 \text{ ft.}$$

$$= 17{\cdot}49 \text{ ft.}$$

To find the length of the tie, the length EC is required first.

But $EC = AD$, and $\dfrac{AD}{AC} = \cos 50°$, so

$$\begin{aligned} AD &= 15 \cos 50° \\ &= 15 \times 0{\cdot}6428 \\ &= 9{\cdot}64 \text{ ft.} \end{aligned}$$

Numbers	Logs
9·64	0·9841
0·9397	$\bar{1}$·9730
10·26	1·0111

In the triangle ECB, $\dfrac{EC}{CB} = \cos 20°$, i.e.

$$\frac{9{\cdot}64}{CB} = \cos 20°,$$

$$\begin{aligned} \therefore \quad CB &= \frac{9{\cdot}64}{\cos 20°} = \frac{9{\cdot}64}{0{\cdot}9397} \\ &= 10{\cdot}26 \text{ ft.} \end{aligned}$$

Length of the tie = 10·26 ft.

Example. In order to find the height of a hill, two points A and B are chosen, 880 yd. apart, on level ground, so that the line AB is in the direction of the hill. The elevations of the summit from A and B respectively are 18° and 32°. Find the height of the hill above the level of AB.

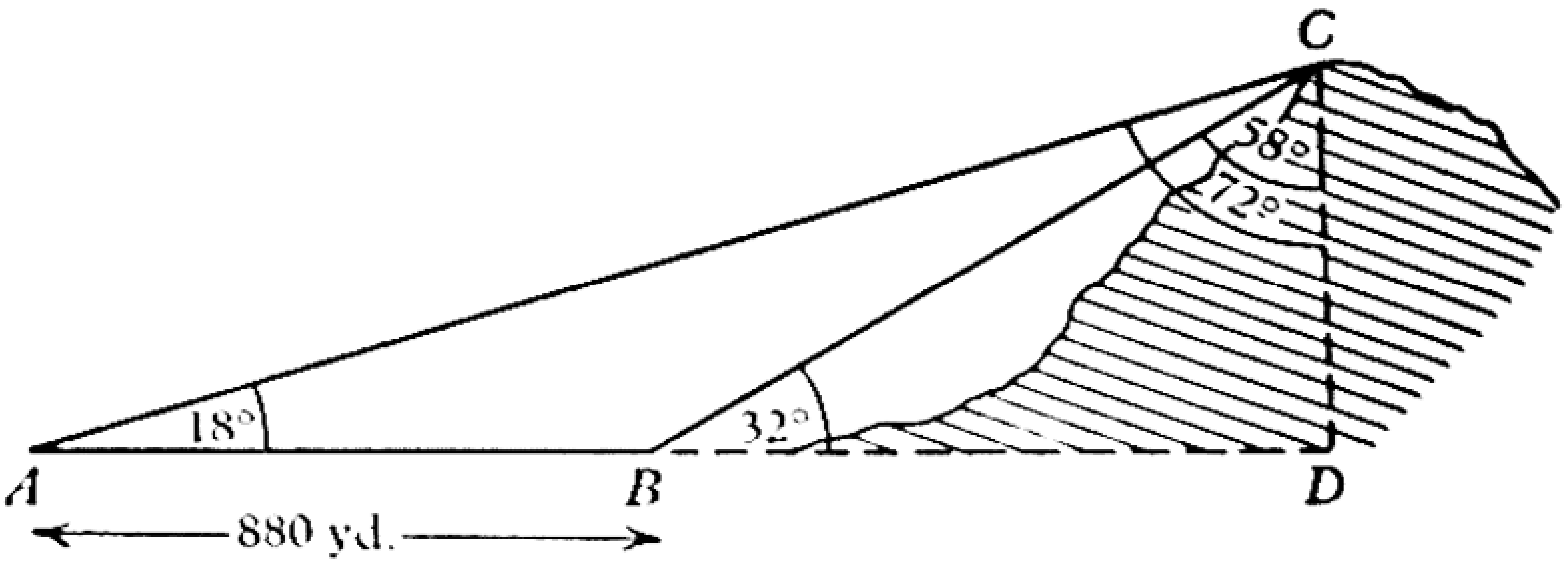


Fig. 166

Let CD (fig. 166) be the height of the hill.

In the triangle ACD,

$$\frac{AD}{CD} = \tan 72°, \quad \text{or} \quad AD = CD \tan 72°. \tag{1}$$

In the triangle BCD,

$$\frac{BD}{CD} = \tan 58°, \quad \text{or} \quad BD = CD \tan 58°. \tag{2}$$

Now $AB = AD - BD$, i.e.

$$\begin{aligned} 880 &= AD - BD \\ &= CD\tan 72^\circ - CD\tan 58^\circ \\ &\qquad\qquad \text{(from (1) and (2))} \\ &= CD(\tan 72^\circ - \tan 58^\circ) \\ &= CD(3{\cdot}0777 - 1{\cdot}6003) \\ &= CD \times 1{\cdot}4774. \end{aligned}$$

Numbers	Logs
880	2·9445
1·4774	0·1693
596·0	2·7752

Therefore

$$\begin{aligned} CD &= \frac{880}{1{\cdot}4774} \quad \text{(dividing both sides by } 1{\cdot}4774) \\ &= 596{\cdot}0 \text{ yd.} \end{aligned}$$

Height of hill = 1788 ft.

Examples for Chapter 34

1. (*a*) The hypotenuse AC of a right-angled triangle is 4 in. long, and the angle A is 44°. Write the lengths of AB and BC.

(*b*) In a triangle ABC, right-angled at B, the side AB is K units in length, and the angle A is θ°. Write the lengths of AC and BC. (*Chap.* 34)

2. (*a*) A is a point on a straight road running due north. From A, the bearing of a building 4 miles away is N. 30° W. Find how far the building is away from the road.

(*b*) A straight path runs in a direction N. 40° E. from the building to join the main road. Find the length of the path. (*Chap.* 34)

3. The elevations of a tower from two points, one due east of the tower, and the other due west of it, are 18° and 20° respectively, and the two points of observation are 1000 ft. apart. Find the height of the tower. (*Chap.* 34)

4. Find the lengths of the diagonals, and the area of a rhombus of side 3 in., and containing an angle of 50°. (*Chap.* 34)

5. Find the total length of girders in the framework in fig. 167 (p. 328), if $CB = BD = 10$ ft. (*Chap.* 34)

6. An aircraft is flying on a straight course at a height of 3700 ft. After it has passed over the head of an observer he measures its elevation as 70°. Ten seconds later the elevation is 50°. Show that the aircraft is flying at approximately 120 m.p.h. (*Chap.* 34)

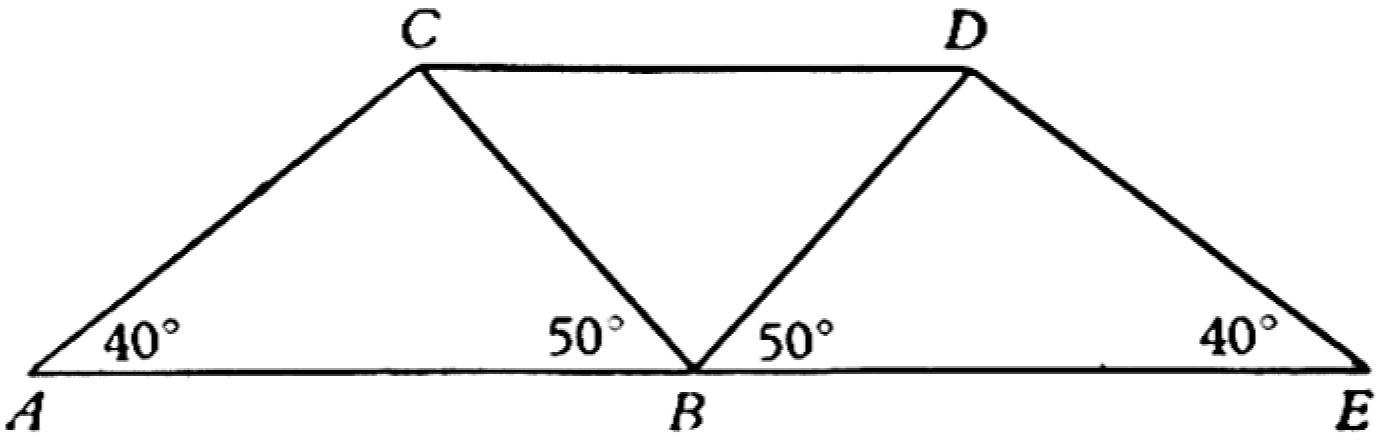


Fig. 167

7. A piece of wire 1 m. long, and of average diameter 1 mm., weighs 6·677 g. Find the density of the metal. (*Chap.* 31)

8. Plot $y = x^3 + x^2 - 2x$ from $x = -3$ to $x = +3$ in half-units. From the graph read off the values of x for which

$$x^3 + x^2 - 2x = 0. \qquad (Chap.\ 28)$$

9. Plot the points $(-2, 4)$, $(-2, 0)$, $(3, 6)$ and $(3, 2)$ on the same axes. Write the equations of the diagonals of the parallelogram which has these four corners, and show that the equations may be expressed as $5y + 2x = 16$, $5y - 6x = 12$.

Show, by substitution, that the point $(\frac{1}{2}, 3)$ lies on both lines, and verify this by reading the co-ordinates of their point of intersection from the graph. (*Chap.* 27)

10. Draw any right-angled triangle containing an angle of 50°. From the drawing calculate sin 50°, cos 50° and tan 50°. Calculate also sin 40°, cos 40° and tan 40°. Verify by division that the sine of an angle divided by the cosine of the same angle is equal to the tangent of the angle. (*Chap.* 32)

Answers to Examples for Chapter 34

1. (*a*) $AB = 2{\cdot}8772$ in.; $BC = 2{\cdot}7788$ in.
   (*b*) $AC = K\sec\theta$, $BC = K\tan\theta$.

2. (*a*) 2 miles. (*b*) 3·11 miles.

3. 171·5 ft. [Let distance from west point to foot of tower be x, then distance from east point to tower is $(1000-x)$. Now height $= x \tan 20°$, also height $= (1000-x)\tan 18°$. Equate right-hand sides and solve for x. Then use this result in right-angled triangle to find height.]

4. 2·5356 in. and 5·4378 in. Area 6·895 sq.in.

5. 87·8 ft.

7. 8·5 g. per c.c.

8. $x = 0$, $x = -2$, $x = 1$.

CHAPTER 35

TRIGONOMETRIC IDENTITIES

We have seen in Chapter 11 that an identity is an equality which is true under all conditions, and have illustrated there a few algebraic identities which are true whatever values are given to the letter or letters concerned. Trigonometric identities are similar statements of equality which are true whatever values are given to the angles mentioned in them. Such identities will prove useful in later work when complex expressions require to be simplified, so, while their practical use is not immediately apparent, they should not be neglected.

It is customary to represent a general angle by θ (*theta*). Other Greek letters such as ϕ (*phi*), α (*alpha*), β (*beta*), γ (*gamma*) may be used, or merely ordinary capital letters according to choice.

Four fundamental identities will be proved first. The truth of them may then be assumed in problems.

To prove that $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{and} \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}.$$

Thus

$$\frac{\sin\theta}{\cos\theta} = \frac{\dfrac{\text{opposite}}{\text{hypotenuse}}}{\dfrac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\text{opposite}}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\text{opposite}}{\text{adjacent}}$$

$$= \tan\theta. \qquad \text{Q.E.D.}$$

Also, since $\quad \cot\theta = \dfrac{1}{\tan\theta}, \quad \cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

To prove that $\sin^2\theta + \cos^2\theta = 1$. [Note that $\sin^2\theta$ is the correct way to write $(\sin\theta)^2$. A trigonometric ratio raised to any positive power has the index written after the word expressing the ratio.]

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \text{therefore} \quad \sin^2\theta = \frac{(\text{opposite})^2}{(\text{hypotenuse})^2},$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \text{therefore} \quad \cos^2\theta = \frac{(\text{adjacent})^2}{(\text{hypotenuse})^2}.$$

Therefore

$$\sin^2\theta+\cos^2\theta = \frac{(\text{opposite})^2}{(\text{hypotenuse})^2}+\frac{(\text{adjacent})^2}{(\text{hypotenuse})^2}$$

$$= \frac{(\text{opposite})^2+(\text{adjacent})^2}{(\text{hypotenuse})^2}$$

$$= \frac{(\text{hypotenuse})^2}{(\text{hypotenuse})^2} \quad \text{(by the theorem of Pythagoras)}$$

$$= 1. \qquad \text{Q.E.D.}$$

To prove that $\sec^2\theta = 1+\tan^2\theta$.

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}, \quad \text{therefore} \quad \sec^2\theta = \frac{(\text{hypotenuse})^2}{(\text{adjacent})^2},$$

$$\sec^2\theta = \frac{(\text{adjacent})^2+(\text{opposite})^2}{(\text{adjacent})^2}$$

(by the theorem of Pythagoras)

$$= \frac{(\text{adjacent})^2}{(\text{adjacent})^2}+\frac{(\text{opposite})^2}{(\text{adjacent})^2}$$

$$= 1+\tan^2\theta. \qquad \text{Q.E.D.}$$

To prove that $\operatorname{cosec}^2\theta = 1+\cot^2\theta$.

$$\operatorname{cosec}\theta = \frac{\text{hypotenuse}}{\text{opposite}}, \quad \text{therefore} \quad \operatorname{cosec}^2\theta = \frac{(\text{hypotenuse})^2}{(\text{opposite})^2},$$

$$\operatorname{cosec}^2\theta = \frac{(\text{opposite})^2+(\text{adjacent})^2}{(\text{opposite})^2} = \frac{(\text{opposite})^2}{(\text{opposite})^2}+\frac{(\text{adjacent})^2}{(\text{opposite})^2}$$

$$= 1+\cot^2\theta. \qquad \text{Q.E.D.}$$

Problems on identities are solved by the use of these facts. It is neither necessary nor desirable to use the lengths of sides when once the fundamental identities have been accepted.

In addition to the identities above, the student must be prepared to use the facts that $\sec\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$ are reciprocals of $\cos\theta$, $\sin\theta$ and $\tan\theta$ respectively.

For convenience, and for reference, a full list of the identities to be known is given and numbered below.

(1) $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$, (2) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$,

(3) $\cos\theta = \dfrac{1}{\sec\theta}$, (4) $\sec\theta = \dfrac{1}{\operatorname{cosec}\theta}$,

(5) $\tan\theta = \dfrac{1}{\cot\theta}$, (6) $\cot\theta = \dfrac{1}{\tan\theta}$,

(7) $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$, (8) $\dfrac{\cos\theta}{\sin\theta} = \cot\theta$,

(9) $\sin^2\theta+\cos^2\theta = 1$, (10) $\sec^2\theta = 1+\tan^2\theta$,

(11) $\operatorname{cosec}^2\theta = 1+\cot^2\theta$.

To prove an identity it is usual to make use of any of the above facts coupled with any algebraic processes necessary to make the left-hand side identical with the right-hand side. When the left-hand side has been shown to be identical with the right-hand side, the identity is proved.

Example. Prove that $\sin\theta = \sqrt{(1-\cos^2\theta)}$.

Left-hand side (L.H.S.) = $\sin\theta$. Also

$$\sin^2\theta+\cos^2\theta = 1 \quad \text{(from (9))},$$
$$\therefore\ \sin^2\theta = 1-\cos^2\theta,$$
$$\therefore\ \sin\theta = \sqrt{(1-\cos^2\theta)},$$
$$\therefore\ \text{L.H.S.} = \text{R.H.S.} \quad \text{Q.E.D.}$$

Example. Prove that $\dfrac{\cos\theta}{1-\cos^2\theta} = \cot\theta\operatorname{cosec}\theta$.

$$\text{L.H.S.} = \frac{\cos\theta}{1-\cos^2\theta}$$
$$= \frac{\cos\theta}{\sin^2\theta} \quad \text{(from (9))}$$
$$= \frac{\cos\theta}{\sin\theta}\frac{1}{\sin\theta}$$
$$= \cot\theta\operatorname{cosec}\theta \quad \text{(from (8) and (2))}$$
$$= \text{R.H.S.} \quad \text{Q.E.D.}$$

Example. Prove that $\dfrac{\sin^2\theta\cos^2\theta+\sin^4\theta}{\cos^2\theta} = \sec^2\theta\sin^2\theta$.

L.H.S. $= \dfrac{\sin^2\theta\cos^2\theta + \sin^4\theta}{\cos^2\theta}$

$= \dfrac{\sin^2\theta(\cos^2\theta + \sin^2\theta)}{\cos^2\theta}$ (Extracting common factor)

$= \dfrac{\sin^2\theta \times 1}{\cos^2\theta}$ (From (9), $\sin^2\theta + \cos^2\theta = 1$)

$= \sec^2\theta\sin^2\theta$ (from (4))

= R.H.S. Q.E.D.

Example. Prove that $\dfrac{\sec\theta}{\sin\theta} - \dfrac{1}{\tan\theta} = \tan\theta$.

L.H.S. $= \dfrac{\sec\theta}{\sin\theta} - \dfrac{1}{\tan\theta}$

$= \dfrac{1}{\cos\theta\sin\theta} - \dfrac{\cos\theta}{\sin\theta}$ (from (4) and (7))

$= \dfrac{1-\cos^2\theta}{\cos\theta\sin\theta}$

$= \dfrac{\sin^2\theta}{\cos\theta\sin\theta}$ (from (9))

$= \dfrac{\sin\theta}{\cos\theta} = \tan\theta =$ R.H.S. Q.E.D.

Example. Prove that $\operatorname{cosec}^2\theta\sec^2\theta = \operatorname{cosec}^2\theta + \sec^2\theta$.

L.H.S. $= \operatorname{cosec}^2\theta\sec^2\theta$

$= (1+\cot^2\theta)(1+\tan^2\theta)$ (from (10) and (11))

$= 1+\cot^2\theta+\tan^2\theta+\cot^2\theta\tan^2\theta$

$= 1+\cot^2\theta+\tan^2\theta+1$ (from (6))

$= 2+\cot^2\theta+\tan^2\theta$

$= 2+(\operatorname{cosec}^2\theta-1)+(\sec^2\theta-1)$ (from (10) and (11))

$= 2+\operatorname{cosec}^2\theta-1+\sec^2\theta-1$

$= \operatorname{cosec}^2\theta+\sec^2\theta =$ R.H.S. Q.E.D.

To find any trigonometric ratio of an angle, given one ratio

If any one trigonometric ratio of an angle is given, then the ratio of two sides of a right-angled triangle is known. It is a simple matter to use the theorem of Pythagoras to determine the third side of the triangle, and hence to write any other ratio of the angle. (All the angles used will be acute.)

Example. If $\cos\theta = 0{\cdot}8$, find $\sin\theta$ and $\tan\theta$ without tables.

$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{0{\cdot}8}{1} = \dfrac{4}{5}$ (These represent the sides BC and AB in fig. 168).

Now $AB^2 = BC^2 + AC^2$,

i.e. $25 = 16 + AC^2$,

$\therefore\ AC^2 = 9$ or $AC = 3$.

Hence $\sin\theta = \dfrac{AC}{AB} = \dfrac{3}{5} = 0{\cdot}6$,

$\tan\theta = \dfrac{AC}{BC} = \dfrac{3}{4} = 0{\cdot}75.$

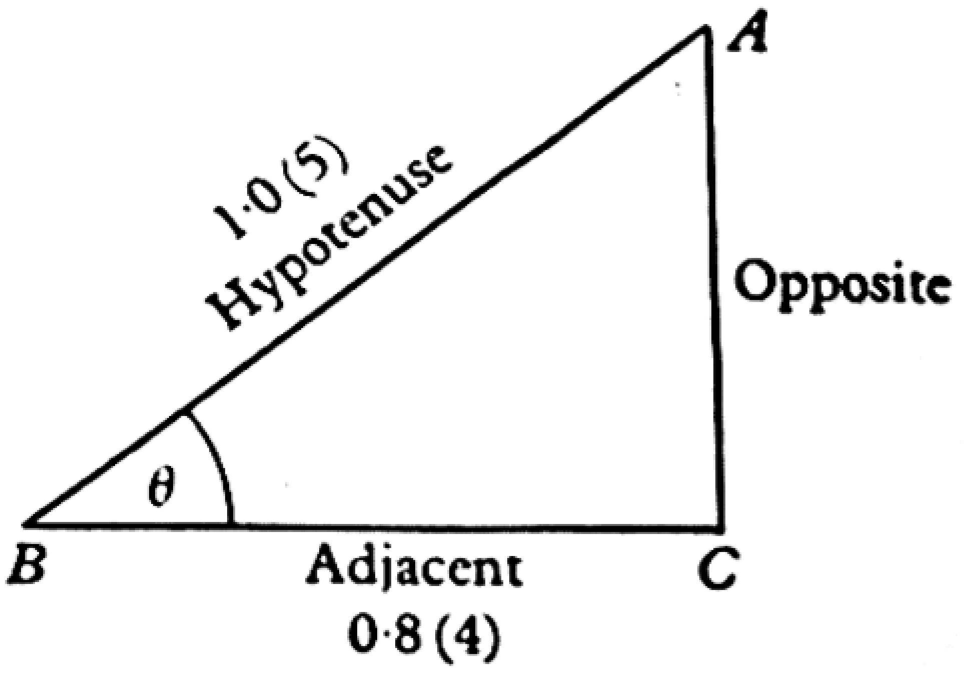


Fig. 168

Example. If $\sec\theta = 1{\cdot}2$, find $\tan\theta$ and $\sin\theta$.

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1{\cdot}2}{1} = \frac{6}{5}.$$

Now $(\text{hypotenuse})^2 = (\text{adjacent})^2 + (\text{opposite})^2$,

i.e. $36 = 25 + (\text{opposite})^2$,

$\therefore\ (\text{opposite})^2 = 11$,

$\therefore\ \text{opposite} = \sqrt{(11)}$.

Thus $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{\sqrt{(11)}}{5}$, $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{\sqrt{(11)}}{6}$.

(These may be evaluated decimally if required.)

Example. If $\tan\theta = \dfrac{2K}{K^2-1}$, find the value of $\cos\theta$.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{2K}{K^2-1}.$$

$$\begin{aligned}\text{Now } (\text{hypotenuse})^2 &= (\text{opposite})^2+(\text{adjacent})^2\\ &= (2K)^2+(K^2-1)^2\\ &= 4K^2+K^4-2K^2+1\\ &= K^4+2K^2+1\\ &= (K^2+1)^2 \quad (\text{Compare with } (a+b)^2),\\ \therefore\ \text{hypotenuse} &= K^2+1.\end{aligned}$$

Hence $$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{K^2-1}{K^2+1}.$$

Example. Given $\cot\theta = \frac{7}{24}$, evaluate $3\sin\theta+4\cos\theta$.

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{7}{24}.$$

$$\begin{aligned}\text{Now } (\text{hypotenuse})^2 &= (\text{opposite})^2+(\text{adjacent})^2\\ &= 24^2+7^2\\ &= 576+49\\ &= 625,\\ \therefore\ \text{hypotenuse} &= 25.\end{aligned}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{24}{25},$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{7}{25}.$$

$$\therefore\ 3\sin\theta+4\cos\theta = 3\times\tfrac{24}{25}+4\times\tfrac{7}{25} = \tfrac{72}{25}+\tfrac{28}{25} = \tfrac{100}{25} = 4.$$

Examples for Chapter 35

1. Prove the following identities:

(*a*) $\cos\theta = \sqrt{(1-\sin^2\theta)}$. (*b*) $\tan\theta\sec\theta = \dfrac{\sin\theta}{1-\sin^2\theta}$.

(c) $\sin^4\theta-\cos^4\theta = \sin^2\theta-\cos^2\theta$.

(*d*) $\tan\theta+\cot\theta = \dfrac{1}{\sin\theta\cos\theta}$. (*Chap.* 35)

2. Prove the following identities:

(*a*) $\sin^2\theta+\sec^2\theta = 2+\tan^2\theta-\cos^2\theta$.

(*b*) $\dfrac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta} = 1-\sin\theta\cos\theta$.

(c) $(\sin\theta+\cos\theta)^2 = 1+2\sin\theta\cos\theta$.

(*d*) $(\tan\theta-\sec\theta)^2 = 2\sec^2\theta(1-\sin\theta)-1$. (*Chap.* 35)

3. Prove the following identities:

(*a*) $\operatorname{cosec}^2\theta + \tan^2\theta = \sec^2\theta + \cot^2\theta$.

(*b*) $\sqrt{\{(1-\sin^2\theta)(\operatorname{cosec}^2\theta - 1)\}} = \dfrac{\cos^2\theta}{\sin\theta}$.

(*c*) $(\tan\theta - 1)(1+\cot\theta) = \tan\theta - \cot\theta$.

(*d*) $1-\sin^4\theta = \cos^2\theta(1+\sin^2\theta)$. (*Chap.* 35)

4. (*a*) Given $\cos\theta = \frac{12}{13}$, find the values of $\sin\theta$ and $\tan\theta$.

(*b*) Given $\sin\theta = 0{\cdot}2$, find the values of $\sec\theta$ and $\cot\theta$.

(*c*) If $\tan\theta = \frac{1}{3}$, evaluate $\cos^2\theta - \sin^2\theta$. (*Chap.* 35)

5. (*a*) If $\tan\theta = \frac{2}{3}$, evaluate $2\sin^2\theta + 3\cos^2\theta + \dfrac{2}{\sqrt{(13)}}\sin\theta$.

(*b*) Given $\tan\theta = \dfrac{2t}{1-t^2}$, show that $\sin\theta = \dfrac{2t}{1+t^2}$, and $\cos\theta = \dfrac{1-t^2}{1+t^2}$. (*Chap.* 35)

6. (*a*) A force of 10 lb. wt. is inclined to the horizontal at an angle θ, where $\cos\theta = \frac{2}{7}$. Prove that the horizontal and vertical components of the force are $\frac{20}{7}$ lb. wt. and $\dfrac{30\sqrt{5}}{7}$ lb. wt.

(*b*) If $\tan\theta = \dfrac{2ab}{a^2-b^2}$, find expressions for $\sin\theta$ and $\cos\theta$. (*Chaps.* 35 *and* 33)

7. A girder jib is in the shape of a rhombus of side 10 ft., containing angles of 20° at C and D. A member joins A and B, the two nearest corners. Find the length of AB, and the length CD. Find also the area of the rhombus. (*Chap.* 34)

8. An open trough has vertical ends shaped as trapezia, and sloping sides equally inclined to the vertical. Its depth is 2 ft., widths at top and bottom 3 ft. and 2 ft. respectively, and length 6 ft. Find (*a*) the weight of water it will hold when full, taking 1 cu.ft. of water to weigh 62·3 lb., (*b*) the area of the wetted surface. (*Chaps.* 30 *and* 31)

9. The areas of all rectangles which have a perimeter of 36 in. are given by area $= 18x - x^2$, where x is the length of one of the sides of the rectangle. Plot area against x from $x = 0$ to $x = 18$. Find for what value of x the area is a maximum and find the maximum area. (*Chap.* 28)

10. Values for AC, BC and angle A are given for a triangle. Is it possible to draw two triangles satisfying these conditions when (*a*) BC is greater than AC, (*b*) BC is less than AC?

Let B and B' be the two possible positions when two triangles can be drawn. Prove that the angles B and B' in the two triangles are supplementary. (*Chap.* 23)

Answers to Examples for Chapter 35

4. (*a*) $\sin\theta = \frac{5}{13}$, $\tan\theta = \frac{5}{12}$.

(*b*) $\sec\theta = \dfrac{5}{\sqrt{(24)}}$, $\cot\theta = \sqrt{(24)}$. (*c*) $\frac{4}{5}$.

5. (*a*) 3.

6. (*b*) $\sin\theta = \dfrac{2ab}{a^2+b^2}$; $\cos\theta = \dfrac{a^2-b^2}{a^2+b^2}$.

7. 3·472 ft.; 19·696 ft.; 34·2 sq.ft.

8. (*a*) 1869 lb. (*b*) 46·744 sq.ft.

9. $x = 9$; area = 81 sq.in.

10. Possible in (*b*) only.

CHAPTER 36

SOLUTION OF TRIANGLES BY THE SINE RULE

Measurements of heights and distances which cannot be performed directly are based on the lengths of sides of triangles. In surveying work triangles do not necessarily contain a right angle; moreover, they may contain an obtuse angle. There are thus two problems to be solved before we can attempt the solution of triangles which do not contain an angle of 90°. First we must understand the meaning of the sine of an obtuse angle, and secondly, we must prove a rule which applies to all types of triangle. The latter is called the sine rule.

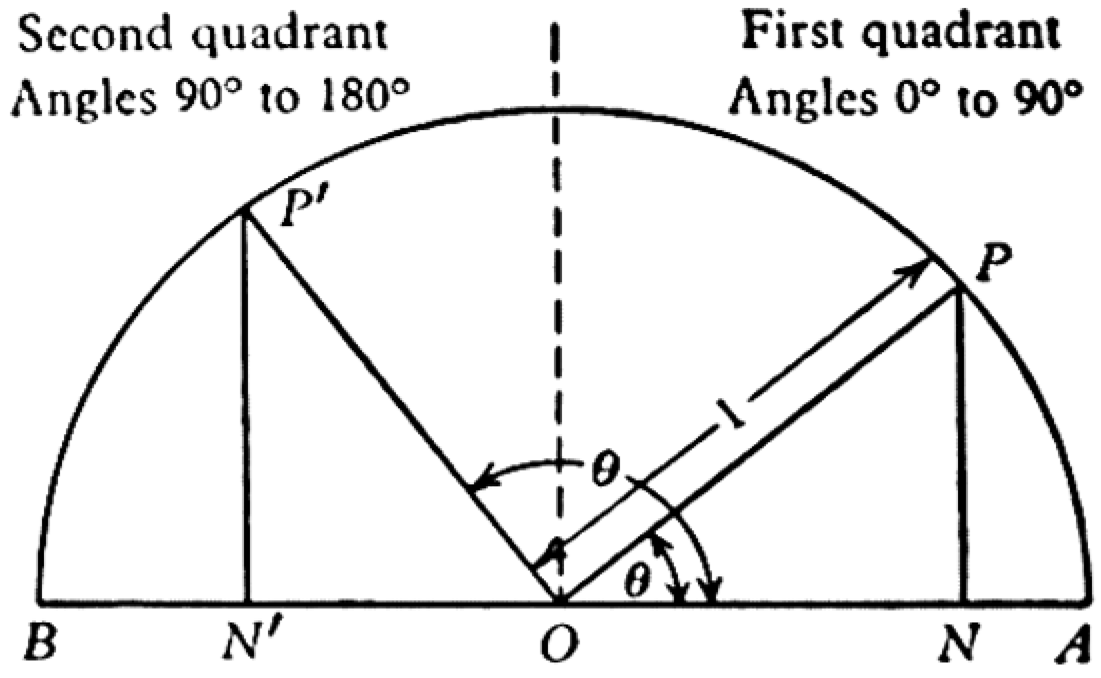


Fig. 169

Sines of angles from 0° *to* 180°

To deal with angles from 0° to 180°, a semicircle of unit radius is drawn as in fig. 169. The radius moves round from the position OA, where the angle θ is zero, to the position B, where the angle θ is 180°. Since OP, the radius, is unity, the sine of the angle θ is the length PN of the perpendicular from P to the line AB. The greatest length PN can have is 1, when θ is 90°. As θ increases beyond 90°, the length of PN decreases until it becomes zero again when P is at B.

It is obvious that the length of PN is the same when angle $AOP = 30°$ as when angle $BOP = 30°$ (i.e. when $AOP = 150°$). Similarly, $\sin 60° = \sin 120°$, $\sin 40° = \sin 140°$.

In general, the sine of an angle is the same as the sine of ($180°$ – that angle), or $\sin\theta = \sin(180° - \theta)$.

Thus, to find sin 125° from the tables we look up sin (180° − 125°) or sin 55°, and to find sin 95° we look up sin (180 − 95°) or sin 85°.

The sine of an angle is equal to the sine of its supplement.

The sine rule

Consider fig. 170, which contains two triangles ABC, one acute-angled, and the other obtuse-angled at C. It is usual to refer to the sides by small letters, as shown. Thus the side a is the side *opposite* the angle A, similarly for b and c. AD is drawn perpendicular to BC or BC produced.

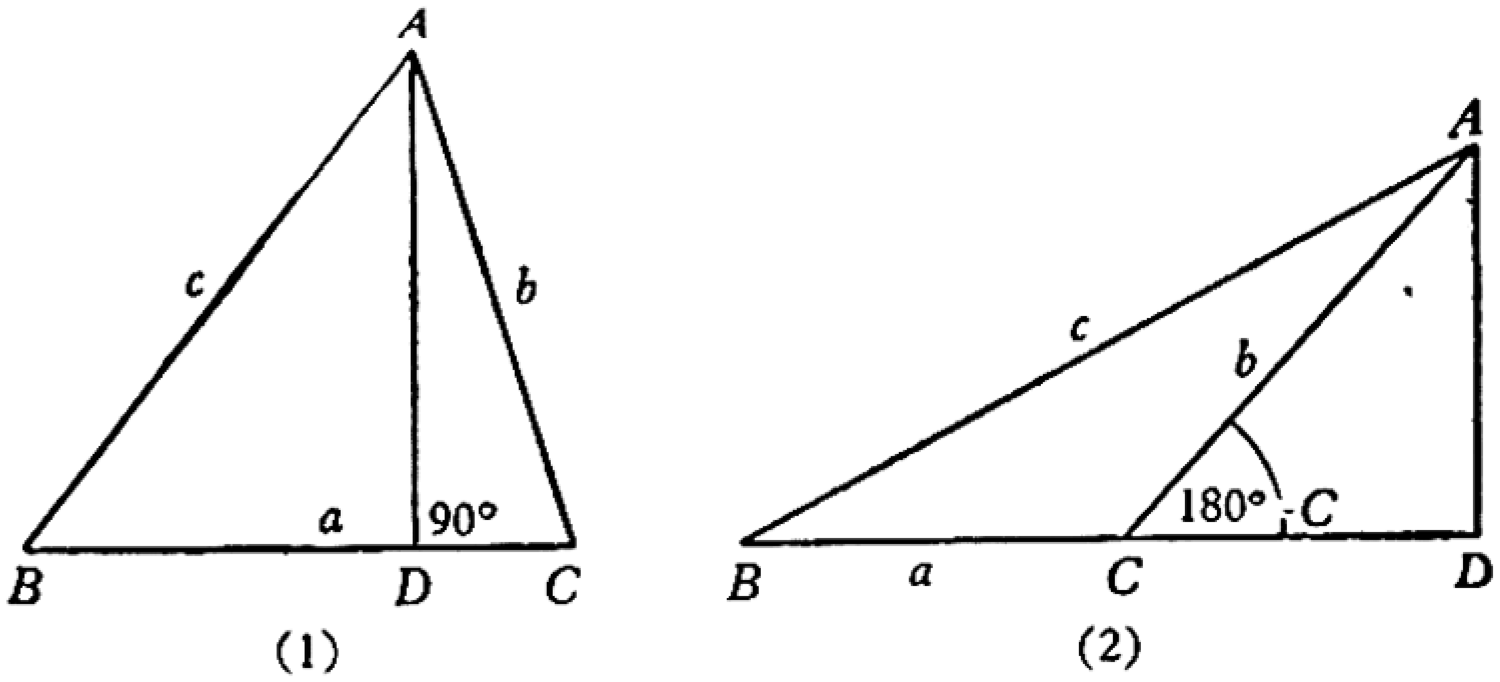


Fig. 170

In triangle (1),

$$\frac{AD}{c} = \sin B \quad \text{or} \quad AD = c\sin B;$$

also

$$\frac{AD}{b} = \sin C \quad \text{or} \quad AD = b\sin C.$$

$$\therefore\ c\sin B = b\sin C \quad \text{or} \quad \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Similarly, it could be shown by drawing a perpendicular from C to AB that $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$.

Thus

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

In triangle (2), $\dfrac{AD}{c} = \sin B$ or $AD = c\sin B$;

also $\dfrac{AD}{b} = \sin(180° - C) = \sin C$ or $AD = b\sin C$.

$\therefore\ c\sin B = b\sin C$, which is the same result as before.

Hence the relationship $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ is true for all triangles, whether or not they are acute-angled. This is the sine rule, and may be expressed in words as: 'The lengths of the sides of any triangle are proportional to the sines of the angles opposite those sides.'

The rule is inadequate to solve triangles under certain conditions, but may always be used to solve a triangle when the values of two angles are known, together with the length of one side.

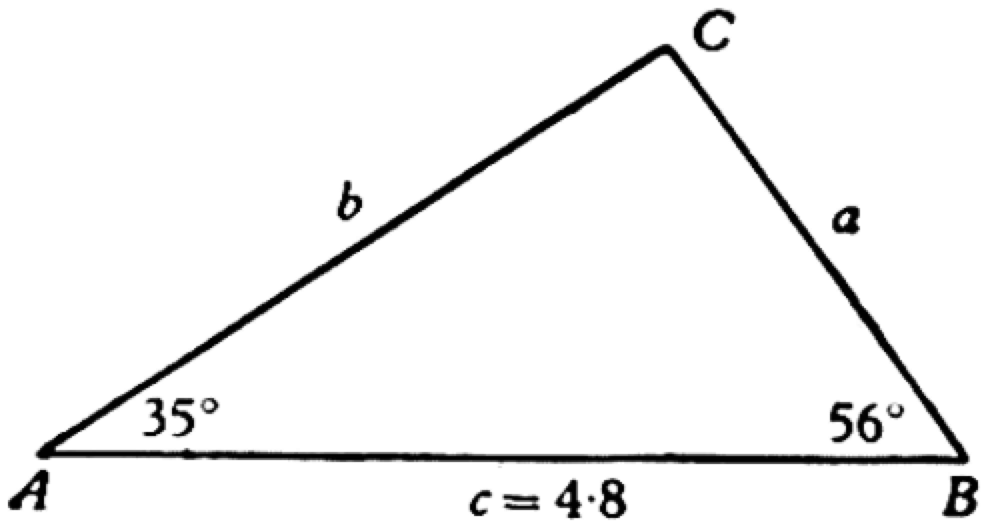


Fig. 171

Example. Solve completely the triangle ABC in which $A = 35°$, $B = 56°$ and $c = 4{\cdot}8$.

Solving the triangle means determining the lengths of a and b, and the value of C (see fig. 171).

Angle $C = 180 - (35° + 56°) = 89°$.

From the sine rule,

$$\frac{c}{\sin C} = \frac{b}{\sin B} \quad \text{(This will give the length of } b\text{),}$$

i.e.

$$\frac{4{\cdot}8}{\sin 89°} = \frac{b}{\sin 56°},$$

$$\therefore\ b = \frac{4{\cdot}8 \sin 56°}{\sin 89°}$$

$$= \frac{4{\cdot}8 \times 0{\cdot}8290}{0{\cdot}9998}$$

$$= \underline{3{\cdot}98 \text{ units.}}$$

Numbers	Logs
4·8	0·6812
0·8290	$\bar{1}$·9186
	0·5998
0·9998	$\bar{1}$·9999
3·98	0·5999

From the sine rule,

$$\frac{c}{\sin C} = \frac{a}{\sin A} \quad \text{(This will give the length of } a\text{),}$$

i.e.

$$\frac{4{\cdot}8}{\sin 89°} = \frac{a}{\sin 35°},$$

$$\therefore\ a = \frac{4{\cdot}8 \sin 35°}{\sin 89°}$$

$$= \frac{4{\cdot}8 \times 0{\cdot}5736}{0{\cdot}9998}$$

$$= \underline{2{\cdot}754 \text{ units.}}$$

4·8	0·6812
0·5736	$\bar{1}$·7587
	0·4399
	$\bar{1}$·9999
2·754	0·4400

$$\underline{\text{Hence, } C = 89°,\ a = 2{\cdot}754,\ b = 3{\cdot}980.}$$

Example. The jib of a crane is inclined at 38° to the vertical, and the tie makes an angle of 28° with the jib. If the jib is 18 ft. long, find (i) the length of the tie, (ii) the vertical height of the top of the jib above its base.

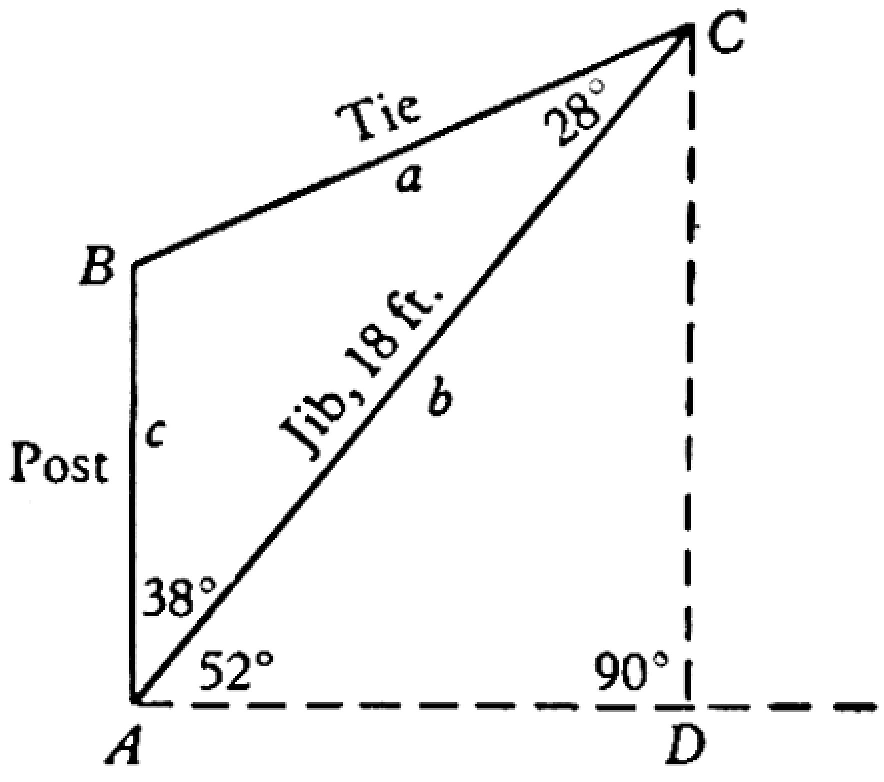


Fig. 172

In fig. 172, AC is the jib, and BC is the tie. Angle B is

$$180° - (28° + 38°) = 114°.$$

Applying the sine rule to the triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$ (This will give a since other values are known),

i.e. $$\frac{a}{\sin 38°} = \frac{18}{\sin 114°}$$ (Now $\sin 114° = \sin(180° - 114°)$ $= \sin 66°$),

$$\therefore\ a = \frac{18 \sin 38°}{\sin 66°}$$

$$= \frac{18 \times 0{\cdot}6157}{0{\cdot}9135}$$

$$= \underline{12{\cdot}13\ \text{ft.}}$$

Numbers	Logs
18	1·2553
0·6157	$\bar{1}$·7894
	1·0447
0·9135	$\bar{1}$·9607
12·13	1·0840

For the second answer the sine rule need not be used, because the triangle CAD contains a right angle.

In the right-angled triangle CAD, $\dfrac{CD}{AC} = \sin 52°$.

$$\therefore\ CD = AC \sin 52°$$
$$= 18 \times 0{\cdot}7880$$
$$= \underline{14{\cdot}184\ \text{ft.}}$$

Thus, length of tie = 12·13 ft., and height of top of jib = 14·18 ft.

Example. From the top of a vertical cliff the angles of depression of two buoys in line with an observer are 31° and 14° respectively. The buoys are 200 ft. apart. Find (i) the height of the cliff, (ii) the distance of the nearer buoy from the foot of the cliff.

In fig. 173, B and C are the buoys, AD is the cliff. Since the water surface is horizontal, angle $ACD = 14°$, and angle $ABD = 31°$.

The height of the cliff cannot be determined until the length of one side of a triangle which contains AD is known.

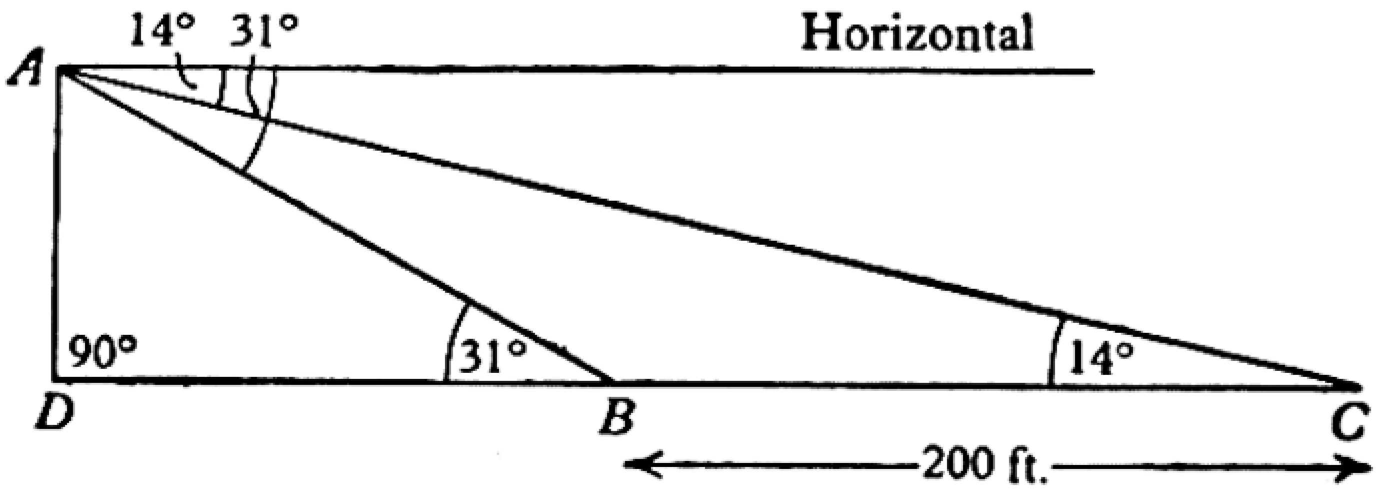


Fig. 173

Let us find the length of AB first.

In the triangle ABC,

$$\frac{AB}{\sin 14°} = \frac{BC}{\sin 17°} \quad (\text{angle } BAC = 31° - 14° = 17°)$$

or

$$\frac{AB}{\sin 14°} = \frac{200}{\sin 17°}.$$

$$\therefore\ AB = \frac{200 \sin 14°}{\sin 17°}$$

$$= \frac{200 \times 0{\cdot}2419}{0{\cdot}2924}$$

$$= 165{\cdot}4 \text{ ft.}$$

In the right-angled triangle ADB,

$$\frac{AD}{AB} = \sin 31°,$$

$$AD = 165{\cdot}4 \sin 31°$$

$$= 165{\cdot}4 \times 0{\cdot}5150$$

$$= \underline{85{\cdot}19 \text{ ft.}}$$

Numbers	Logs
200	$2{\cdot}3010$
$0{\cdot}2419$	$\bar{1}{\cdot}3836$
	$1{\cdot}6846$
$0{\cdot}2924$	$\bar{1}{\cdot}4660$
$165{\cdot}4$	$2{\cdot}2186$
$165{\cdot}4$	$2{\cdot}2186$
$0{\cdot}5150$	$\bar{1}{\cdot}7118$
$85{\cdot}19$	$1{\cdot}9304$
$165{\cdot}4$	$2{\cdot}2186$
$0{\cdot}8572$	$\bar{1}{\cdot}9331$
$141{\cdot}8$	$2{\cdot}1517$

Also, to determine DB from the same triangle,

$$\frac{DB}{AB} = \cos 31°.$$

$$\therefore\ DB = 165{\cdot}4 \times 0{\cdot}8572$$

$$= \underline{141{\cdot}8 \text{ ft.}}$$

Thus, height of cliff = 85·2 ft.; distance of nearer buoy = **141·8 ft.**

[This problem could have been solved by the method of Chapter 34, without using the sine rule. The student should attempt it for an exercise.]

Examples for Chapter 36

1. (*a*) Write the values of sin 95°, sin 124°, sin 176°.

(*b*) Solve completely the triangle ABC in which $a = 15$, $B = 30°$, $C = 26°$. (*Chap.* 36)

2. (*a*) If A, B and C are the angles of a triangle, show that $\sin(A+B) = \sin C$.

(*b*) Solve completely the triangle ABC in which $A = 110°$, $B = 42°$, $b = 17{\cdot}8$. (*Chap.* 36)

3. A straight road runs north-east. From a point A, the bearing of a power station is N. 7° E., and from a point B on the same road, distant 880 yd. from A, the bearing of the power station is S. 88° W. Find the distance of B from the power station, and of the power station from the road. (*Chap.* 36)

4. In order to find the height of a hill two points are chosen on level ground in line with the hill. The angles of elevation of the summit from the two points are 41° and 28° respectively, and the points are 400 ft. apart. Find the height of the hill. (*Chap.* 36)

5. The resultant of two forces is a force of 15 lb. wt. and makes angles 25° and 62° with the components. Find the components and the area of the parallelogram. (*Chaps.* 36 *and* 33)

6. Calculate the total length of the girders in the framework in fig. 174. The framework is symmetrical about a vertical line through B. (*Chap.* 36)

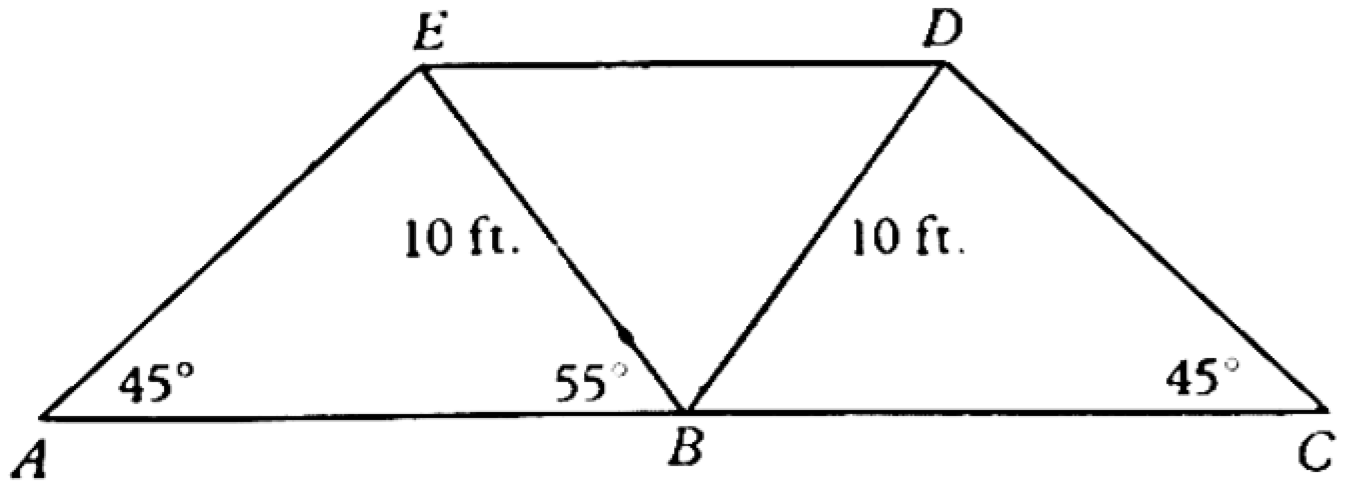


Fig. 174

7. Ten spheres, radius $\frac{1}{2}$ in., are melted down and recast into one sphere. Find its radius. (*Chap.* 31)

8. The table gives diameters of copper wires and corresponding fusing currents. Plot current (I amperes) against diameter (d in.) and from the graph find the fusing currents for copper wires of diameters 0·021 in. and 0·025 in.:

Diameter (d)	0·005	0·008	0·01	0·012	0·015	0·018	0·022	0·028
Current (I)	3·4	6·8	11·5	14·1	18·4	24·7	33·4	48

(*Chap.* 26)

9. (a) Prove the following identities:

(i) $\sec\theta(1+\tan\theta) = \sec^2\theta(\cos\theta+\sin\theta)$.

(ii) $\sec^2\theta-\cos^2\theta = \sin^2\theta+\tan^2\theta$.

(iii) $\tan\theta-\cot^2\theta = \dfrac{(\sin\theta-\cos\theta)(1+\sin\theta\cos\theta)}{\sin^2\theta\cos\theta}$.

(b) If $\tan\theta = 2$, find the values of $\sin\theta$ and $\cos\theta$.

(*Chap.* 35)

10. The ordinates of an indicator diagram, including the end ordinates, are 1·85, 2·0, 1·85, 1·65, 1·45, 1·2, 0·95, 0·70, 0·65, 0·45, 0·35 in. The length of the diagram is 4 in. Find the area of the diagram, and the mean pressure if the scale of pressure is 80 lb. per sq.in. to 1 in. (*Chap.* 25)

Answers to Examples for Chapter 36

1. (a) 0·9962; 0·8290; 0·0698.

(b) $A = 124°$; $b = 9·045$; $c = 7·93$.

2. (b) $C = 28°$; $c = 12·48$; $a = 25$.

3. 548·6 yd.; 374·1 yd.

4. 548 ft.

5. 13·26 lb. wt.; 6·348 lb. wt.; 84·07 units.

6. 82·49 ft.

7. 1·077 in.

8. $d = 0·021$, $I = 30$; $d = 0·025$, $I = 40$.

9. (b) $\sin\theta = \dfrac{2}{\sqrt{5}}$; $\cos\theta = \dfrac{1}{\sqrt{5}}$.

10. 4·8 sq.in.; 96 lb. per sq.in.

CHAPTER 37

CIRCULAR MEASURE. ANGULAR VELOCITY

This chapter will introduce another unit of angular measure which is sometimes preferable to the system of degrees, minutes and seconds. The unit to be defined is called a *radian.*

Mark off along the circumference of any circle an arc equal in length to the radius, and from the ends of this arc draw radii. The angle at the centre of the circle, formed by the two radii, is one radian.

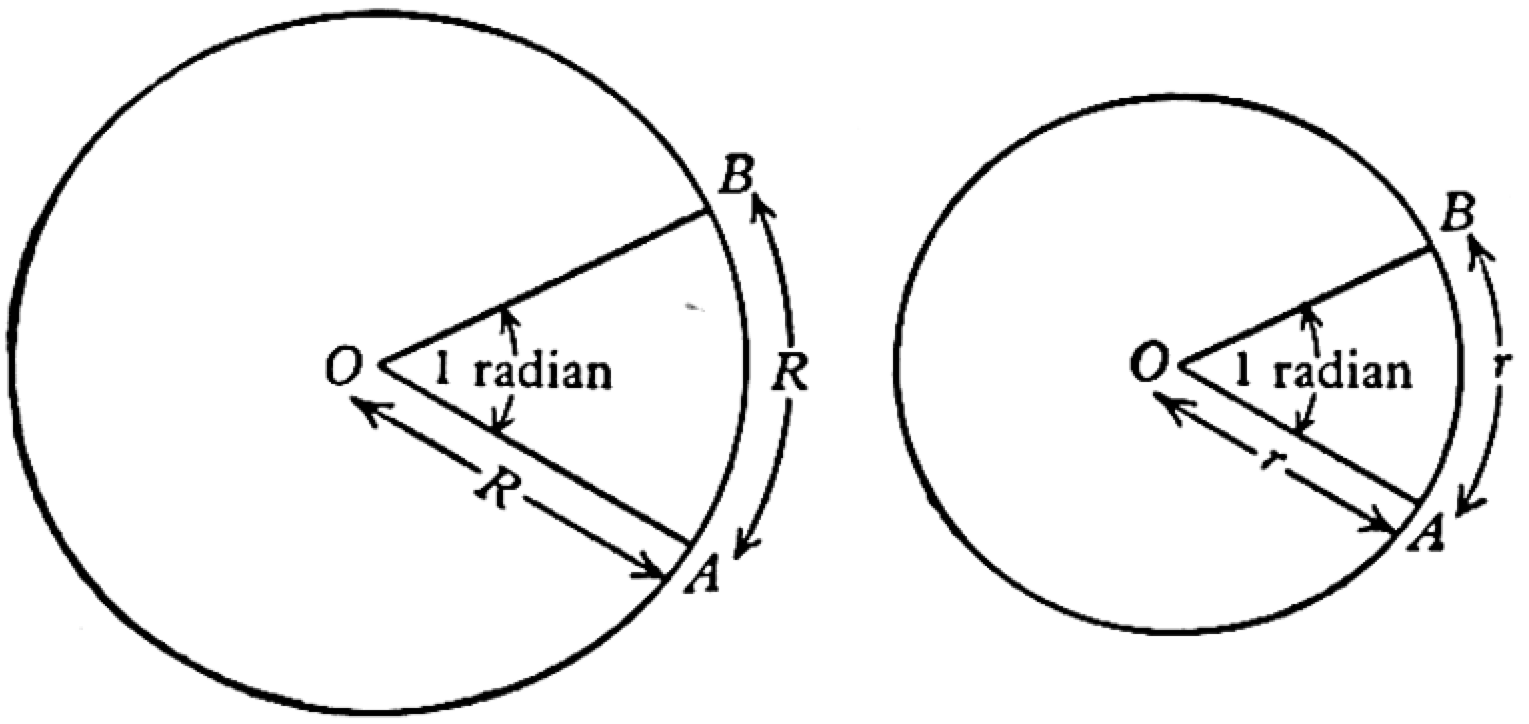


Fig. 175

Fig. 175 shows two circles with radii drawn in this manner, illustrating that the value of a radian is constant, and independent of the size of circle drawn. An angle which is formed by drawing straight lines from two points to a third point is said to be the angle *subtended* by the two points at the third point. A radian is therefore defined as the angle subtended at the centre of a circle by an arc of the circle equal in length to the radius.

The number of radians in a complete circle is the number of times an arc equal in length to the radius can be marked off along the circumference of the circle. But we know that

$$\frac{\text{Circumference}}{\text{Radius}} = 2\pi,$$

so the number of radians in a complete circle is 2π.

$$2\pi \text{ radians} = 360^\circ,$$

$$\pi \text{ radians} = 180^\circ,$$

$$\frac{\pi}{2} \text{ radians} = 90^\circ,$$

$$\frac{\pi}{3} \text{ radians} = 60^\circ,$$

$$\therefore\ 1 \text{ radian} = \frac{180^\circ}{\pi} = \frac{180^\circ}{3{\cdot}142} = 57{\cdot}3^\circ \text{ (approx.)}.$$

The last statement supplies a convenient conversion factor. To convert radians to degrees multiply by 57·3, and to convert degrees to radians divide by 57·3.

When the angle in degrees is some simple multiple or fraction of 180°, then the answer in radians is better expressed as a multiple or fraction of π. Thus

$$22\tfrac{1}{2}^\circ = \frac{\pi}{8} \text{ radians}, \quad 270^\circ = \frac{3\pi}{2} \text{ radians}.$$

Length of a circular arc

Let r be the radius of the circle; then if the length of arc is r, the angle subtended by the arc at the centre is 1 radian. If the length of the arc is $2r$, the angle is 2 radians. Thus,

length of arc = radius × angle subtended at the centre in radians,

$$l = r\theta \quad \text{(where } \theta \text{ is in radians)}.$$

Example. Find the length of an arc which subtends an angle of 2·3 radians at the centre of a circle of radius 5 in.

$$\begin{aligned} \text{Length of arc} &= \text{radius} \times \theta \\ &= 5 \times 2{\cdot}3 \text{ in.} \\ &= 11{\cdot}5 \text{ in.} \end{aligned}$$

Example. Find the angle, in radians, subtended at the centre of a circle radius 6 in., by an arc of the circumference, 11·2 in. in length.

$$l = r\theta.$$

$$\therefore\ \theta = \frac{l}{r} = \frac{11{\cdot}2}{6} = 1{\cdot}867 \text{ radians}.$$

Angular velocity

When a flywheel is revolving it is apparent that different parts of the wheel are moving with different speeds. All points on the rim have the same speed, while a point half-way between the axle and the circumference has only half the speed of a point on the circumference. Points closer to the axle have smaller speeds still. The property that is common to all points in and on the flywheel is that for a particular number of revolutions in a given time, the *angular* velocity is the same. Thus the number of revolutions in a specified time is the measure of the angular velocity of a rotating body. For certain engineering purposes it is preferable to express angular velocity in *radians per second.* Just as θ has been used to express a general angle, so ω (*omega*) is used to express a general angular velocity in radians per second.

To find the linear velocity of a point in a revolving body. Let P, fig. 176, be a point in a body revolving about the point O, then, whatever shape the body has, the point P will describe a circle radius OP about O. Let the angular velocity of the body be ω radians per second. Then in one second the radius OP will have moved through an angle of ω radians. Let P move to P' in one second, so that the angle $POP' = \omega$ radians. Then the distance moved through space by P in one second is the arc PP'.

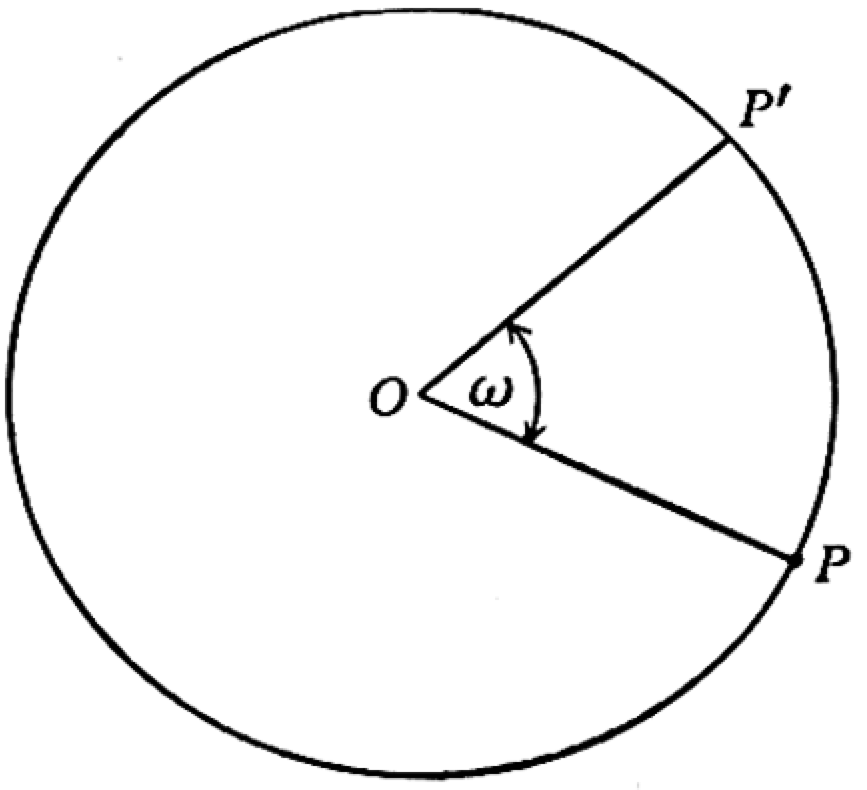


Fig. 176

Length of arc = radius × angle in radians.

$$\therefore \; PP' = \text{radius} \times \omega.$$

But PP', being the distance moved by P in one second, may be taken as a measure of the linear velocity of the point P.

$\therefore$ Linear velocity = radius × angular velocity (in radians per sec.),

i.e. $$v = r\omega,$$

or $$\omega = \frac{v}{r} \quad \text{(if } r \text{ is in feet, and } v \text{ is in feet per sec.).}$$

To convert an angular velocity in revolutions per min. to radians per sec. Let N be the number of revolutions per min. Then the number of revolutions per sec. is $\frac{N}{60}$. But 1 revolution is equal to 2π radians; thus the number of radians per sec. is $\frac{2\pi N}{60}$.

Example. A flywheel radius 2 ft. 6 in. is revolving at 3 revolutions per sec. Find the linear velocity of (a) a point on its circumference, (b) a point 6 in. distant from the centre of the axle.

Angular velocity in radians per sec. $= 3 \times 2\pi = 6\pi$ radians per sec.

$$\begin{aligned} \text{(i) Linear velocity} &= \text{radius} \times \text{angular velocity} \\ &= 2{\cdot}5 \times 6\pi \\ &= 15\pi \text{ ft. per sec.} \\ &= 47{\cdot}13 \text{ ft. per sec.} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \qquad v &= \text{radius} \times \omega \\ &= \tfrac{1}{2} \times 6\pi \\ &= 3\pi \text{ ft. per sec.} \\ &= 9{\cdot}246 \text{ ft. per sec.} \end{aligned}$$

Thus, a point on the circumference has a speed of **47·13 ft. per sec.**, while the speed 6 in. from the centre of the axle is 9·246 ft. per sec.

Example. Two gears having 40 teeth and 15 teeth respectively are meshed. The larger one is revolving at 25 revolutions per sec. Find the angular velocity of the smaller one (a) in revolutions per sec., (b) in radians per sec.

Since the gears have 40 and 15 teeth, their circumferences, and hence their radii, are in the ratio $\frac{40}{15}$ or $\frac{8}{3}$.

Also 40 teeth of the smaller gear must have passed the point of contact while the larger gear has completed one revolution. Therefore the angular velocity of the smaller is greater than the angular velocity of the larger in the ratio $\frac{8}{3}$.

$$\begin{aligned} \text{Angular velocity of smaller gear} &= 25 \times \tfrac{8}{3} \text{ revolutions per sec.} \\ &= \underline{66\tfrac{2}{3} \text{ revolutions per sec.}} \\ &= 66\tfrac{2}{3} \times 2\pi \text{ radians per sec.} \\ &= \underline{418{\cdot}9 \text{ radians per sec.}} \end{aligned}$$

Example. A belt connects two pulleys of radii 6 in. and 1 ft. 3 in., and the smaller pulley is revolving 100 times per min. Find the speed of the belt in ft. per sec., and the angular velocity of the larger pulley in radians per sec.

Angular velocity of smaller pulley

$$\begin{aligned} &= 100 \text{ revolutions per min.} \\ &= \tfrac{100}{60} \text{ revolutions per sec.} \\ &= \frac{2\pi \times 100}{60} \text{ radians per sec.} \\ &= \frac{10\pi}{3} \text{ radians per sec.} \end{aligned}$$

Now velocity (v) of belt = radius × angular velocity

$$\begin{aligned} &= \frac{1}{2} \times \frac{10\pi}{3} \text{ ft. per sec.} \\ &= \underline{5{\cdot}237 \text{ ft. per sec.}} \end{aligned}$$

Since the belt passes round the larger pulley, the velocity of its circumference is also $\frac{1}{2} \times \frac{10\pi}{3}$, or $\frac{5\pi}{3}$ ft. per sec.

Now $\qquad v = r\omega$

or $\qquad \omega = \frac{v}{r} = \frac{5\pi}{3} \div 1\tfrac{1}{4}$

$$\begin{aligned} &= \frac{5\pi}{3} \times \frac{4}{5} = \frac{4\pi}{3} \text{ radians per sec.} \\ &= \underline{4{\cdot}189 \text{ radians per sec.}} \end{aligned}$$

Thus, the speed of the belt is 5·237 ft. per sec., and the angular velocity of the larger pulley is 4·189 radians per sec.

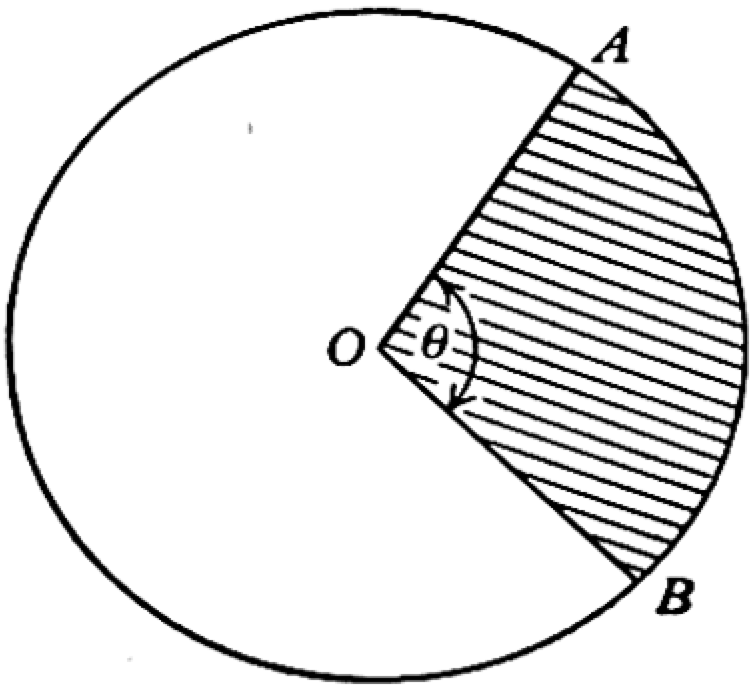


Fig. 177

Area of a sector of a circle when the angle is expressed in radians

Consider a sector of a circle containing an angle θ radians, as in fig. 177. The area is the same fraction of the area of the complete circle as the angle θ is of a complete revolution. Since θ is expressed in radians, this fraction is $\theta/2\pi$.

Thus, area of sector $= \pi r^2 \times \frac{\theta}{2\pi} = \frac{1}{2} r^2\theta.$

Examples for Chapter 37

1. (*a*) Convert the following angles to radians, expressing them as fractions of π:

$$270^\circ, \quad 30^\circ, \quad 18^\circ, \quad 54^\circ, \quad 75^\circ.$$

(*b*) Convert the angles in radians, to degrees:

$$4\pi, \quad \frac{\pi}{8}, \quad \frac{\pi}{6}, \quad \frac{5\pi}{4}.$$

(*c*) Convert the following to radians:

$$24{\cdot}5^\circ, \quad 176{\cdot}2^\circ, \quad 38^\circ, \quad 100^\circ.$$

(*Chap.* 37)

2. (*a*) What is the length of an arc of a circle radius 4·5 in., subtending an angle of 2·5 radians at the centre?

(*b*) A cyclist is riding at 30 m.p.h. round a circular track of radius 50 yd. Find his angular velocity about the centre in radians per sec. (*Chap.* 37)

3. A weight of 0·5 lb. is being revolved on the end of a string 2·5 ft. long at a speed of 40 revolutions per min. Find the speed of the weight in ft. per sec.

Find the tension in the string, given that

$$\text{Tension (lb. wt.)} = \frac{\text{weight}}{32} \times \frac{(\text{velocity})^2}{\text{radius.}}.$$

(*Chap.* 37)

4. A pulley driving a belt is revolving 75 times in one minute. The radius of the pulley is 9 in. Find the speed of the belt, and if the belt is driving a pulley radius 4 in., find the angular velocity of the driven pulley in radians per sec. (*Chap.* 37)

5. (*a*) A circular disc, radius 4 in., has a sector of area 10 sq.in. cut from it; find the angle of the sector in radians.

(*b*) Show that $\text{radius} = \sqrt{\left(\dfrac{2 \times \text{area of sector}}{\text{angle of sector in radians}}\right)}.$

Hence find the radius of a circle from which a sector containing an angle of 3·54 radians has an area of 31·22 sq.in. (*Chap.* 37)

6. A gear wheel containing 50 teeth revolves 40 times per sec. It is required that this gear should drive two gears meshed with it at speeds of 20π radians per sec. and 25π radians per sec. Find the numbers of teeth on the driven gears. (*Chap.* 37)

7. (*a*) Derive a formula expressing the radius of a cone in terms of its weight, density, and height.

(*b*) Find the radius of a cone cast from a block of metal weighing 144 g., of density 7·2 g. per c.c., the cone to be 4 cm. in height.

(For manipulation of formulae see Chapter 16.)

(*Chaps.* 29 *and* 31)

8. (*a*) Plot $y = x^2 + 2x - 3$ from $x = -5$ to $x = +4$ and write the abscissae of the points at which the curve intersects the axis of x.

(*b*) Find the equation of the line passing through (0, 3) and $(-3, 0)$. Plot this line on the same paper as the curve in (*a*), and write the abscissae of the points of intersection of the line and the curve. (*Chaps.* 28 *and* 27)

9. (*a*) Draw any triangle ABC, and draw AD perpendicular to BC. Hence show that the area of the triangle ABC is $\frac{1}{2}BC\,.\,AB\sin B$.

(*b*) A triangle ABC has $BC = 6$ units, $B = 26°$, $C = 32°$. Consider AD perpendicular to BC, and hence show that

$$BD\tan 26° = DC\tan 32°.$$

Calling $BD = x$, and $DC = 6 - x$, find the lengths of BD and DC. (*Chap.* 34)

10. In any triangle ABC, a point D is chosen in the side AB such that $AD = DB$, and DE is drawn parallel to BC to cut AC in E. What is the ratio of the lengths BC and DE?

Where would the point D be placed if $\dfrac{BC}{DE} = \dfrac{3}{2}$? (*Chap.* 23)

Answers to Examples for Chapter 37

1. (*a*) $\frac{3\pi}{2}$, $\frac{\pi}{6}$, $\frac{\pi}{10}$, $\frac{3\pi}{10}$, $\frac{5\pi}{12}$. (*b*) 720°, $22\frac{1}{2}$°, 30°, 225°.

   (*c*) 0·4276, 3·075, 0·6632, 1·745.

2. (*a*) 11·25 in. (*b*) 0·2934 radian per sec.

3. 10·47 ft. per sec., 0·69 lb. wt.

4. 5·89 ft. per sec., 17·67 radians per sec.

5. 1·25 radians., 4·2 in.

6. 200 teeth, 160 teeth.

7. (*a*) $r = \sqrt{\frac{3W}{\pi h d}}$. (*b*) 2·185 cm.

8. (*a*) $x = -3$, $x = 1$. (*b*) $y = x+3$, $x = 2$, $x = -3$.

9. $BD = 3{\cdot}368$, $DC = 2{\cdot}632$.

10. $\frac{BC}{DE} = \frac{2}{1}$. $\frac{2}{3}$ of the distance from A to B.

CHAPTER 38

SIMULTANEOUS EQUATIONS. (GRAPHICAL METHOD)

A straight line in Cartesian co-ordinates may always be expressed in the form $y = mx+c$, where m is the gradient, and c is the intercept on the y axis. When the coefficient of x and c are integers this is possibly the most convenient form of the equation, but when either of these happens to be a fraction, the equation may be multiplied throughout by the common denominator in order to remove fractions. The line $y = -\frac{3x}{4}+\frac{1}{2}$ may be written $4y = -3x+2$ (multiplying by 4) or $4y+3x-2 = 0$, and the line $y = x-1$ may be written $y-x+1 = 0$. It is convenient to remember that the equation of a straight line may not always be presented in the form $y = mx+c$. Any equation in x and y, or one of these only, which contains no terms of the second degree (i.e. terms in x^2, or y^2, or xy), represents a straight line. Just as the general equation of the parabola may be written

$$y = ax^2+bx+c,$$

so the straight line may be written $ax+by+c = 0$.

To convert a linear equation from the form $ax+by+c = 0$ to the form $y = mx+c$, the equation must be divided throughout by the coefficient of y, and then all terms except y removed to the right-hand side.

Thus $$3x-2y+5 = 0$$

becomes $$-\frac{3x}{2}+y-\frac{5}{2} = 0,$$

or $$y = \frac{3x}{2}+\frac{5}{2}.$$

Example. Find the gradient of the line $x+3y-4 = 0$, and find its intercept on the y axis.

$$x+3y-4 = 0,$$

$$\therefore \quad \frac{x}{3}+y-\frac{4}{3} = 0,$$

$$\therefore \quad y = -\frac{x}{3}+\frac{4}{3}.$$

The gradient is $-\frac{1}{3}$ and the intercept is $\frac{4}{3}$.

Consider the equations: $3x+2y = 9,$ (1)

$4x-y = 1.$ (2)

Equation (1) may be written $2y = 9-3x$, or $y = \dfrac{9}{2}-\dfrac{3x}{2}$.

Equation (2) may be written $y = 4x-1$.

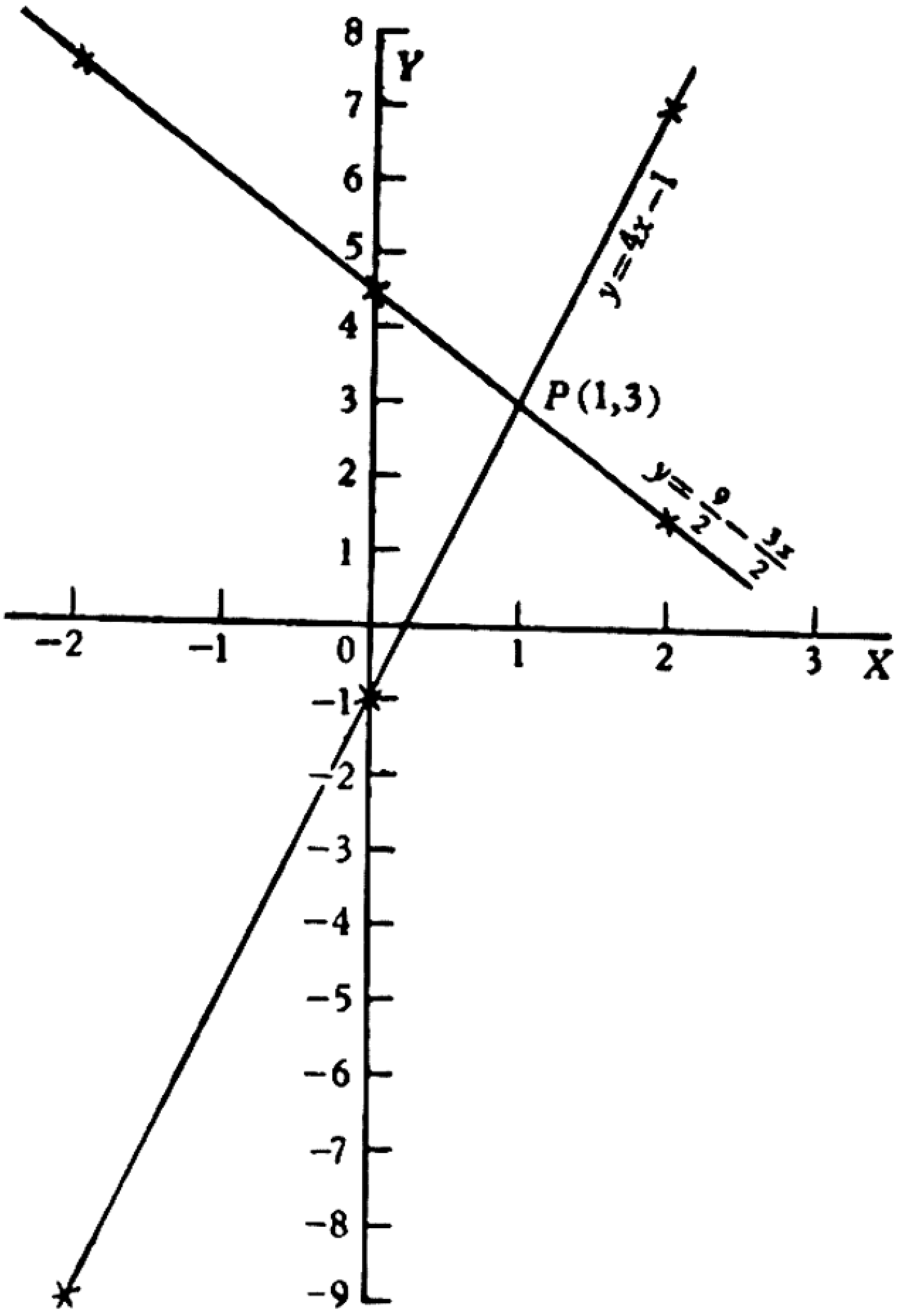


Fig. 178

$y = 4x-1$:

x	-2	0	2
y	-9	-1	7

$y = \dfrac{9}{2}-\dfrac{3x}{2}$:

x	-2	0	2
y	$7\frac{1}{2}$	$4\frac{1}{2}$	$1\frac{1}{2}$

The graphs are shown in fig. 178, intersecting at the point $P\,(1, 3)$. The point P is the only point which lies on both lines, so the only value of x which satisfies both equations is 1, and the only value of y to satisfy both equations is 3.

The solutions to a pair of simple simultaneous equations are the co-ordinates of a point at which the lines represented by both equations intersect. We have, therefore, an alternative method of solving simultaneous linear equations. (Compare with the method of Chapter 17.)

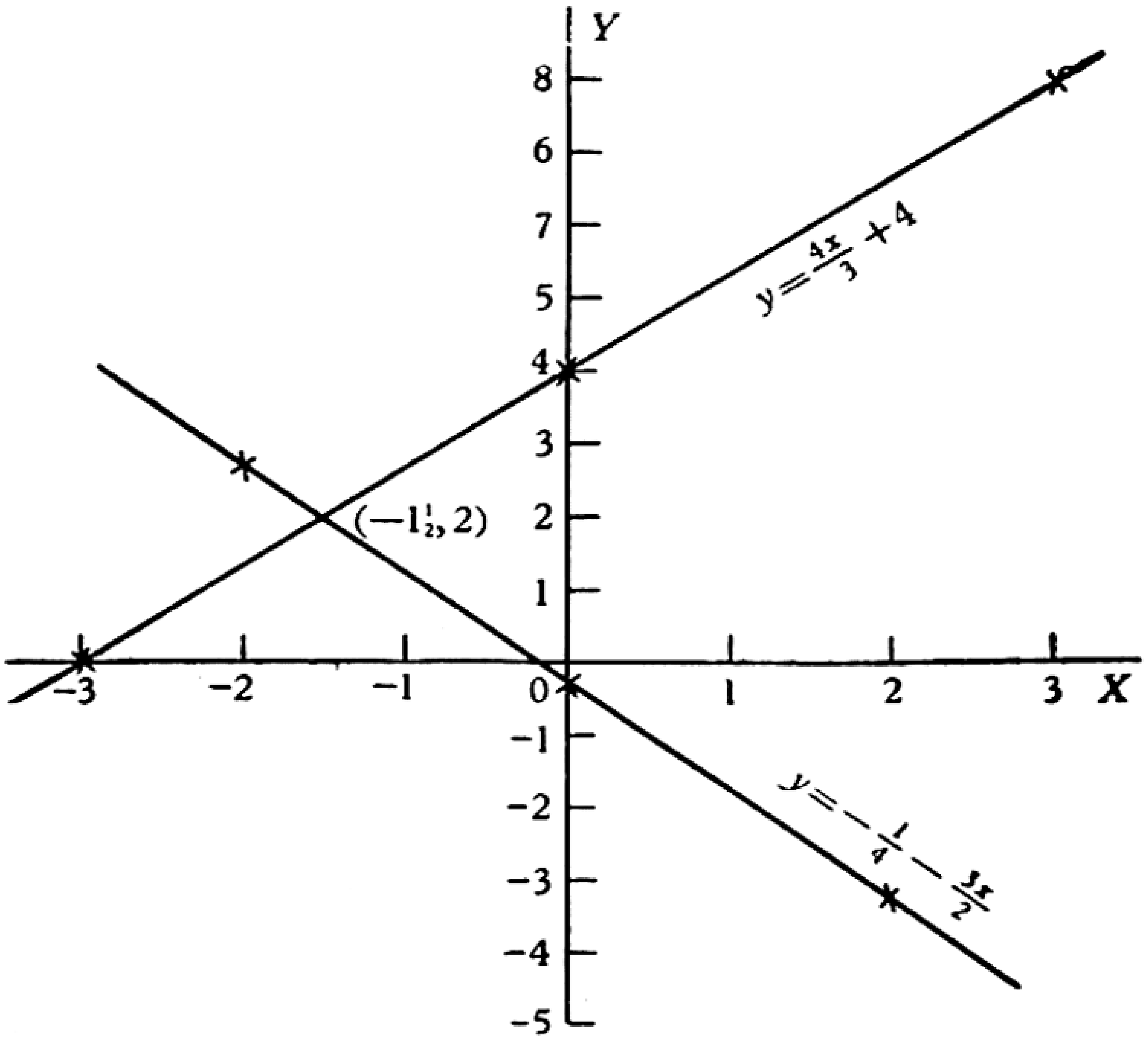


Fig. 179

Example. Solve graphically for x and y: $6x+4y+1 = 0$, $3y-4x = 12$.

$$6x+4y+1 = 0, \qquad (1)$$

$$3y-4x = 12. \qquad (2)$$

Equation (1) may be written $\frac{6x}{4}+y+\frac{1}{4} = 0$, or $y = -\frac{3x}{2}-\frac{1}{4}$.

Equation (2) may be written $y-\frac{4x}{3} = 4$, or $y = \frac{4x}{3}+4$.

$y = -\frac{1}{4}-\frac{3x}{2}$:

x	-2	0	2
y	$2\frac{3}{4}$	$-\frac{1}{4}$	$-3\frac{1}{4}$

$y = \frac{4x}{3} + 4$:

x	-3	0	$+3$
y	0	4	8

(Choosing -3 and $+3$ to avoid awkward fractions).

Fig. 179 shows that the lines intersect in the point $(-1\frac{1}{2}, 2)$. These are the only values of x and y to satisfy both equations.

$$\text{Hence } x = -1\tfrac{1}{2},\ y = 2.$$

Time and distance problems

The graphical method is a convenient way of solving problems on uniform speeds. The distance-time graph of a train travelling at constant speed is a straight line, because equal distances are covered in equal intervals of time.

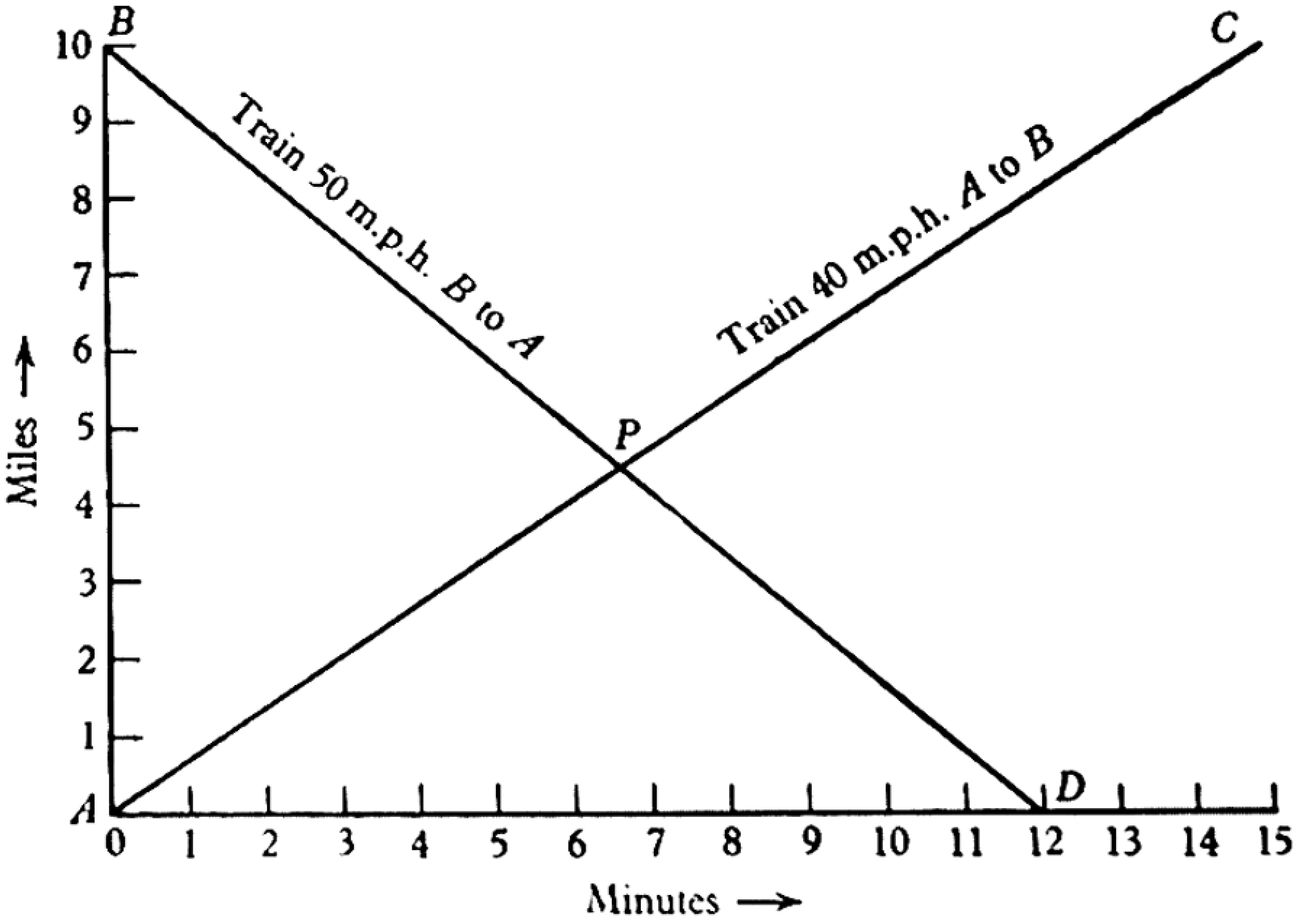


Fig. 180

Consider two stations A and B, 10 miles apart. A train leaves A travelling towards B with a constant speed of 40 m.p.h. At the same time another train leaves B travelling towards A at a constant speed of 50 m.p.h. To show these facts graphically, choose a vertical axis 10 units long to represent distance, and a horizontal axis to represent time long enough for the slower train to complete the journey, i.e. 15 min. (Fig. 180).

The line AC is the distance-time graph for the train travelling from A to B, plotted by a knowledge of two points, the point (0, 0) and the point (10 miles, 15 min.), because when the time is zero the train has not left A, and when the time is 15 min., it has reached B.

The line BD is the distance-time graph for the other train, plotted from points (0, 10) and (12, 0), for when the time is zero the train is 10 miles from A, and when the time is 12 min. it has arrived at A. These lines intersect at P, so the co-ordinates of P represent the times the trains pass and the distance from A or B. The time of passing is apparently just over 6·6 min., at a distance of just over 4·4 miles from A. (Calculation shows that these values are actually $6\frac{2}{3}$ min., and $4\frac{4}{9}$ miles.)

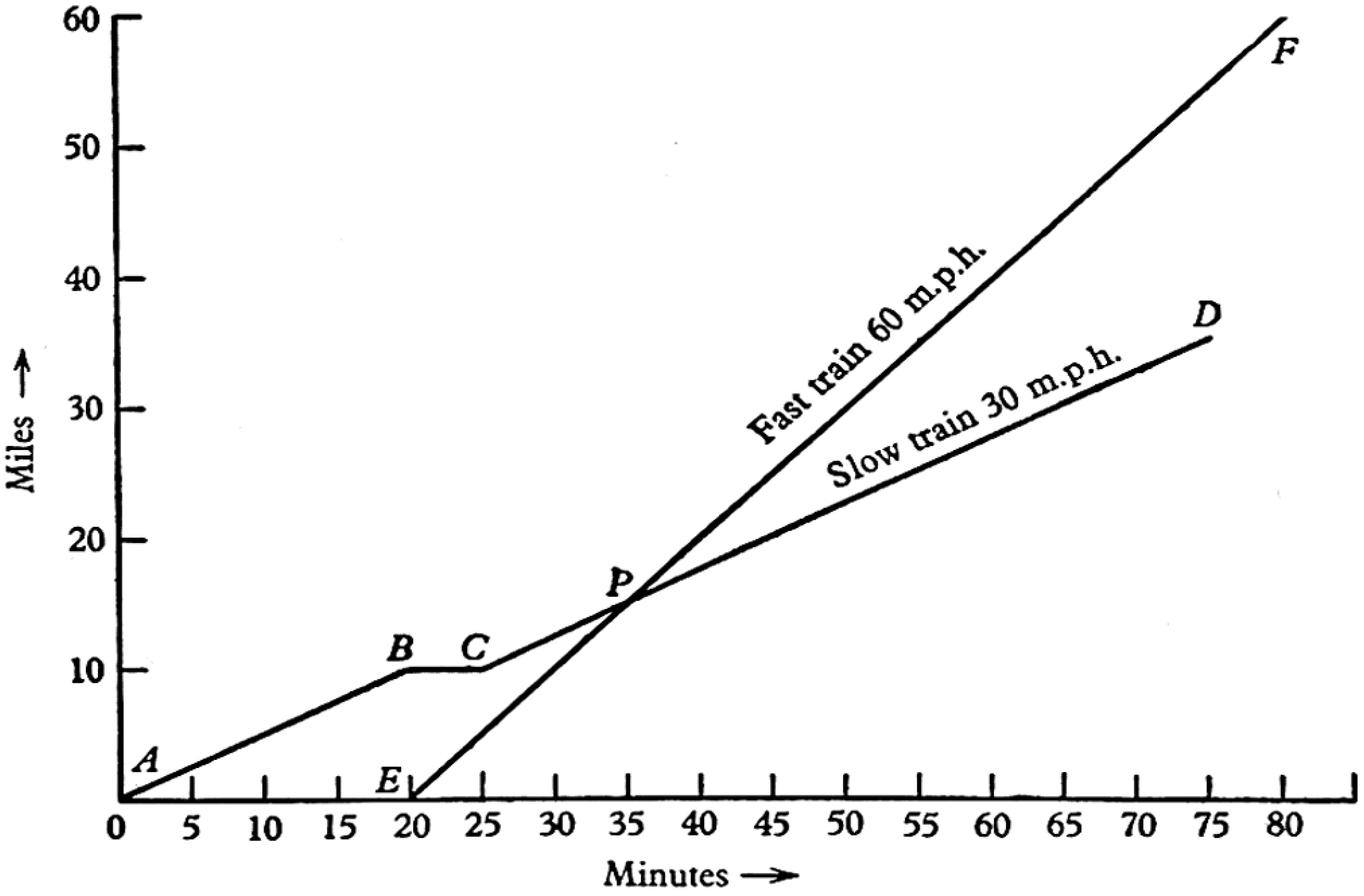


Fig. 181

Example. A slow train leaves a station X and travels at 30 m.p.h. It stops for 5 min. at a station Y, 10 miles from X, and then proceeds at 30 m.p.h. Twenty minutes after it has left X, an express leaves X and travels at 60 m.p.h. Find when and where the express overtakes the slow train.

Distance is measured vertically and time horizontally in fig. 181. The line $ABCD$ is the distance-time graph of the slow train. AB has a slope of 30 m.p.h., BC represents the time spent in the station Y, and CD also has a slope of 30 m.p.h. The line EF is the distance-time graph of the express, starting at the same point (as regards distance) as the slow train, but at the point 20 min. along the time axis, and having a slope of 60 m.p.h. Then P gives the time and place of overtaking.

From the graph, the express overtakes the slow train 15 miles from X, 35 min. after the latter has left X.

Examples for Chapter 38

1. Solve graphically for x and y:

$$5x+3y = 13,$$
$$3y-2x+1 = 0.$$
(*Chap.* 38)

2. Solve graphically for x and y:

$$2x-y+3 = 0,$$
$$6x+2y-1 = 0.$$
(*Chap.* 38)

3. Solve the simultaneous equations (*a*), (*b*) and (*c*), by plotting three graphs on the same axes:

(*a*) $3y+x = 4,\ y-3x+2 = 0.$
(*b*) $y-3x+2 = 0,\ y+2x+2 = 0.$
(*c*) $y+2x+2 = 0,\ 3y+x = 4.$ (*Chap.* 38)

4. Two stations X and Y are 50 miles apart. A train leaves X at 2 o'clock and travels at 35 m.p.h. towards Y. At 2.10 a train leaves Y travelling at 30 m.p.h. towards X, stopping for 5 min. at a station Z, 20 miles from Y. Show graphically that the trains pass in the station Z, and give the time of passing. (*Chap.* 38)

5. A motorist leaves home at 8.00 a.m. and travels at 25 m.p.h. After 40 min. he stops for 10 min. and then proceeds at 25 m.p.h. A second motorist starts from the same point as the first at 8.20 a.m. and travels in the same direction at 30 m.p.h. Find the time and place at which he overtakes the first motorist.
(*Chap.* 38)

6. A to B is 10 miles, B to C is 10 miles, C to D is 10 miles. A motorist X leaves A at noon travelling at 30 m.p.h. through B and C to D.

Find graphically:

(*a*) At what time should a motorist Y leave B, travelling at 40 m.p.h. in order to arrive at D at the same time as X?

(*b*) If a motorist left B at 12.40 p.m., at what speed would he require to travel to reach D at the same time as X?

(*c*) A motorist Z at C can travel at 40 m.p.h. and wishes to arrive in B as X is passing. At what time should he leave C?

(*Chap.* 38)

7. The cross-section of some steel girders, each 10 ft. long, is a letter L, of external lengths 6 in., and thickness 1 in. A lorry whose load must not exceed 3 tons is to carry these. Find how many it can load at one time if 1 cu.ft. of steel weighs 480 lb.

(*Chap.* 29)

8. (*a*) Prove that $\cos A + \sin A \tan A = \sec A$.

(*b*) Find the height of a flagstaff whose elevation from a point distant 60 ft. from its base is 42°.

(*c*) If $\tan\theta = 2{\cdot}5$, show that $2\sin^2\theta + 3\cos^2\theta = 2\frac{4}{29}$.

(*Chaps.* 35 *and* 34)

9. In a trapezium $ABCD$, AB is parallel to CD, and $AD = BC$. Prove that the diagonals are equal in length. (*Chap.* 23)

10. (*a*) Express in radians, using fractions of π, 36°, $22\frac{1}{2}$°, 135°.

(*b*) Express in degrees, $\frac{5\pi}{3}$ radians, $0{\cdot}7\pi$ radians, $\frac{\pi}{15}$ radians.

(*c*) The angular velocity of an electric motor is 50 revolutions per sec. Find the velocity of the surface of the commutator if its radius is 3 in. (*Chap.* 37)

Answers to Examples for Chapter 38

1. $x = 2$, $y = 1$. **2.** $x = -\frac{1}{2}$, $y = 2$.

3. (*a*) $x = 1$, $y = 1$. (*b*) $x = 0$, $y = -2$. (*c*) $x = -2$, $y = 2$.

4. 2 hr. 51 min. **5.** 9.10 a.m. 25 miles from home.

6. (*a*) 12.30 p.m. (*b*) 60 m.p.h. (*c*) 12.5 p.m.

7. 18. **8.** (*b*) 54 ft.

10. (*a*) $\frac{\pi}{5}, \frac{\pi}{8}, \frac{3\pi}{4}$; (*b*) 300°, 126°, 12°; (*c*) 78·55 ft. per sec.

CHAPTER 39

DETERMINATION OF LAWS

We have seen in Chapter 18 that the law of the straight line plays an important part in engineering problems. Experimental work frequently consists in taking readings of corresponding values of two variables which are known to be related by a linear law, or a law which may be modified to give a linear graph, and then finding the two constants from the graph furnished by these readings.

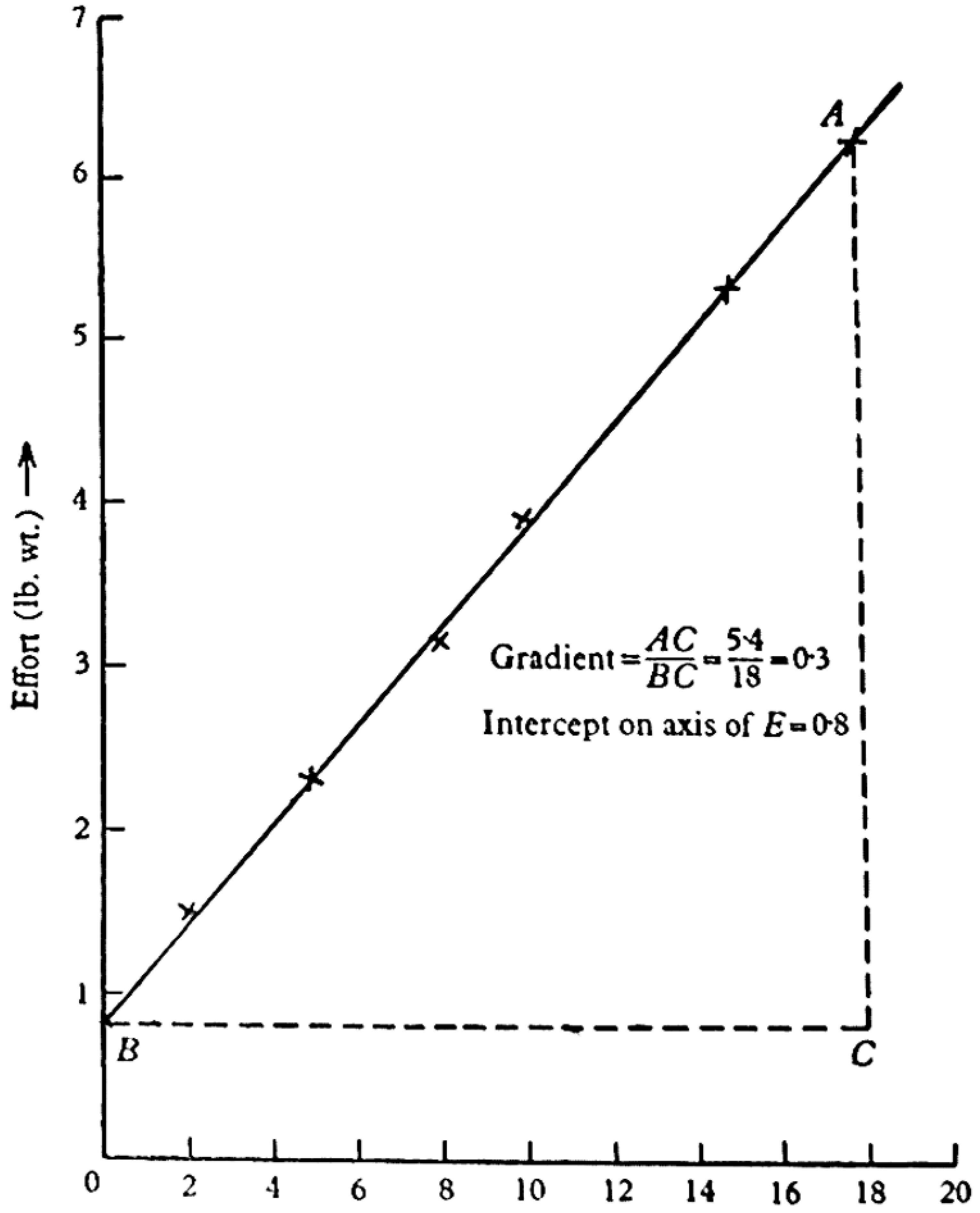


Fig. 182

(1) *Method when the true origin is used*

The readings given in the table were obtained for loads (W lb.) and corresponding efforts (E lb.) for a simple machine.

It is expected that the results should conform to a law of the form $E = aW + b$, where a and b are constants.

Load (W lb. wt.)	2	5	8	10	15	18
Effort (E lb. wt.)	1·5	2·3	3·15	3·9	5·3	6·2

It is important that the horizontal and vertical axes should be chosen correctly, otherwise a different form of the law will be obtained. The variable which stands alone must be plotted vertically, and the variable on the right-hand side is plotted horizontally. The reason will be clear if we compare with the form $y = mx + c$ for the equation of a straight line. The constants m and c represent gradient and intercept on the y axis respectively. Thus in the case of our expected law, a represents the gradient, and b represents the intercept on the vertical axis, which must be E, so E and W take the places of y and x.

These values are plotted in fig. 182, where, it will be noticed, the origin is the point (0, 0), i.e. the true origin. Also, there are three points which do not lie on the straight line drawn, but certain errors are expected in practical work, and, if the points form a straight line approximately, any small deviations may be ascribed to experimental errors. In this case, the best straight line is drawn.

The intercept on the axis of E gives the constant b; thus $b = 0{\cdot}8$.

To find the gradient. Choose two points actually on the line drawn, not necessarily points given in the table, and preferably points some distance apart. Let these be $A\,(18, 6{\cdot}2)$ and $B\,(0, 0{\cdot}8)$. The gradient is $\frac{AC}{BC}$, and is positive.

Now $\frac{AC}{BC} = \frac{5{\cdot}4}{18} = 0{\cdot}3$. Hence $a = 0{\cdot}3$.

The law of the machine is $E = 0{\cdot}3W + 0{\cdot}8$.

Example. From the experimental figures given, find the relation between Y and X.

X	6	9	12	16	20
Y	7·6	6·7	6	5	4

The graph is shown in fig. 183. The best straight line misses two of the points in the table. Writing the relationship in the form $Y = aX + b$, then $b = 9$.

For the gradient, points $A(0, 9)$ and $B(20, 4)$ are taken. Hence $a = -0{\cdot}25$.

The equation is therefore $\underline{Y = -0{\cdot}25X + 9.}$

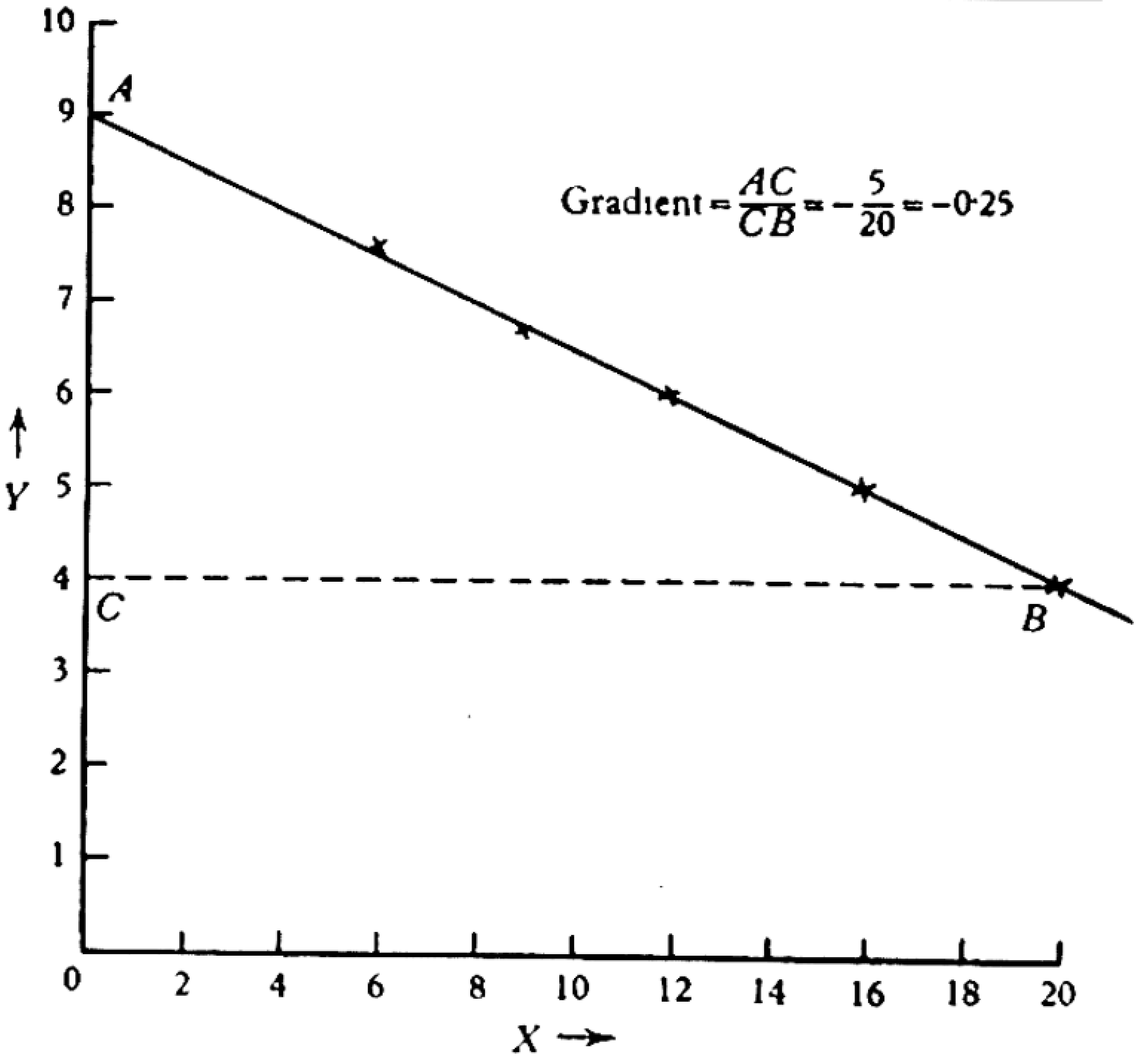


Fig. 183

(2) *Method when it is not convenient to use the true origin*

If the origin is not zero on the horizontal axis, it is incorrect to take the vertical intercept as one of the constants, for obviously, the intercept would have a different value if the origin were moved. The gradient is not affected by the position of the origin, so this is read as before, but the 'intercept' constant must now be obtained by a simple equation.

Example. Readings for the weight of fuel oil (W) consumed by an engine in a measured time, and the corresponding values of brake horse-power (B.H.P.) are given. Find a relation of the form

$$W = a\,(\text{B.H.P.}) + b \quad (a \text{ and } b \text{ constants}).$$

B.H.P.	8·0	8·5	9·0	9·5	10·0
W	6·8	7·2	7·6	8·0	8·4

W is plotted vertically and B.H.P. horizontally in fig. 184. If we had chosen $(0, 0)$ as the origin, the points would have been grouped together in the top right-hand corner of the graph, so, for greater accuracy, the origin has been taken at 8 on the B.H.P. axis, and 6 on the W axis. The points all lie on a straight line, and from the points marked A and B, the gradient is found to be 0·8.

Hence $W = 0{\cdot}8\,\text{B.H.P.} + b$ (where b is to be determined).

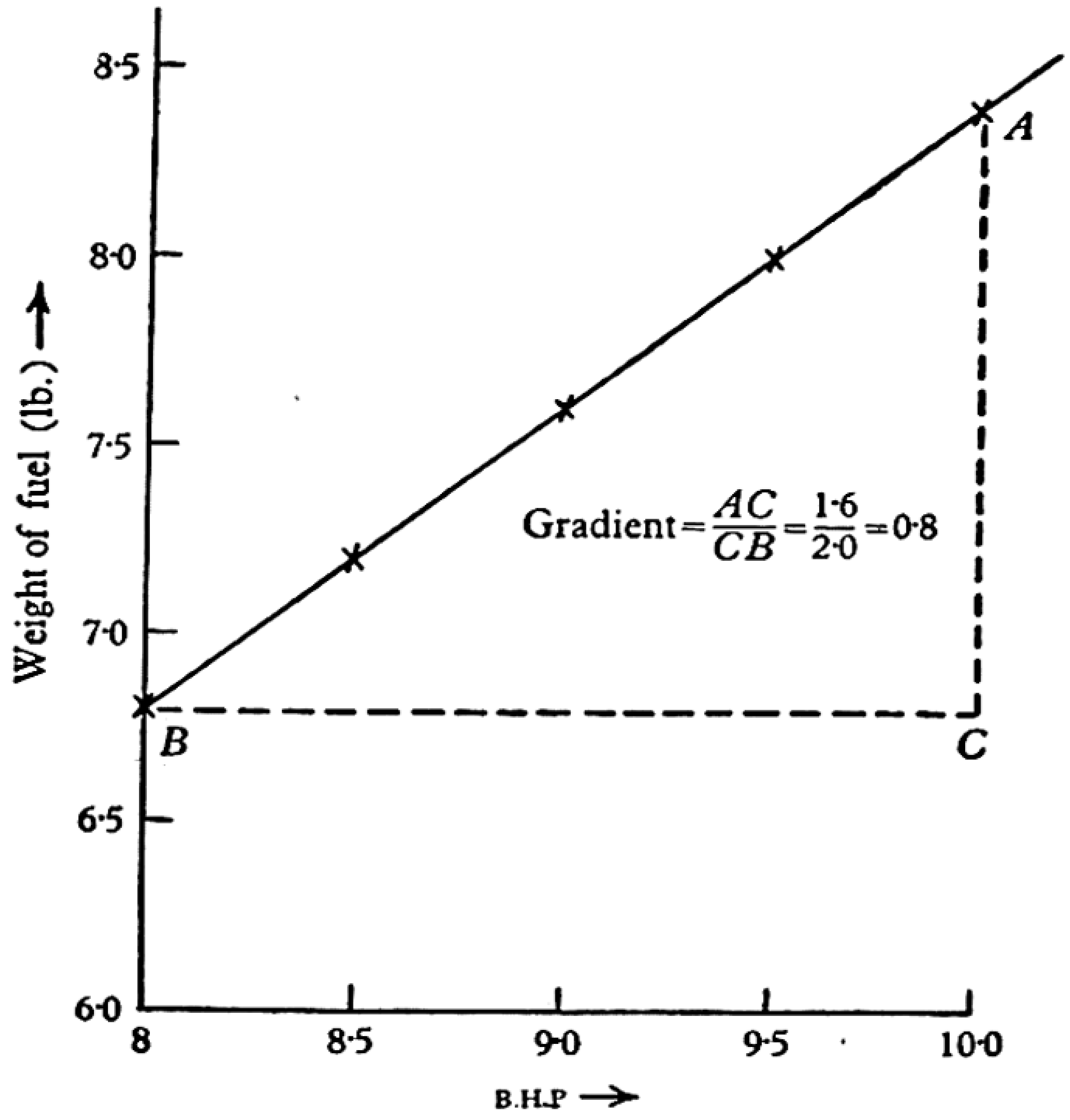


Fig. 184

To solve for b, choose any point which actually lies on the line, substitute values for W and B.H.P. in the equation, leaving b as the only unknown. From point B we have

$$6{\cdot}8 = 0{\cdot}8 \times 8 + b,$$
$$6{\cdot}8 = 6{\cdot}4 + b,$$
$$\therefore\ b = 0{\cdot}4.$$

The relationship is $\underline{W = 0{\cdot}8\,\text{B.H.P.} + 0{\cdot}4.}$

(3) *Method of plotting linear graph for relationships which are not linear*

An experiment was performed to find the relation between the heat produced (H calories) by various currents (I amperes) in the same times. It is expected that a law of the form $H = aI^2$ will fit the facts, if a is a constant. Now, plotting H against I, we know, should give a parabola (compare with $y = ax^2$) but a is not easily determined from such a graph. We decide, therefore, that if each value of I is squared, then the graph of H against I^2 should be linear if the relationship assumed is correct. The gradient of this line will be a, for if we write X for I^2, the law becomes $H = aX$, which is linear. A third line I^2 is added to the table of values given.

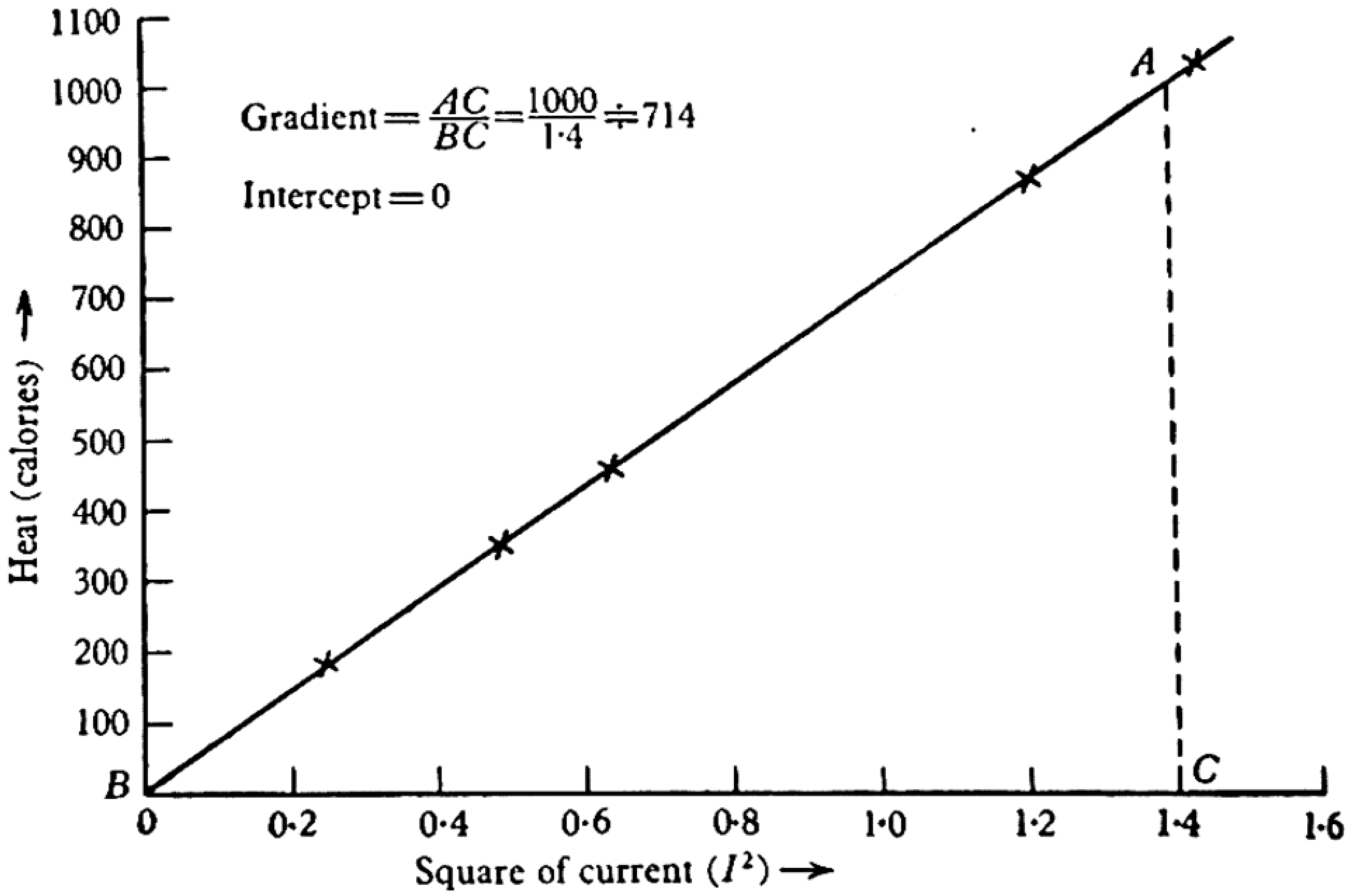


Fig. 185

I	0·5	0·7	0·8	1·1	1·2
H	180	350	460	865	1030
I^2	0·25	0·49	0·64	1·21	1·44

H is plotted against I^2 in fig. 185, giving a linear graph whose slope is approximately 714. The graph also passes through the origin, which is to be expected, because the assumed relationship contains only one constant.

Example. Various pressures (P cm. of mercury) were applied to an enclosed mass of gas, and the corresponding volumes (V c.c.) were read. It is expected that a law of the form $P = \frac{K}{V} + b$

(K and b constants) relates P and V. Find, from a linear graph, the values of K and b.

Volumes (V c.c.)	30	28	25	24	22
Excess pressures (P cm.)	4	9·7	20	24	33·1

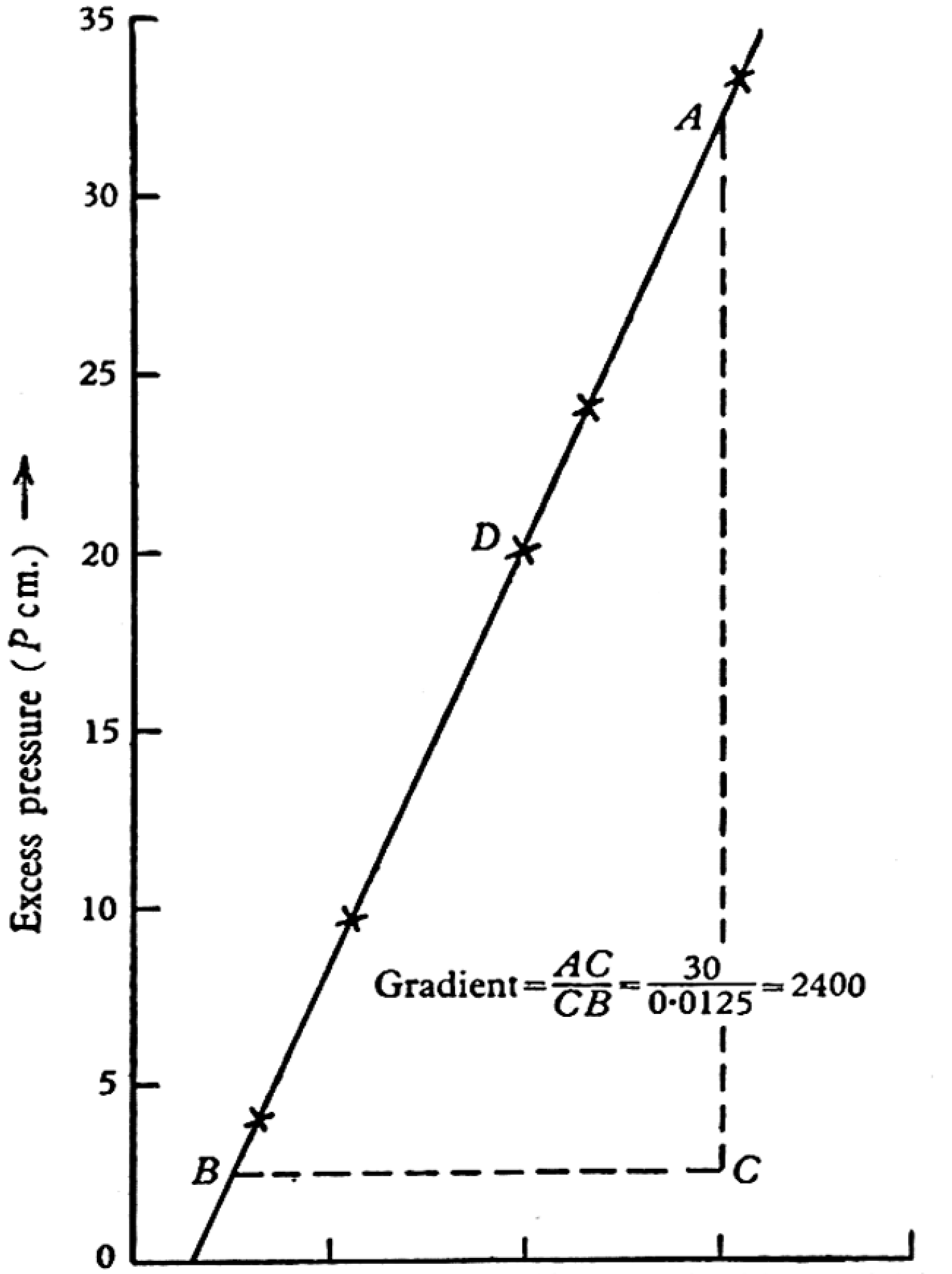


Fig. 186

It is obvious that a graph of P against V will not give a straight line, but writing $P = K\left(\frac{1}{V}\right) + b$, then it is apparent that values of P against $\frac{1}{V}$ should be linear. Values of $\frac{1}{V}$ are added as a third line to the table, and these are plotted horizontally against P vertically.

A straight line is obtained from these values in fig. 186, but the origin is not (0, 0).

$\frac{1}{V}$	0·0333	0·0357	0·0400	0·0417	0·0455
P	4	9·7	20	24	33·1

The gradient from points A and B is 2400. Hence,

$$P = \frac{2400}{V} + b.$$

From point D (0·04, 20), $\frac{1}{V} = 0{\cdot}04$, i.e. $V = 25$.

Substituting for P and V in the original equation,

$$20 = \tfrac{2400}{25} + b,$$

$$\therefore\ b = -76.$$

The required relation is $P = \frac{2400}{V} - 76.$

Examples for Chapter 39

1. The length of a coil spring is measured when the spring is lying horizontally. It is then suspended from one end and weights are hung on it. The table gives loads and corresponding extensions. Show by a graph that the load (W lb. wt.) and the extension (e in.) are connected by a law of the form $e = aW + b$, where a and b are constants, and find the law.

W (lb.)	0·5	1·0	1·25	2·0	2·5
e (in.)	0·38	0·55	0·64	0·9	1·08

(*Chap.* 39)

2. A machine gave the following readings for loads (W lb. wt.) and corresponding efforts (E lb. wt.). Show that the two variables are connected by a law of the form $W = aE + b$, where a and b are constants, and find the values of a and b.

W (lb.)	5	10	15	20	25	30
E (lb.)	1·13	1·8	2·38	3·0	3·6	4·25

(*Chap.* 39)

3. The table shows weights of steam (W lb.) used in a given time by a steam engine, and the corresponding values of the indicated horse-power (I.H.P.). Find a relation of the form

$$W = a\,(\text{I.H.P.}) + b.$$

I.H.P.	150	180	200	250	280
W	2650	3250	3650	4650	5250

(*Chap.* 39)

4. The resistance of a wire (R ohms) was measured at different temperatures (t° C.). Find graphically a law of the form

$$R = a + bt,$$

where a and b are constants.

t° C.	20	40	60	80	100
R ohms	11·02	11·83	12·65	13·46	14·28

(*Chap.* 39)

5. The distances (s ft.) covered by a falling body in various times (t sec.) are given in the table. Show by a linear graph that a relation of the form $s = at^2$ connects the variables.

Times (t sec.)	1·5	3·2	4·8	5·0	6·5
Distances (s ft.)	36	164	369	400	676

(*Chap.* 39)

6. The readings in the table below are expected to conform to a law of the form $Y = \dfrac{A}{X} + B$. Test if this is the case, by drawing a linear graph, and find the law.

X	2	5	7	9	12	16
Y	7·8	3·3	2·44	1·97	1·55	1·24

(*Chap.* 39)

7. (*a*) Solve graphically for x and y,

$$4x - 6y = 9, \quad 8x - 14y = 17.$$

(*b*) Two towns A and B are 80 miles apart. A motorist leaves A at 10 o'clock, and travels towards B at 35 m.p.h. After 1 hr. he rests for 15 min. and then proceeds at the same speed as before. Another motorist leaves B at 10.30 and travels towards A at 40 m.p.h. Find graphically when and where they pass.

(*Chap.* 38)

8. (*a*) ABC is any acute-angled triangle, and AD is perpendicular to BC. Prove that

$$a = c \cos B + b \cos C,$$

and write the length of AD, when $a = b = c$.

(*b*) The legs of a pair of compasses are 4 in. long. Find the distance between the point and the pencil when the angle between the legs is 35°.

(*Chaps.* 34 *and* 33)

9. A cylinder, radius R, and height h is to be cast from the lead obtained by melting n spheres each of radius r. Write a formula for R in terms of n, r and h. (For transformation of formulae, see Chapter 16.)

Find R when $n = 150$, $r = 0{\cdot}2$ and $h = 25$. (*Chap.* 31)

10. The areas of cross-section at equal depths of a ship's hold 15 ft. deep are as follows: 1100, 1080, 1060, 1040, 1020, 1000, 970, 940, 930, 890, 860, 840, 800, 770, 730, 700 sq.ft., where the areas 1100 and 700 refer to the top and bottom respectively. Find the capacity of the hold.

(*Chap.* 39)

Answers to Examples for Chapter 39

1. $e = 0{\cdot}35W + 0{\cdot}2$.
2. $W = 8E - 4$.
3. $W = 20\,(\text{I.H.P.}) - 350$.
4. $R = 10{\cdot}2 + 0{\cdot}041t$.
5. $s = 16t^2$.
6. $Y = \dfrac{15{\cdot}1}{X} + 0{\cdot}2$.
7. (*a*) $x = 3$, $y = \frac{1}{2}$. (*b*) 42 miles from A at 11.27.
8. (*a*) $\dfrac{a\sqrt{3}}{2}$. (*b*) 2·406 in.
9. $R = 2\sqrt{\dfrac{nr^3}{3h}}$, 0·253 cm.
10. 13,830 cu.ft.

CHAPTER 40

SOLUTION OF QUADRATIC EQUATIONS BY GRAPHICAL METHODS

We have seen that the point at which two straight lines intersect gives the solutions for the simultaneous equations in x and y which represent the straight lines (Chapter 38).

In general, the point or points in which two graphs intersect give the answers for the equations of the graphs solved simultaneously.

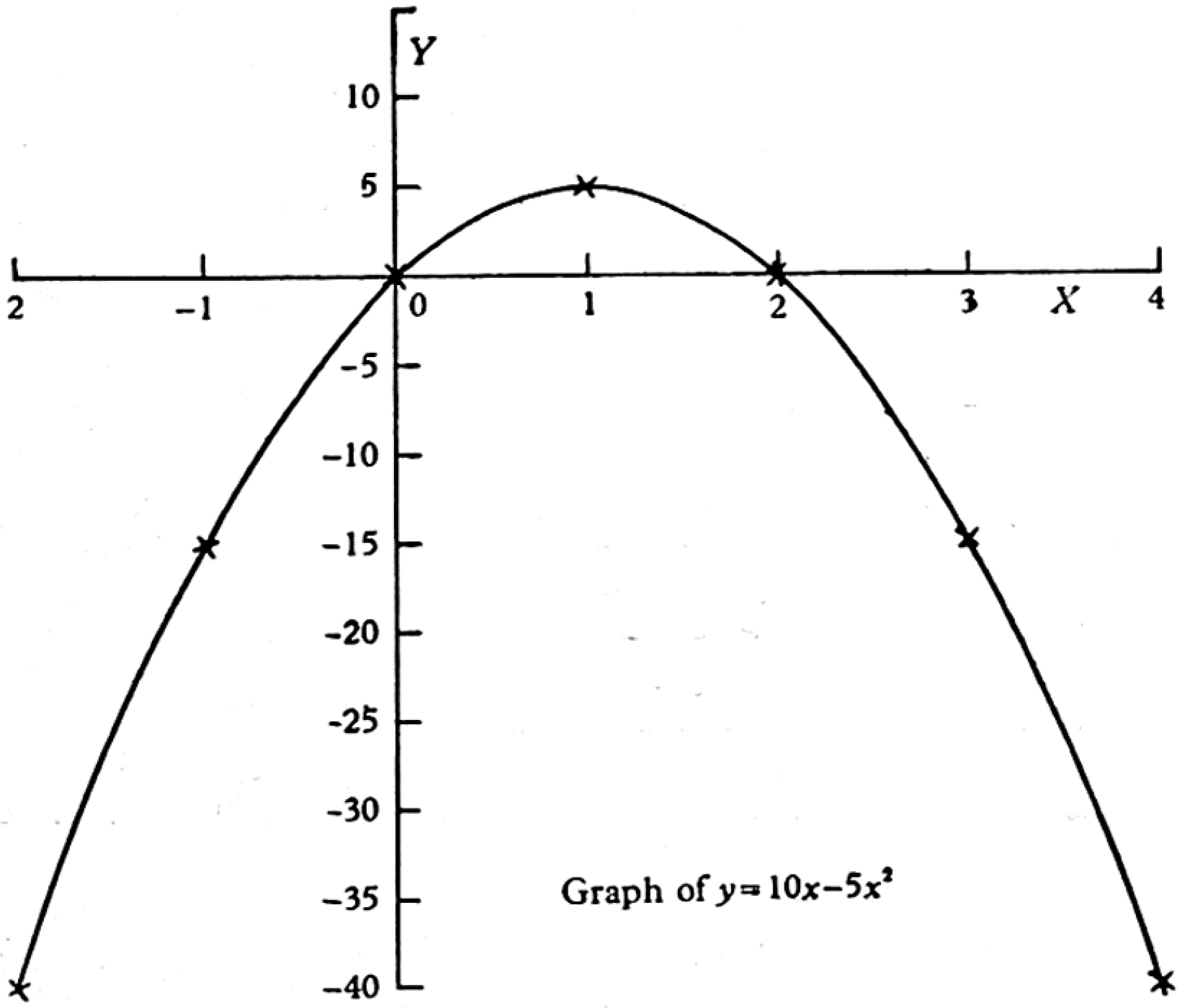


Fig. 187

(1) *Solution of quadratic equation by intersection of parabola and axis of x*

Consider the graph of $y = 10x - 5x^2$.

x	-2	-1	0	1	2	3	4	A
x^2	4	1	0	1	4	9	16	
$-5x^2$	-20	-5	0	-5	-20	-45	-80	B
$+10x$	-20	-10	0	10	20	30	40	C
$y = 10x - 5x^2$	-40	-15	0	5	0	-15	-40	$D = B + C$

Plotting values of line A horizontally against line D vertically the graph in fig. 187 is obtained. It is seen to intersect the axis of x at points where $x = 0$ and $x = 2$.

Now these points are the intersections of two graphs, namely, $y = 10x - 5x^2$ and $y = 0$ (the axis of x), and if we replace the value of y in the first equation by the value of y in the second equation we have $0 = 10x - 5x^2$, which is a quadratic equation in x. Its solutions, or roots are $x = 0$ and $x = 2$, that is, the values of x in $y = 10x - 5x^2$, corresponding to $y = 0$, are 0 and 2.

Hence, to solve graphically an equation $ax^2 + bx + c = 0$, plot $y = ax^2 + bx + c$, and the abscissae of the points at which the parabola intersects the x axis are the roots of the required equation.

Example. Plot $y = x^2 - 3x + 1$ from $x = -1$ to $x = 3$, and from the graph obtain the roots of the equation $x^2 - 3x + 1 = 0$.

$$y = x^2 - 3x + 1.$$

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
x^2	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	$2\frac{1}{4}$	4	$6\frac{1}{4}$	9
$-3x$	3	$1\frac{1}{2}$	0	$-1\frac{1}{2}$	-3	$-4\frac{1}{2}$	-6	$-7\frac{1}{2}$	-9
$+1$	1	1	1	1	1	1	1	1	1
$y = x^2 - 3x + 1$	5	$2\frac{3}{4}$	1	$-\frac{1}{4}$	-1	$-1\frac{1}{4}$	-1	$-\frac{1}{4}$	1

Fig. 188 is plotted from these figures and is seen to intersect the x axis at two points A and B. At A, $x = 0{\cdot}38$ (approx.) and at B, $x = 2{\cdot}61$ (approx.).

The roots of the equation $x^2 - 3x + 1 = 0$ are $x = 0{\cdot}38$ and $x = 2{\cdot}61$.

(2) *Solution of* $ax^2 + bx + c = 0$ *by plotting* $y = ax^2$ *and* $y = -bx - c$.

The same results may be obtained, and a little time saved, if two graphs are plotted instead of one.

Consider the solution of $ax^2 + bx + c = 0$. We have seen that the intersections of $y = ax^2 + bx + c$ and $y = 0$ give the roots; that is, by equating the values of y for two simultaneous equations, an equation in x is solved. Instead of plotting

$$y = ax^2 + bx + c,$$

plot $y = ax^2$ and $y = -bx-c$, then for the two points where the graphs intersect, the values of y will be equal, so that at these points of intersection,

$$ax^2 = -bc-c,$$

or

$$ax^2+bx+c = 0.$$

Hence the abscissae of the points of intersection of $y = ax^2$ and $y = -bx-c$ are the roots of the equation $ax^2+bx+c = 0$.

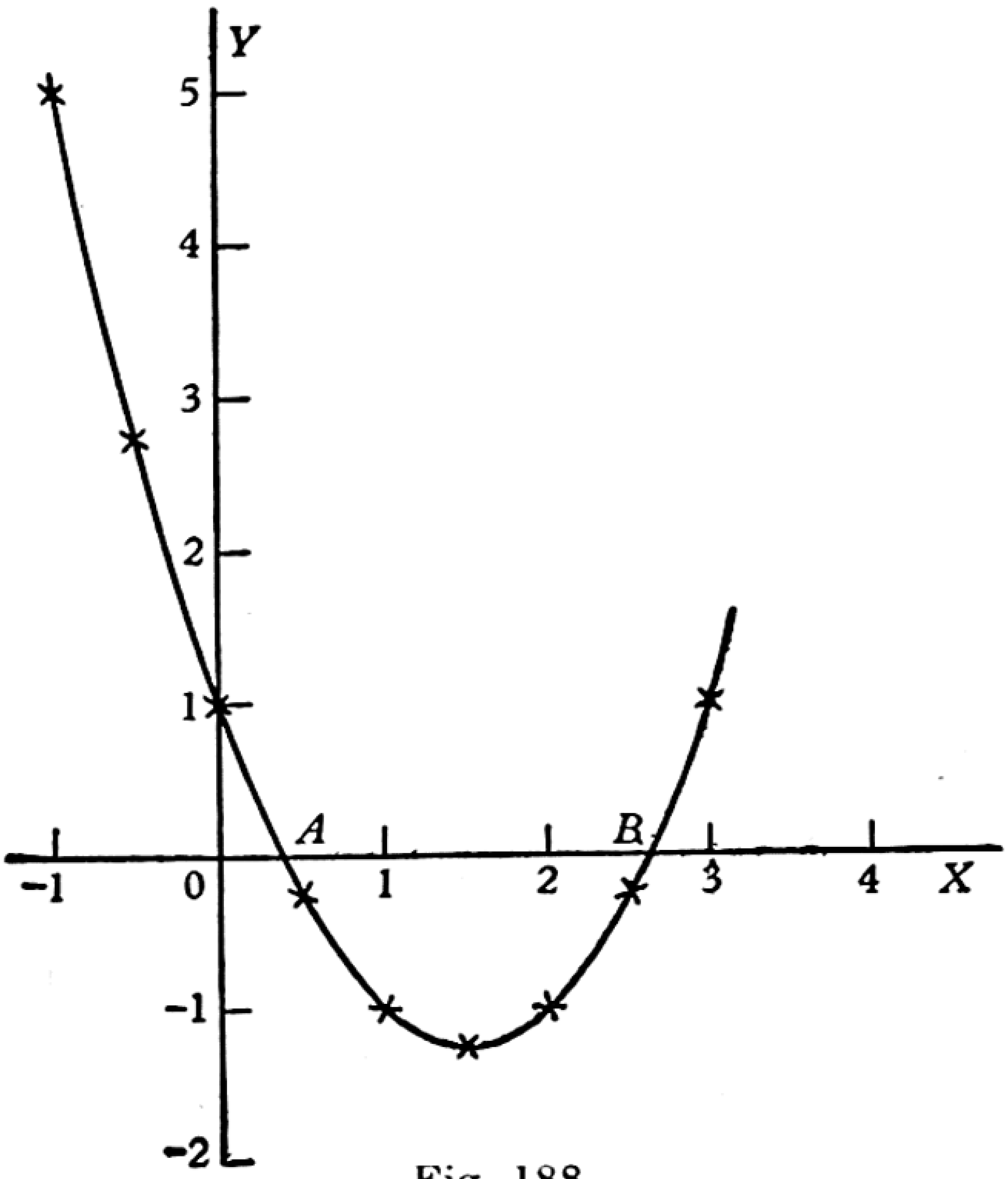


Fig. 188

Example. Solve the equation $2x^2-x-3 = 0$ by plotting $y = 2x^2$ between $x = -2$, and $x = +2$, and a linear graph.

If $2x^2-x-3$ represents ax^2+bx+c, then the required graphs are $y = 2x^2$ and $y = -(-x)-(-3)$, or $y = x+3$.

I.e. the points of intersection represent

$$2x^2 = x+3,$$

or

$$2x^2-x-3 = 0.$$

$$y = 2x^2.$$

x	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
x^2	4	$2\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	$2\frac{1}{4}$	4
$2x^2$	8	$4\frac{1}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	$4\frac{1}{2}$	8

The graph of $y = 2x^2$ is shown in fig. 189.

To plot $y = x + 3$, choose three values for x.

x	-2	0	$+2$
$y = x + 3$	1	3	5

This line is shown also in fig. 189, and is seen to intersect $y = 2x^2$ at points A and B, whose abscissae are -1 and $+1\frac{1}{2}$.

Hence, the roots of $2x^2 - x - 3 = 0$ are $x = -1$ and $x = 1\frac{1}{2}$.

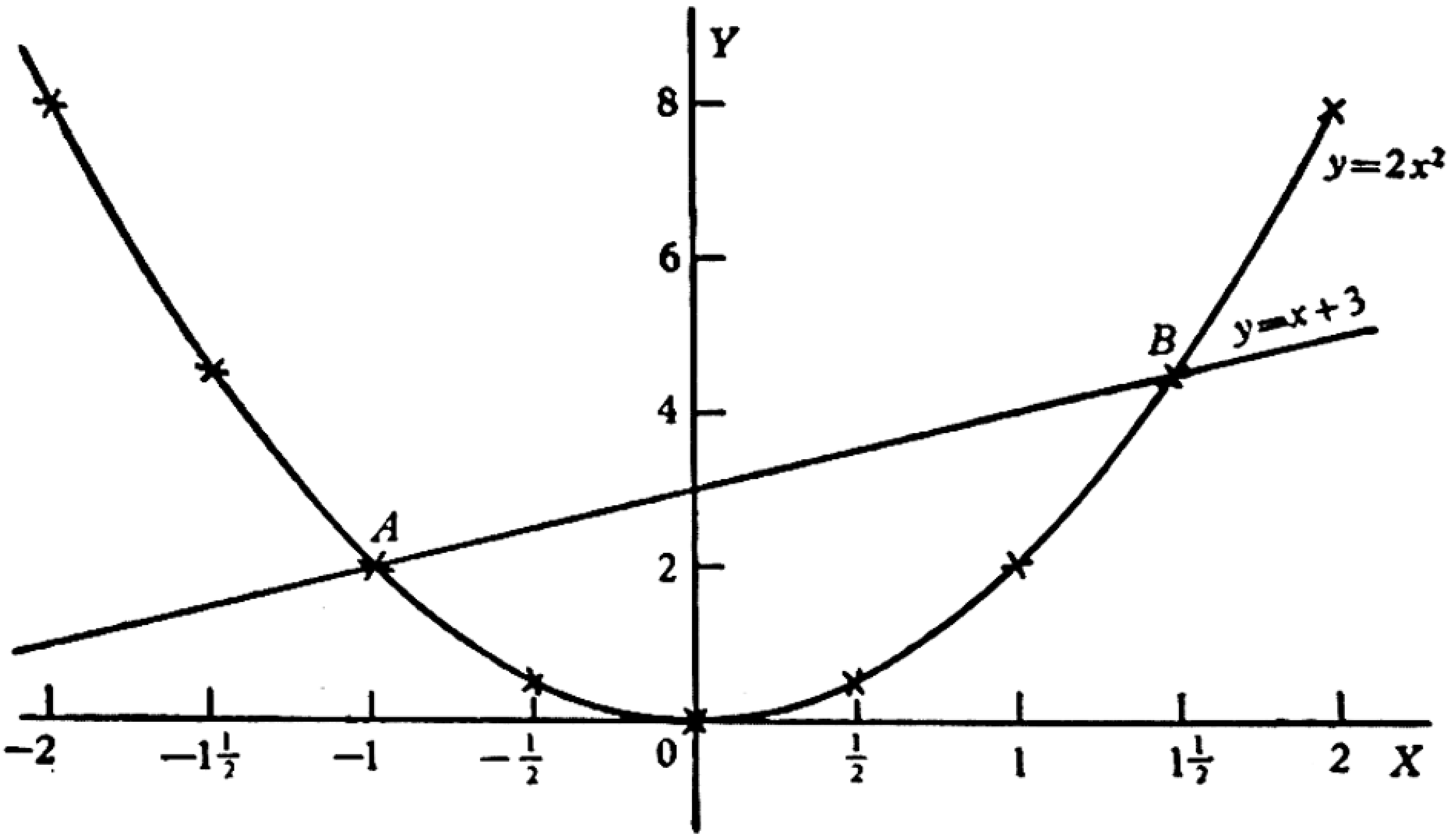


Fig. 189

(3) *Solution of quadratic equation from parabola intersected by a horizontal line*

Example. Plot $y = 3 - 4x - 4x^2$ from $x = -2$ to $x = +1$, and from the graph solve the following equations:

(i) $3 - 4x - 4x^2 = 0$, (ii) $4x^2 + 4x + 1 = 0$, (iii) $2x^2 + 2x - 3 = 0$.

$$y = 3 - 4x - 4x^2.$$

x	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
x^2	4	$2\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1
$-4x^2$	-16	-9	-4	-1	0	-1	-4
$-4x$	8	6	4	2	0	-2	-4
$+3$	3	3	3	3	3	3	3
$y = 3 - 4x - 4x^2$	-5	0	3	4	3	0	-5

The graph is shown in fig. 190.

(i) The roots of $3-4x-4x^2=0$ are the abscissae of the points where the graph intersects the axis of x (A and B)

$$\underline{x=-1\tfrac{1}{2} \quad \text{and} \quad x=\tfrac{1}{2}.}$$

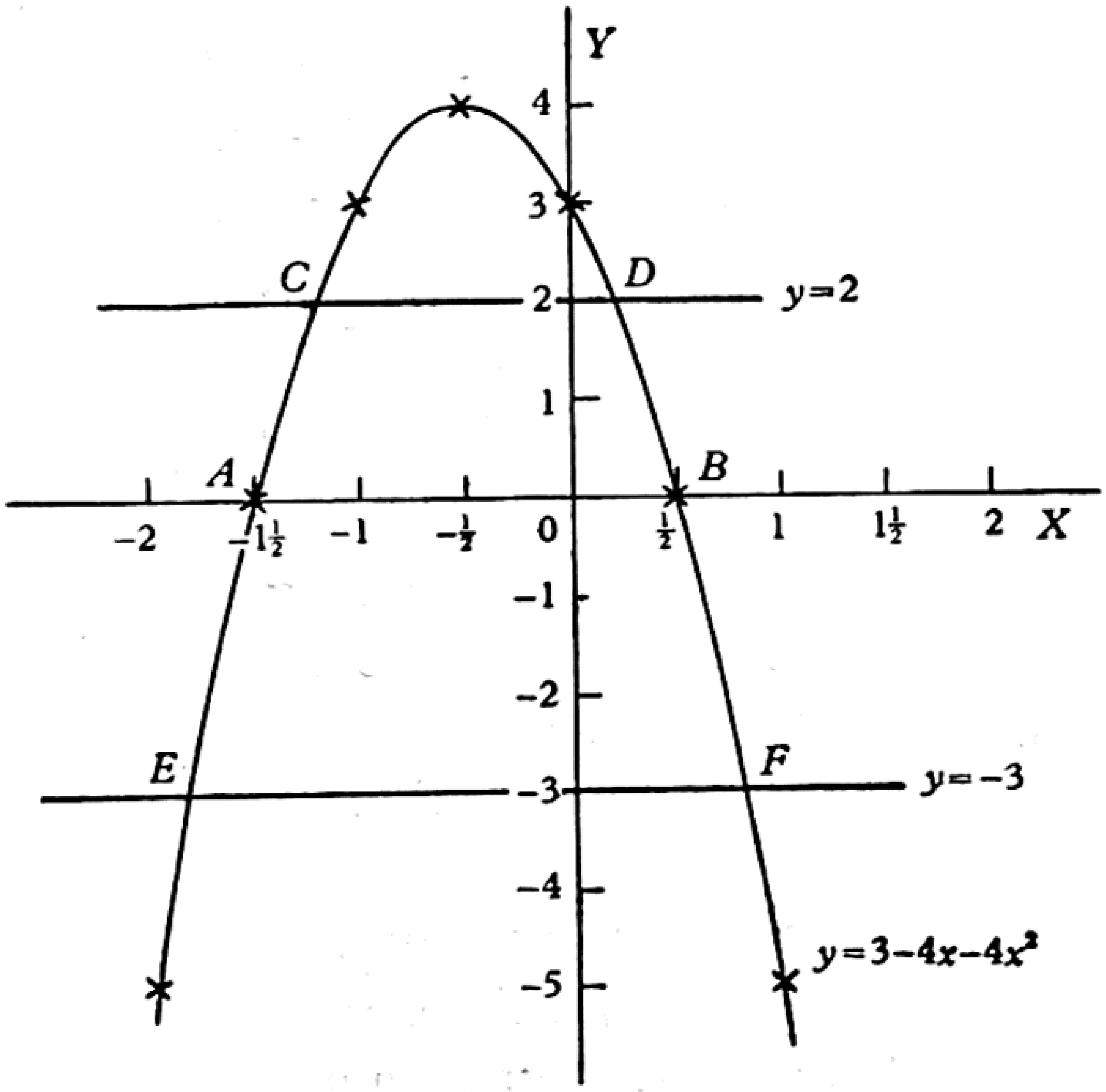


Fig. 190

(ii) $4x^2+4x-1=0$ may be written $1-4x-4x^2=0$ (multiplying by -1) in order to have the same coefficients of x^2 and x as the equation to the curve.

Now the points of intersection of $y=3-4x-4x^2$ and $y=c$ (a horizontal line) give the roots of the equation $c=3-4x-4x^2$ or $3-c-4x-4x^2=0$.

But the equation to be solved is $1-4x-4x^2=0$; therefore $3-c=1$, i.e. $c=2$. The horizontal line to be drawn is $y=2$.

The points of intersection of $y=3-4x-4x^2$ and $y=2$ are C and D whose abscissae are $-1{\cdot}2$ and $0{\cdot}2$ (approximately).

The roots of $4x^2+4x-1=0$ are $-1{\cdot}2$ and $0{\cdot}2$ (approx.).

(iii) The coefficients of x^2 and of x in the equation

$$2x^2+2x-3=0$$

will be the same as those in the equation of the curve if the former equation is multiplied by -2, when it becomes

$$6-4x-4x^2=0.$$

The abscissae of the points of intersection of $y = 3-4x-4x^2$ and $y = c$ are the roots of the equation $c = 3-4x-4x^2$, or

$$3-c-4x-4x^2 = 0.$$

Therefore $3-c = 6$, whence $c = -3$.

The horizontal line to be plotted is $y = -3$.

This line intersects the curve $y = 3-4x-4x^2$ at E and F giving values of x, $-1{\cdot}82$ and $0{\cdot}82$ (approx.).

Thus the roots of $2x^2+2x-3 = 0$ are $-1{\cdot}82$ and $0{\cdot}82$ approx.

(4) *Types which require a line of the form* $y = mx+c$

When the equation of the graph and the equation to be solved differ in the coefficient of x, then the straight line to be drawn is of the form $y = mx+c$. Note that the coefficients of x^2 in the graph and the equation to be solved must be identical, or made identical by multiplying the equation to be solved (not the graph) by some constant.

The method is illustrated in the following example.

Example. Plot $y = x^2-2x+2$ from $x = -2$ to $x = 4$ and, by means of the graph and linear graphs solve (i) $x^2-3x+2 = 0$, (ii) $3-x-2x^2 = 0$.

$$y = x^2-2x+2.$$

x	-2	-1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$-2x$	4	2	0	-2	-4	-6	-8
$+2$	2	2	2	2	2	2	2
$y = x^2-2x+2$	10	5	2	1	2	5	10

The graph is shown in fig. 191.

(i) To solve $x^2-3x+2 = 0$. (Coefficient of x^2 identical with that in graph, but coefficient of x different.)

Let $y = mx+c$ be the straight line to be drawn.

Where $y = x^2-2x+2$ and $y = mx+c$ intersect, it follows that

$$x^2-2x+2 = mx+c \quad \text{(equating values of } y\text{)},$$

$$x^2-(2+m)x+(2-c) = 0. \tag{X}$$

The equation which is to be solved is (X), thus

$$x^2-(2+m)x+(2-c) = 0 \quad \text{and} \quad x^2-3x+2 = 0$$

are identical, therefore

$$-(2+m) = -3 \quad \text{(equating coefficients of } x\text{)}$$

and

$$2-c = +2 \quad \text{(equating constant terms),}$$

whence $-2-m = -3$, or $m = 1$ and $c = 0$.

The straight line is therefore $y = x+0$ or $y = x$.

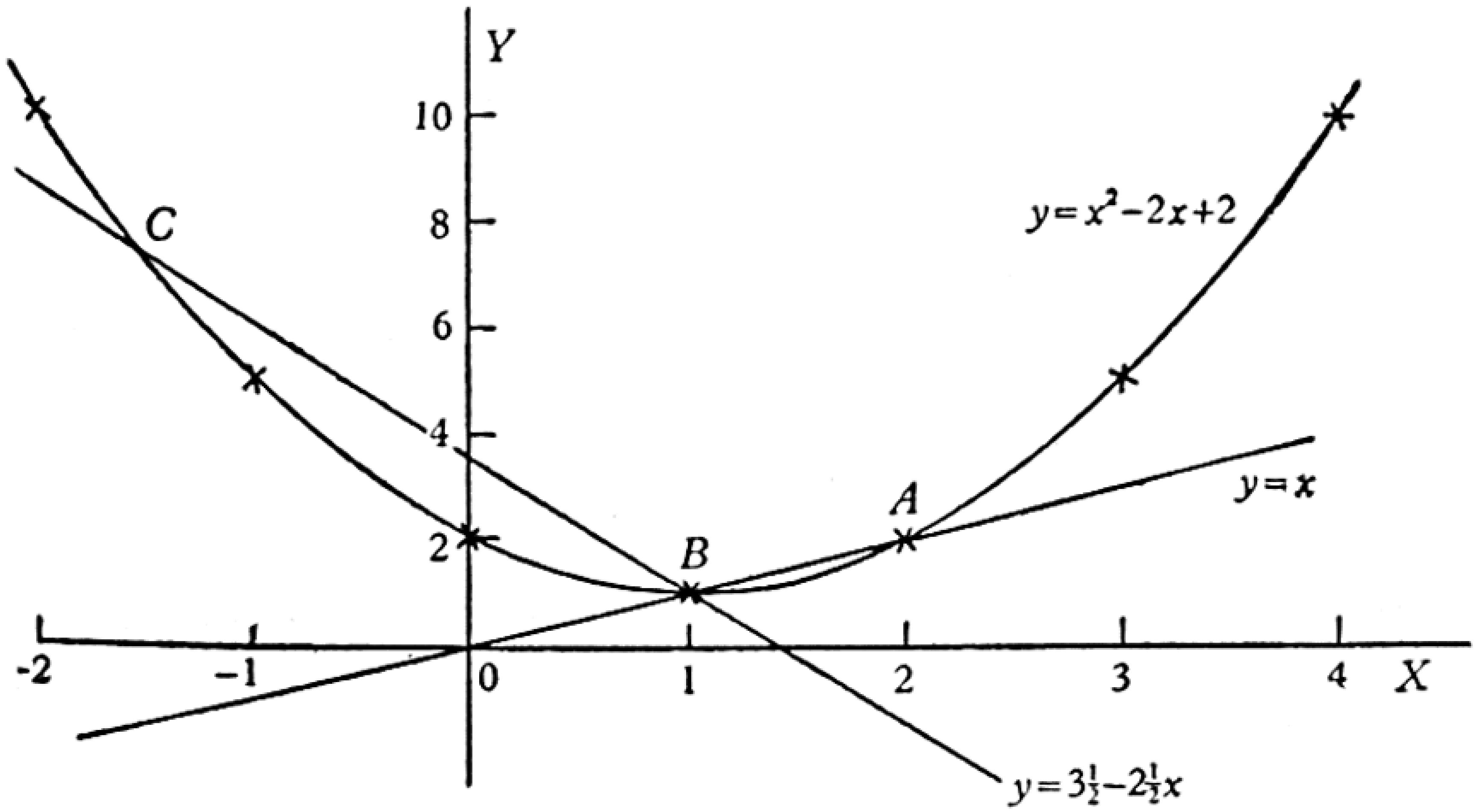


Fig. 191

The points of intersection of $y = x$ and $y = x^2-2x+2$ are A and B, which give $\underline{x = 2 \text{ and } x = 1}$ as the $\underline{\text{roots of } x^2-3x+2 = 0}$.

(ii) $3-x-2x^2 = 0$. Dividing by -2, this equation becomes $x^2+\frac{x}{2}-\frac{3}{2} = 0$, and the coefficient of x^2 is the same as that in the equation of the curve.

Let $y = mx+c$ be the straight line required, then the points of intersection of line and curve give the roots of

$$x^2-2x+2 = mx+c,$$

or

$$x^2-(2+m)x+(2-c) = 0. \tag{X}$$

The equation to be solved is (X) which is therefore identical with $x^2+\frac{x}{2}-\frac{3}{2} = 0$, so

$$-(2+m) = \tfrac{1}{2} \quad \text{(equating coefficients of } x\text{)}$$

and

$$2-c = -\tfrac{3}{2} \quad \text{(equating constant terms),}$$

i.e. $-2-m = \frac{1}{2}$ or $m = -\frac{3}{2}$, also $c = 3\frac{1}{2}$.

The straight line required is therefore $y = -\frac{3x}{2} + 3\frac{1}{2}$.

This line is seen to intersect the curve at points B and C, whose abscissae are 1 and $-\frac{3}{2}$.

Hence the roots of $3-x-2x^2 = 0$ are $x = 1$ and $x = -\frac{3}{2}$.

Examples for Chapter 40

1. (*a*) Solve $x^2-5x+6 = 0$ by plotting $y = x^2-5x+6$ from $x = 0$ to $x = 5$.

(*b*) Solve the equation in (*a*) by plotting $y = x^2$ and a linear graph. Show that the results agree by the two methods. (*Chap.* 40)

2. (*a*) Solve $0 = 1-3x-2x^2$ by plotting $y = 1-3x-3x^2$ from $x = -4$ to $x = +3$.

(*b*) Solve the same equation by plotting $y = -2x^2$ and a linear graph. (*Chap.* 40)

3. Plot $y = x^2-x-1$ from $x = -3$ to $x = 4$, and from the graph find the roots of (i) $x^2-x-1 = 0$, (ii) $x^2-x-3 = 0$. (*Chap.* 40)

4. Plot $y = 2x^2-x+1$ from $x = -1\frac{1}{2}$ to $x = 2\frac{1}{2}$. Show that the equation $2x^2-x+1 = 0$ has no roots which may be expressed as ordinary numbers. From the graph solve (i) $2x^2-x-1 = 0$, (ii) $x^2-\frac{1}{2}x-1 = 0$. (*Chap.* 40)

5. Plot $y = 3x^2+x-4$ from $x = -3$ to $x = 3$, and from the graph solve

(i) $3x^2+x-4 = 0$, (ii) $3x^2+2x-1 = 0$, (iii) $x^2-2x-1 = 0$. (*Chap.* 40)

6. Plot $y = 2-x-x^2$ from $x = -4$ to $x = +3$, and solve the following equations from the graph:

(i) $x^2+x-2 = 0$, (ii) $x^2-2x-3 = 0$, (iii) $8x^2-2x-3 = 0$. (*Chap.* 40)

7. Express an angular velocity of 5·2 revolutions per sec. in radians per sec., and express an angular velocity of 28·5 radians per sec. in revolutions per min.

The kinetic energy of a rotating flywheel is given by

$$\text{K.E.} = \frac{1}{2} \cdot \frac{MR^2}{2} \cdot (\omega)^2,$$

where M is the mass of the wheel in lb., R is the radius in feet, and ω is the angular velocity in radians per sec.; the energy is then given in foot-poundals. Calculate the kinetic energy of a flywheel of mass 56 lb., radius 10 in., revolving at 80 revolutions per min. (*Chap.* 37)

8. An experiment on a screw jack gave the following figures for effort (E lb. wt.) and load (W lb. wt.).

Find the law connecting effort and load in the form

$$W = a + bE \quad (a \text{ and } b \text{ constants}).$$

E (lb. wt.)	11·8	13·5	15·2	16·8	18·5
W (lb. wt.)	100	120	140	160	180

(*Chap.* 39)

9. Two trees A and B stand on one bank of a straight portion of a river. A third tree C stands on the opposite bank between A and B. If $AB = 120$ ft., angle $CAB = 28°$, angle $CBA = 40°$, find the width of the river. (*Chaps.* 34 *and* 36)

10. Find the weight in pounds of 1 gross of steel tubes each 1 ft. long, outer radius 2·5 in., inner radius 2·25 in., if the specific gravity of steel is 7·6, and 1 cu.ft. of water weighs 62·3 lb. (*Chap.* 31)

Answers to Examples for Chapter 40

1. (*a*) $x = 2$, $x = 3$. (*b*) Straight line is $y = 5x - 6$.

2. (*a*) $x = -1{\cdot}78$, $x = 0{\cdot}28$. (*b*) Straight line is $y = 3x - 1$.

3. (i) $x = 1{\cdot}62$, $x = -0{\cdot}62$.
   (ii) $x = 2{\cdot}3$, $x = -1{\cdot}3$. Straight line is $y = 2$.

4. (i) $x = 1$, $x = -\frac{1}{2}$. Straight line is $y = 2$.
   (ii) $x = 1{\cdot}28$, $x = -0{\cdot}78$. Straight line is $y = 3$.

5. (i) $x = 1$, $x = -1\frac{1}{3}$.
   (ii) $x = -1$, $x = \frac{1}{3}$. Straight line is $y = -x-3$.
   (iii) $x = 2{\cdot}41$ and $x = -0{\cdot}41$. Straight line is $y = 7x-1$.

6. (i) $x = -2$, $x = 1$.
   (ii) $x = 3$, $x = -1$. Straight line is $y = -3x-1$.
   (iii) $x = \frac{3}{4}$, $x = -\frac{1}{2}$. Straight line is $y = \dfrac{13}{8}-\dfrac{5x}{4}$.

7. 32·68 radians per sec., 0·0756 revolution per min. 683·4 ft. poundals.

8. $W = 12E-42$.

9. 39 ft.

10. 1767 lb.

REVISION EXAMPLES ON PART 1

SET 1

1. (*a*) $\frac{3}{4}(\frac{4}{5}$ of $\frac{1}{2}+\frac{2}{3}$ of $1\frac{1}{11})$.

(*b*) $3{\cdot}1(4{\cdot}7-2{\cdot}8+4{\cdot}3)$.

(*c*) $3{\cdot}78\times10^2+0{\cdot}9\times10^4+52{\cdot}3\times10^{-2}$.

(*d*) Express as decimals, $\frac{2}{9}$, $\frac{2}{7}$, $\frac{3}{17}$, and find their sum correct to four significant figures. (*Chaps.* 1 *and* 2)

2. (*a*) Taking $\sqrt{3}$ as $1{\cdot}732$, find $\sqrt{432}$.

(*b*) Show that $(2+\sqrt{2})^2=6+4\sqrt{2}$.

(*c*) If $v^2=10816$, find v. (*Chap.* 4)

3. (*a*) The alloy manganin consists of 84 % copper, 12 % manganese, and 4 % nickel, by weight. Find the weight of copper and of manganese in a sample of manganin which contains 7·2 g. of nickel.

(*b*) A column of air is enclosed in a tube of uniform bore, and its length is 25 cm. when the pressure is 84 cm. of mercury. It is allowed to expand according to Boyle's Law until the pressure is 76 cm. of mercury. Find the length of the air column. (*Chap.* 3)

4. (*a*) A hopper weighing Y tons, and carrying X tons of ore, is raised to a height of s yd. in 3 min. The total work done, in foot-pounds, is equal to the weight lifted (in pounds), multiplied by the distance raised (in feet). Write an expression for the work done per minute. (*Chap.* 5)

(*b*) Rewrite the formula with Y as subject, and evaluate Y when $X=4$, $s=35$, and Work $=548{,}800$ ft.lb. (*Chap.* 16)

5. (*a*) Evaluate (i) $(ax-by+cz)(a-b)$. (ii) $(a-b)^3$. (*Chap.* 7)

(*b*) A formula for lenses is $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$. Rewrite the formula with f as subject, and evaluate f when $u=20$, and $v=-10$. (*Chap.* 16)

6. (*a*) Evaluate by logs: $4{\cdot}28 \times \sqrt{21{\cdot}6 \times 5410}$.

(*b*) Evaluate A from the formula $A = P\left(1+\frac{r}{100}\right)^n$ when $P = 550$, $r = 2{\cdot}5$, $n = 4$. (*Chap.* 9)

7. (*a*) Resolve into their simplest factors:

(i) $a^2 - ab - 6b^2$.

(ii) $ab - bc + 2ad - 2cd$.

(iii) $1 - \frac{a^3}{125}$.

(*b*) Divide $4d^4 + 7d^3 - 6d^2 - 7d + 2$ by $4d^2 + 7d - 2$, and hence factorise $4d^4 + 7d^3 - 6d^2 - 7d + 2$. (*Chaps.* 11 *and* 12)

8. (*a*) Solve for x: (i) $3x + 15 = 7 - 5x$.

(ii) $(3x-2)(x-1) = x(3x-4)$. (*Chap.* 14)

(*b*) A train averages x miles per hour for the first hour, $1{\cdot}3x$ miles per hour for the next $1\frac{1}{2}$ hours, and $2x$ miles per hour for the next half-hour. The total distance covered is $118\frac{1}{2}$ miles. Find x. (*Chap.* 15)

9. (*a*) Solve for x and y:

$$3x + 4y = 5,$$
$$4(x+y) + 3(y-x) = 4{\cdot}5.$$

(*Chap.* 17)

(*b*) The length of a bar of stainless steel (L_t) at a temperature t° C. is given by

$$L_t = L_0(1 + \alpha t),$$

where L_0 is the length of the bar at 0° C., and α is the coefficient of linear expansion of the metal. At 100° C. the length of a given bar is 100·11 cm., and at 200° C. its length is 100·22 cm. Find the coefficient of linear expansion of stainless steel, and the length of the bar at 0° C. (*Chap.* 18)

10. (*a*) If x varies inversely as y, and $x = 12{\cdot}2$ when $y = 3$, find y when $x = 5$.

(*b*) The resistance to the motion of a body is proportional to the square of its speed. When the speed is 40 ft. per sec., the resistance is 20 lb. wt. Find the resistance when the speed is 60 miles per hour. (*Chap.* 19)

Answers to Revision Examples, Set 1

1. (*a*) $\frac{93}{110}$. (*b*) 19·22. (*c*) 9378·523. (*d*) 0·6844.

2. (*a*) 20·784. (*c*) $v = 104$.

3. (*a*) Copper 151·2 g., manganese 21·6 g. (*b*) 27·63 cm.

4. (*a*) $2240s(X+Y)$ ft.lb. per min.

(*b*) $Y = \dfrac{\text{Work}}{2240s} - X$; $Y = 3$.

5. (*a*) (i) $a^2x - aby + acz - abx + b^2y - bcz$.

(ii) $a^3 - 3a^2b + 3ab^2 - b^3$.

(*b*) $f = \dfrac{uv}{u-v}$; $f = -6\frac{2}{3}$.

6. (*a*) 107,600. (*b*) 607.

7. (*a*) (i) $(a+2b)(a-3b)$. (ii) $(a-c)(b+2d)$.

(iii) $\left(1-\dfrac{a}{5}\right)\left(1+\dfrac{a}{5}+\dfrac{a^2}{25}\right)$.

(*b*) $(4d-1)(d+2)(d-1)(d+1)$.

8. (*a*) (i) $x = -1$; (ii) $x = 2$. (*b*) $x = 30$.

9. (*a*) $x = 1$, $y = \frac{1}{2}$. (*b*) $\alpha = 11 \times 10^{-6}$, $L_0 = 100$ cm.

10. (*a*) $y = 7\cdot32$. (*b*) 96·8 lb. wt.

SET 2

1. (*a*) Add $x+3y-4z$, $3z+4y-2x$, $z-3y$.

(*b*) From $3a-4b+2c$ subtract $3c-2b+a$.

(*c*) $a^2b^2c \times a^3bc \times bc^3$.

(*d*) $\dfrac{x^2y}{z^2} \div \dfrac{xy^2}{z^3}$.

(*e*) Express as a single fraction, $\dfrac{a^2b}{a+b} + \dfrac{ac}{a-b}$.

(*Chaps.* 6, 7 *and* 13)

2. (*a*) Given $\log 2 = 0{\cdot}3010$, $\log 30 = 1{\cdot}4771$, write $\log 12$, $\log 54$, $\log 12{\cdot}8$ without tables.

(*b*) If $\log X = a$, $\log Y = b$, $\log Z = c$, write $\log X^2YZ$ and $\log \dfrac{XY}{\sqrt{Z}}$ in terms of a, b and c. (*Chap.* 8)

3. Evaluate H from the formula

$$H = \frac{2\pi n a^2 i}{\sqrt{(x^2+a^2)^3}},$$

where $\pi = 3{\cdot}142$, $n = 25$, $a = 3{\cdot}5$, $i = 0{\cdot}095$, $x = 7{\cdot}2$. (*Chap.* 10)

4. Factorise the following:

(i) $b^2x - c^2x + ac^2 - ab^2$.

(ii) $4p^2 - 13pq + 3q^2$.

(iii) $x^2 - y^2 + 2xz + z^2$.

(iv) $x^3 + 2x^2 - x - 2$. (*Chaps.* 12 *and* 13)

5. (*a*) Solve for x: (i) $x^2 - 7x = 0$. (ii) $2x^2 - 7x + 3 = 0$.

(*b*) The distance, s ft., covered by a body moving with constant acceleration, is given by

$$s = ut + \tfrac{1}{2}ft^2,$$

where t represents the time during which the body has been in motion. Find t when $s = 192$, $u = 16$, $f = 16$. (*Chap.* 20)

6. The weight of a metal cylinder is given by

$$W = \pi(R^2 - r^2)\,ld.$$

Write a formula for r in terms of the other letters, and evaluate r when $\pi = 3{\cdot}142$, $R = 4{\cdot}5$, $W = 11190$, $l = 50$, $d = 8{\cdot}9$.

(*Chaps.* 16 *and* 9)

7. If $y = x^2 + 8x$, evaluate y when $x = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, 4\frac{1}{2}, 5\frac{1}{2}, 6\frac{1}{2}, 7\frac{1}{2}$, and find the average value of y in this range. (*Chap.* 4)

8. (*a*) Solve for x, $\dfrac{x+4}{3x+5} = 0{\cdot}5$. (*Chap.* 14)

(*b*) One end of a solid cylinder is hollowed out to form a hemispherical depression of the same radius as the cylinder. If the volume of metal left is equal to the volume of metal removed, show that the height (h) of the cylinder is given by $h = \frac{4r}{3}$, where r is the radius. $\left(\text{Volume of hemisphere} = \frac{2\pi r^3}{3}.\right)$ (*Chap.* 15)

9. (a) A straight line passes through the points (1, 3) and (−2, 5). Find its equation.

(b) The Effort (E) required to balance a load (W) was measured for a certain machine in two cases. When $E = 11{\cdot}4$, $W = 12$, and when $E = 15{\cdot}4$, $W = 32$. Find the law of the machine, assuming it to be of the form $E = aW + b$, where a and b are constants. (*Chap.* 18)

10. (a) Express as decimals, correct to three significant figures, $\frac{1}{17}$, 34·26 %, $\frac{2}{3} \times 10^{-2}$.

(b) Write 3·235 and 3·244 correct to two decimal places. The length and breadth of a rectangle are given as 3·24 and 2·42 units, correct to two decimal places. Using three places of decimals, find the difference between the greatest and least values of the area. (*Chap.* 2)

Answers to Revision Examples, Set 2

1. (a) $4y - x$. (b) $2a - 2b - c$. (c) $a^5b^4c^5$. (d) $\frac{xz}{y}$.

(e) $\dfrac{a^3b - a^2b^2 + a^2c + abc}{a^2 - b^2}$.

2. (a) 1·0791, 1·7323, 1·1070.

(b) $2a + b + c$; $a + b - \frac{1}{2}c$.

3. 0·037 (to 3 places).

4. (i) $(x-a)(b-c)(b+c)$. (ii) $(p-3q)(4p-q)$.

(iii) $(x+z-y)(x+z+y)$. (iv) $(x-1)(x+1)(x+2)$.

5. (a) (i) $x = 0$, $x = 7$; (ii) $x = 3$, $x = \frac{1}{2}$. (b) $t = 4$.

6. $r = \sqrt{\left(R^2 - \frac{W}{\pi l d}\right)}$; $r = 3{\cdot}5$.

7. $53\frac{1}{4}$.

8. (*a*) $x = 3$.

9. (*a*) $3y + 2x - 11 = 0$. (*b*) $E = 0{\cdot}2W + 9$.

10. (*a*) 0·0588, 0·343, 0·00667. (*b*) 3·24, 3·24; 0·050.

Set 3

1. (*a*) Express the difference between $\frac{5}{64}$ and 7·821 % accurately in decimal form.

(*b*) In a census of passengers at a railway station for one week it was found that $\frac{1}{4}$ of the total travelled on Saturday, $\frac{1}{10}$ on Sunday, $\frac{1}{5}$ on Monday, $\frac{1}{6}$ on Friday, while the ratio of the numbers travelling on Tuesday, Wednesday and Thursday was 5:6:6. If 84 passengers travelled on Tuesday, find the total for the week. (*Chaps.* 1, 2 *and* 3)

2. (*a*) One cubic centimetre of mercury weighs 13·6 g. Find the weight of 1 cu.ft. of mercury in lb. wt. given 453·6 g. = 1 lb. and 2·54 cm. = 1 in.

(*b*) Express a pressure of 1034 g. wt. per sq.cm. in lb. wt. per sq.in. (*Chaps.* 2 *and* 9)

3. The velocity ratio of a differential pulley, consisting of pulleys whose radii are R and r, is equal to twice the radius of the larger pulley, divided by the difference of the two radii. Write a formula for this velocity ratio (V), and evaluate V when the smaller radius is half the larger radius. What is the approximate value of V when r becomes very small? What happens when the two radii are equal? (*Chap.* 5)

4. (*a*) If $\log 5 = 0{\cdot}6990$, write 50 as a power of 10. Hence write 50 as a power of 100.

(*b*) Evaluate D from the formula

$$D = \frac{Wl^3}{4Ebd^3}$$

when $W = 14$, $l = 180$, $E = 720$, $b = d = 15{\cdot}5$.

(*Chaps.* 8 *and* 10)

5. (*a*) Evaluate $1{\cdot}25^2 - 0{\cdot}25^2$, $1001^2 - 998^2$.

(*b*) Factorise:

$$a^2 - c^2 + b^2 - d^2 - 2ab + 2cd,$$

$$8xp^2 - 4xpq - 12xq^2,$$

$$a^3 + 2a^2 - a - 2.$$

(*Chaps.* 11 *and* 12)

6. (*a*) Solve for x:

$$4(x+2) + 3(5-x) = 4x + 2(10-x) + 2{\cdot}75.$$

(*b*) By increasing the average speed from 35 m.p.h. to 42 m.p.h., the time of a train journey is reduced by 1 hr. Find the length of the journey. (*Chaps.* 14 *and* 15)

7. (*a*) The volume of a sphere is given by $V = \frac{4\pi r^3}{3}$, and its surface area is $A = 4\pi r^2$.

Express the radius in terms of the volume, and express the volume in terms of the radius and the surface area.

(*b*) Find the radius of a sphere whose surface area is 154 sq.in. (Take $\pi = \frac{22}{7}$.) (*Chap.* 16)

8. (*a*) Solve for x and y:

$$x + 2y = \frac{ab+2}{b},$$

$$3y - x = \frac{3-ab}{b}.$$

(*b*) The resistance (R lb. wt.) to the motion of a train may be calculated from the formula $R = A + BV^2$, where A and B are constants, and V is the velocity, in miles per hour. When the speed is 30 m.p.h. the resistance is 26·5, and when the speed is 50 m.p.h. the resistance is 66·5. Find the values of A and B.

(*Chap.* 18)

9. (*a*) Show, by the method of completing the square, that the roots of $ax^2+bx+c=0$ are $x=\dfrac{-b\pm\sqrt{(b^2-4ac)}}{2a}$.

(*b*) Solve for x: $5x^2+7x-6=0$.

(*c*) Find the roots of $5x^2+7x-1=0$ correct to two decimal places. (*Chap.* 20)

10. The number of cubic feet (Q) discharged per hour from a certain orifice is proportional to the square root of the head (h ft.)

When $Q=49$, $h=4$. Find the number of cubic feet discharged when the head is 10 ft., and find the head when 65 cu.ft. are discharged per hour. (*Chap.* 19)

Answers to Revision Examples, Set 3

1. (*a*) 0·000085. (*b*) 1008.

2. (*a*) 848·6 lb. per cu.ft. (*b*) 14·7 lb. per sq.in.

3. $V=\dfrac{2R}{R-r}$; 4; 2; $V=\infty$.

4. (*a*) $50=10^{1\cdot6990}$, $50=100^{0\cdot8492}$. (*b*) 0·4914.

5. (*a*) 1·5, 5997.

(*b*) $(a-b-c+d)(a-b+c-d)$; $4x(2p-3q)(p+q)$;
$(a-1)(a+1)(a+2)$.

6. (*a*) $x=0\cdot25$. (*b*) 210 miles.

7. (*a*) $r=\sqrt[3]{\dfrac{3V}{4\pi}}$, $V=\dfrac{Ar}{3}$. (*b*) $r=3\frac{1}{2}$ in.

8. (*a*) $x=a$, $y=\dfrac{1}{b}$. (*b*) $A=4$, $B=0\cdot025$.

9. (*b*) $x=\frac{3}{5}$, $x=-2$. (*c*) $x=0\cdot13$, $x=-1\cdot53$.

10. 77·49 cu.ft.; 7·037 ft.

SET 4

1. (*a*) Add together $3x+4y-2z$, $4z+2x-3y$, $x+y-z$.

(*b*) From $ab-b^2$ subtract $2a^2-3ab+b^2$.

(*c*) $\frac{ab}{c}\times\frac{2c}{a^2}\times\frac{3a}{c^2}$.

(*d*) $(2a-3b)(4a^2+6ab+9b^2)$.

(*e*) Temperature absolute = temperature Centigrade + 273. Convert to absolute temperatures, 0° C., 273° C., − 121° C.

(*Chaps.* 6 *and* 7)

2. (*a*) Write $\frac{1}{128}$ and $\frac{1}{1024}$ as decimals correct to 5 decimal places.

(*b*) The following series gives a value for $\sqrt{3}$. Give the answer correct to 4 decimal places.

$$2\{1-\tfrac{1}{8}-\tfrac{1}{128}-\tfrac{1}{1024}\}.$$

(*c*) Using this result, calculate $\sqrt{48}$. (*Chaps.* 2 *and* 4)

3. (*a*) Write 64 and 2 as powers of 4, and hence write the logs of 64 and of 2 to the base 4.

(*b*) If $I=\frac{\pi R^4}{2}$, write a formula for $\log I$.

(*c*) If $\log I=\log b+3\log d-1{\cdot}0792$, write a formula for I, and evaluate I when $b=0{\cdot}5$, $d=0{\cdot}65$. (*Chaps.* 8 *and* 10)

4. (*a*) Solve for x: $3x-4=5(4-x)+4$.

(*b*) Solve for x and y:

$$\frac{5}{x}-\frac{3}{y}=1,$$

$$\frac{3}{x}-\frac{5}{y}+9=0.$$

(*c*) Solve for x and y:

$$xy=6,$$

$$x^2y-xy^2=6.$$

(*Chaps.* 14 *and* 17)

5. (*a*) Express d in terms of c and E from the formula

$$c = \sqrt{\frac{E}{d}}.$$

(*b*) The force, in dynes, between two electric charges Q_1 and Q_2, placed d cm. apart, is equal to the product of the charges divided by the square of their distance apart. Write a formula for d in terms of Q_1, Q_2 and F, and calculate d when $F = 16$, $Q_1 = 8$, $Q_2 = 32$.

Two equal charges placed 2 cm. apart exert a force of 9 dynes on each other. Find the magnitude of each charge. (*Chap.* 16)

6. (*a*) Solve for x: $16x - x^2 = 0$.

(*b*) Solve for x: $12 + x - x^2 = 0$.

(*c*) If $ml^2 + kl + d = 0$, and $m = 0{\cdot}25$, $k = 0{\cdot}2$, $d = -1{\cdot}5$, find the positive value of l. (*Chap.* 20)

7. The product of a force in dynes and a distance in cm. is a measure of work in ergs, while the product of a force in lb. wt. and a distance in feet is a measure of work in ft.lb. Express $4{\cdot}2 \times 10^7$ ergs in ft.lb., given that $2{\cdot}54$ cm. = 1 in., 1 g. wt. = 981 dynes, and $453{\cdot}6$ g. wt. = 1 lb. wt. (*Chaps.* 4 *and* 9)

8. (*a*) Express as a single fraction:

$$\frac{2}{9(x-1)} - \frac{2}{9(x+2)} - \frac{11}{3(x^2+4x+4)}.$$

(*b*) Factorise: $36 - 3x - 3x^2$,

$$12 - 6x + 4y - 2xy,$$

$$p^2 - q^2 - p^5 + p^3q^2.$$ (*Chaps.* 12 *and* 13)

9. If the volume of a gas is measured at 0° C. (V_0), and at a temperature t (V_t), the coefficient of expansion of the gas is given by the expansion divided by the product of the volume at 0° C. and the rise in temperature.

Express (i) V_t in terms of the other quantities,

(ii) V_0 in terms of the other quantities.

Find the volume at 56° C. of a gas whose volume at 0° C. is 150 c.c. and whose coefficient of expansion is $\frac{1}{273}$. (Work in fractions.) *(Chaps.* 1 *and* 16)

10. From the Compound Interest formula

$$A = P\left(1+\frac{r}{100}\right)^n,$$

express r in terms of the other quantities, and calculate r when $A = 169{\cdot}6$, $P = 150$, $n = 5$. *(Chaps.* 16 *and* 9)

Answers to Revision Examples, Set 4

1. (*a*) $6x+2y+z$. (*b*) $4ab-2b^2-2a^2$. (*c*) $\frac{6b}{c^2}$.

   (*d*) $8a^3-27b^3$. (*e*) 273°, 546°, 152°.

2. (*a*) 0·00781, 0·00098. (*b*) 1·7324. (*c*) 6·9296.

3. (*a*) $64 = 4^3$, $2 = 4^{\frac{1}{2}}$, $\log_4 64 = 3$, $\log_4 2 = 0{\cdot}5$.

   (*b*) $\log I = 0{\cdot}1962 + 4\log R$. (*c*) $I = \frac{bd^3}{12}$; 0·01144.

4. (*a*) $x = 3{\cdot}5$. (*b*) $x = \frac{1}{2}$, $y = \frac{1}{3}$. (*c*) $x = 3$, $y = 2$.

5. (*a*) $d = \frac{E}{c^2}$. (*b*) $d = \sqrt{\frac{Q_1 Q_2}{F}}$; $d = 4$; $Q_1 = Q_2 = 6$.

6. (*a*) $x = 0$, $x = 16$. (*b*) $x = 4$, $x = -3$. (*c*) $l = 2{\cdot}082$.

7. 3·555 ft.lb.

8. (*a*) $\frac{5-3x}{(x-1)(x+2)^2}$.

   (*b*) $3(4+x)(3-x)$; $(4-2x)(3+y)$;
   $(1-p)(1+p+p^2)(p-q)(p+q)$.

9. (i) $V_t = V_0(1+\gamma t)$. (ii) $V_0 = \frac{V_t}{1+\gamma t}$; $180\frac{210}{273}$ c.c.

10. $r = 100\left\{\sqrt[n]{\left(\frac{A}{P}\right)}-1\right\}$; 2·5.

SET 5

1. (*a*) What is the value of $(2\sqrt{2})^2$? If $\sqrt{2} = 1{\cdot}414$, evaluate $2\sqrt{2}\{3+\sqrt{2}\}$.

(*b*) Calculate the reciprocal of $\sqrt{3}$ correct to 3 decimal places.

(*c*) Calculate $\sqrt{0{\cdot}01423079}$ correct to 4 decimal places.

(*Chaps.* 2 *and* 4)

2. A contract is to be completed in 56 days. After 30 days two-fifths of the work is completed. By what percentage must the labour be increased if the task is to be completed in time, assuming all labourers to be of equal value? (*Chap.* 3)

3. (*a*) Express with positive indices: $\dfrac{1}{\sqrt{x}}, \dfrac{\sqrt[3]{x}}{\sqrt{y}}, \dfrac{a^2\sqrt{b}}{\sqrt{c}}$.

(*b*) If $A = \dfrac{BC^2\sqrt{D}}{E}$, write a formula for $\log A$.

(*c*) If

$$\log t = 0{\cdot}7983 + \tfrac{1}{2}\{\log I - \log M - \log H\},$$

write a formula for t. (*Chaps.* 8 *and* 9)

4. The weight of n spheres, radius r, made of a metal whose weight is d g. per c.c., is W. Write a formula for the radius of one sphere in terms of W, n and d. $\left(\text{Volume of sphere} = \dfrac{4\pi r^3}{3}.\right)$

Calculate r, when $W = 70{\cdot}98$, $n = 50$, $d = 7{\cdot}9$.

(*Chaps.* 10 *and* 16)

5. (*a*) Divide $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ by $x^2 + 2xy + y^2$, and hence write the fourth root of the dividend.

(*b*) From the formula $v^2 - u^2 = 2fs$, evaluate s if $v = 7{\cdot}905$, $u = 6{\cdot}905$, $f = 7{\cdot}405$.

(*c*) Evaluate the dividend in (*a*) if $x = 2$, $y = -1$.

(*Chap.* 11)

6. (*a*) Solve for x:

$$4x - 2y = 3x(2y+1) - 2x - 2y(3x+2y).$$

(*b*) A rectangular sheet of metal, 12 in. by x in., is to have squares of 2 in. side cut from its four corners, so that the remainder may fold into an open box of volume 96 cu.in. Find x.

(*Chap.* 15)

7. The total resistance to the motion of a car consists of the sum of a constant force and a force which varies as the square of the velocity. When the velocity is 25 ft. per sec., the resistance is 96·25 lb. wt., and when the velocity is 30 ft. per sec., the resistance is 110 lb. wt. Find the resistance when the velocity is 40 ft. per sec. (*Chaps.* 18 *and* 20)

8. (*a*) Factorise $a^3 - ab^2 + 2a^2b - 2b^3$.

(*b*) Factorise $a^3 - 6a^2b + 11ab^2 - 6b^3$.

(*c*) Factorise $5l^2 + 7lm - 6m^2$.

(*d*) Solve for x: $5x^2 + 7x - 6 = 0$.

(*e*) Solve for x: $x^3 - 6x^2 + 11x - 6 = 0$.

(*Chaps.* 12, 13 *and* 20)

9. (*a*) If x and y are both positive, solve for x and y:

$$x^2 + 2xy + y^2 = 16,$$
$$x^2 - 2xy + y^2 = 4.$$

(*b*) a and b are positive, and

$$a^2 - b^2 = 32,$$
$$a^2 + 2ab + b^2 = 64.$$

Find the ratio of a to b. (*Chap.* 17)

10. (*a*) Convert a speed of V miles per hour into a speed measured in feet per second.

(*b*) The horse-power of an engine is equal to the force exerted (in lb. wt.) multiplied by the distance in feet per second, divided by 550. Write an expression for the horse-power of an engine exerting a force of W tons wt. travelling at V miles per hour, and evaluate when $W = 10$, $V = 30$. (*Chap.* 5)

Answers to Revision Examples, Set 5

1. (*a*) 8; 12·484. (*b*) 0·577. (*c*) 0·1193.

2. $73\frac{10}{13}$%.

3. (*a*) $\dfrac{1}{x^{\frac{1}{2}}}$, $\dfrac{x^{\frac{1}{3}}}{y^{\frac{1}{2}}}$, $\dfrac{a^2b^{\frac{1}{2}}}{c^{\frac{1}{3}}}$.

   (*b*) $\text{Log}\,A = \log B + 2\log C + \frac{1}{2}\log D - \log E$.

   (*c*) $t = 6{\cdot}284\sqrt{\dfrac{I}{MH}}$.

4. $r = \sqrt[3]{\dfrac{3W}{4\pi nd}}$; 0·35.

5. (*a*) $x+y$. (*b*) $s = 1$. (*c*) 1.

6. (*a*) $x = \dfrac{2y(1-2y)}{3}$. (*b*) $x = 10$.

7. 145 lb. wt.

8. (*a*) $(a+b)(a-b)(a+2b)$. (*b*) $(a-b)(a-2b)(a-3b)$.

   (*c*) $(5l-3m)(l+2m)$. (*d*) $x = \frac{3}{5}$, $x = -2$.

   (*e*) $x = 1$, $x = 2$, $x = 3$.

9. (*a*) $x = 3$, $y = 1$. (*b*) $\dfrac{a}{b} = 3$.

10. (*a*) V m.p.h. $= \dfrac{22V}{15}$ ft. per sec.

    (*b*) H.P. $= \dfrac{2240W}{550} \cdot \dfrac{22V}{15}$; 8960.

REVISION EXAMPLES ON PART 2

SET 1

1. Two ships leave port together. The slower one travels N. 30° E. at 11 m.p.h. and the faster travels N. 30° W. at 22 m.p.h. Find, by drawing and by calculation, their distance apart after 2 hr., and the bearing of the farther ship from the nearer. (*Chaps.* 21 *and* 30)

2. A small glass tank for optical work consists of four sides and a base made of glass $\frac{1}{4}$ in. thick. The external dimensions are base 3 in. square, and height 3 in. Find the weight of the tank if the specific gravity of the glass is 2·5, and 1 cu.ft. of water weighs 62·3 lb. (*Chaps.* 29 *and* 31)

3. (*a*) Draw the straight line which passes through the points (1, 3) and (– 2, – 21) and write its equation.

(*b*) On the same axes plot $y = 3x^2$ from $x = -2$ to $x = +2$ and write the abscissae of the points of intersection of the two graphs. (*Chaps.* 27 *and* 28)

4. (*a*) A vertical post AB is supported by a stay AC, 10 ft. long, inclined at 60° to the ground CB, which is horizontal. A second string CD is fastened to the post at D, such that CD bisects the angle ACB. Calculate the lengths of CD and of DB.

(*b*) The tension in a steel rope supporting a vertical post is 10 lb. wt. Find the horizontal and vertical components of the force if the rope makes an angle of 40° with the post. (*Chap.* 33)

5. From the doorway of a building the elevation of a factory chimney is 54°, while from a window 20 ft. above the doorway the elevation of the chimney is 50°. Find the height of the chimney. (*Chaps.* 34 *or* 36)

6. (*a*) Solve graphically for x and y:

$$3x + 4y = 5,$$
$$2x - 3y = 9.$$

(*b*) On the same axes plot $xy = -3$, and write the values of x and y which satisfy the equations $2x - 3y = 9$, $xy = -3$. (*Chaps.* 38 *and* 28)

7. The resistance to the motion of a body is connected with the velocity by an equation of the form $R = a + bV$, where R represents the resistance and V the velocity, and a and b are constants. Find the law from a linear graph using the following figures:

V	10	20	30	40	50
R	22	30	38	46	54

(*Chap.* 39)

8. $ABCD$ is a trapezium, whose parallel sides AB and DC are 7 in. and 12 in. respectively, and the distance between them is 6 in. DE cuts AB in E so that the triangle ADE is one-third of the area of the trapezium. Find the distance AE. (*Chap.* 24)

9. ABC is any triangle, and E is the mid-point of BC. CD is drawn parallel to AB, to cut AE produced at D. AF is parallel to BC, and DCF is a straight line. Prove that the triangles ABE and DCE are congruent, and that the area of the triangle AFD is four times the area of the triangle ABE. (*Chaps.* 21 *and* 23)

10. (*a*) Prove from first principles that $\sec^2 A = 1 + \tan^2 A$.

(*b*) Prove $2 \tan A + \sec^2 A = (\tan A + 1)^2$.

(*c*) Express an angle of $50\pi t$ radians in degrees, if $t = 0{\cdot}001$. Hence write the value of $100 \sin 50\pi t$. (*Chaps.* 35 *and* 37)

Answers to Revision Examples, **Set 1**

1. 38·104 miles, N. 60° W.

2. 0·8845 lb.

3. (*a*) $y = 8x - 5$. (*b*) $x = \frac{5}{3}$, $x = 1$.

4. (*a*) $CD = \frac{10}{\sqrt{3}}$ ft., $DB = \frac{5}{\sqrt{3}}$ ft.

(*b*) Vertical force = 7·66 lb. wt.; horizontal force = 6·428 lb. wt.

5. 149 ft.

6. (*a*) $x = 3,\ y = -1$. (*b*) $x = 3,\ y = -1,\ x = \frac{3}{2},\ y = -2$.

7. $R = 14 + 0{\cdot}8V$.

8. $AE = 6\frac{1}{3}$ in.

10. (*c*) 9·0018°; 15·64.

Set 2

1. Three circles, each radius 1 in., are drawn so that each one touches the other two. A triangle is drawn round these circles, with each side touching two of the circles. Show that the length of one side of the triangle is $2(1+\sqrt{3})$ in., and that the area of the triangle is given by $\sqrt{3}(1+\sqrt{3})^2$ sq.in. (*Chaps.* 22 *and* 30)

2. The shape of a metal plate is the area between the curve $y = \sqrt{x}$, the x axis, and the ordinates at $x = 0$ and $x = 8$, the units being inches. Calculate the ordinates at every odd half-inch along the x axis and so determine the area of the plate by the mid-ordinate rule.

The true area is given by $\frac{2}{3}\sqrt{x^3}$, where $x = 8$. Show that the mid-ordinate rule gives the area correct to the first decimal place. (*Chap.* 25)

3. The table gives the ranges (R) for a mortar against the corresponding angles of elevation ($\theta°$). Plot R against θ and determine the maximum range of the mortar, and the corresponding angle of elevation.

R yd.	34·2	50	76·6	86·6	93·9	99·3	98·5	89·9	76·6	64·3
$\theta°$	10	15	25	30	35	42	50	58	65	70

(*Chap.* 26)

4. Plot $y = x^2 - 2x - 3$ from $x = -3$ to $x = 4$, and use the graph to solve the following equations:

(i) $x^2 - 2x - 3 = 0$. (ii) $x^2 - 2x + 1 = 0$. (iii) $x^2 - 3x - 2 = 0$.

(*Chap.* 40)

5. (*a*) Write down the length of the hypotenuse of an isosceles triangle containing a right angle and two equal sides of length a.

(*b*) If $\sin A = 0{\cdot}3$, evaluate $\cos^2 A - \sin^2 A$.

(*c*) Find, from the tables, the angle whose sine is 0·3, and the angle whose cosine is the answer to (*b*), and thus verify that

$$\cos 2A = \cos^2 A - \sin^2 A.$$

(*Chaps.* 33 *and* 35)

6. An aircraft flying at 1000 ft. is approaching an observer who finds that in 3 sec. the elevation changes from 32° to 40°. Find the speed of the aircraft in feet per second.

(*Chaps.* 34 *or* 36)

7. A motorist travels at 35 m.p.h. on a journey of 70 miles, but rests for 15 min. after the first hour. A second motorist covers the same journey, starting 30 min. after the first and arriving 15 min. before him. Find graphically (*a*) the speed of the second motorist, (*b*) the time and place at which he overtakes the first motorist.

(*Chap.* 38)

8. (*a*) A line passes through the origin and the point $(-1, 2)$. Write the equation of the line parallel to this one, and passing through $(0, -1)$.

(*b*) Two variables x and y are related by a law of the form $y = A + \frac{B}{x}$, where A and B are constants. Find the law from the following figures by plotting a linear graph:

x	1	3	6	9	12
y	7	5	4·5	4·3	4·25

(*Chap.* 39)

9. Points A, B, C, D on the axes of co-ordinates are respectively $(-2\frac{1}{2}, 0)$, $(2\frac{1}{2}, 0)$, $(2, 5)$ and $(-2, 5)$. AD, DC, CB are joined by straight lines while A and B are joined by a semicircle, radius $2\frac{1}{2}$ units, in the third and fourth quadrants. The figure represents the vertical section through a buoy, the units being feet, the horizontal section of which is circular. Find the volume of the buoy.

(*Chaps.* 27 *and* 31)

10. The side BA, of the triangle BAC, is produced to F. The bisector of angle BAC meets BC in D, while the bisector of angle FAC meets DC produced in E. If angle $BAC = 2x$, write the value of the difference between angle ACE and angle ADC.

What is the sum of angles ADC and AEC? (*Chap.* 22)

Answers to Revision Examples, Set 2

2. 15·1 sq.in.

3. 45°, 100 yd.

4. (i) $x = 3$, $x = -1$. (ii) Line, $y = -4$, $x = 1$, $x = 1$.
(iii) Line, $y = x - 1$, $x = -0{\cdot}56$, $x = 3{\cdot}56$.

5. (*a*) $a\sqrt{2}$. (*b*) 0·82. (*c*) Angles 17° 28′ and 34° 56′.

6. 136·2 ft. per sec.

7. (*a*) 46·67 m.p.h. (*b*) $1\frac{1}{4}$ hr., 35 miles.

8. (*a*) $y + 2x + 1 = 0$. (*b*) $y = 4 + \frac{3}{x}$.

9. 112·6 cu.ft.

10. x; 90°.

SET 3

1. Plot $y = x^3$ from $x = -4$ to $x = +4$ using half-units. On the same axes and with the same scales plot $xy = 16$. Show, by substitution, that the co-ordinates of the points of intersection of the two curves are solutions to the simultaneous equations $y = x^3$, $xy = 16$. (*Chap.* 28)

2. A rectangle of length y and breadth x is cut from a circle of diameter d, so that its corners lie on the circumference. Prove that the area of the rectangle is $x\sqrt{(d^2 - x^2)}$.

If $x = 5$, find the area remaining when the rectangle has been removed, given that $d = 13$. (*Chaps.* 24 *and* 30)

3. The curved surface of a cone is formed from a sector of a circle, radius a, containing an angle θ radians. Show that the volume of the cone so formed is

$$V = \frac{a^3\theta^2}{24\pi^2}\sqrt{(4\pi^2-\theta^2)}.$$

(*Chaps.* 29, 30 *and* 37)

4. (*a*) The jib of a dockside crane is l ft. long and is inclined at an angle θ to the horizontal. What is the height of the top of the jib above the bottom?

(*b*) Calculate θ when $l = 50$ ft., and the height is 35 ft.

(*c*) What is the height if $l = 50$ ft. and $\theta = 54°$?

(*d*) If $\sin\theta = \frac{a}{b}$, write the values of $\cos\theta$ and $\tan\theta$.

(*Chaps.* 34 *and* 35)

5. (*a*) Prove that $\frac{\cos A \tan A}{\operatorname{cosec} A} = 1-\cos^2 A$.

(*b*) Two stays on the same side of a mast are attached to the top of the mast and to two points on the ground in a line with the foot of the mast. The stays make angles of 45° and 50° with the horizontal, and their lower ends are 20 ft. apart. Find the height of the mast. (*Chaps.* 34 *and* 35)

6. (*a*) Join the points $A(3, 2)$, $B(-5, 6)$ and $C(1, -4)$ by straight lines and write the equations of the lines.

(*b*) If D is the mid-point of AB, and E is the mid-point of BC, write the equations of CD and AE. (*Chap.* 27)

7. The following figures are experimental results from a set of pulleys. Load (W) and effort (E) are related by a law of the form $W = AE + B$, where A and B are constants. Determine the law as accurately as possible.

W (lb.)	10	15	22	30	35	42
E (lb.)	6	7·3	9·1	11	12·2	14

(*Chap.* 39)

8. A hollow copper sphere weighs 1263 g. Its outer radius is 5 cm. Find the inner radius if the specific gravity of copper is 8·9. (*Chap.* 31)

9. (*a*) Express in degrees: $\frac{\pi}{12}$ radians, $\frac{3\pi}{4}$ radians, 3·1 radians.

(*b*) A dynamo is revolving 3000 times in 1 min. Express this angular velocity in radians per second.

(*c*) If the voltage is given by $V = A\cos 100\pi t$, what are the highest and lowest values the voltage can attain?

(*Chaps.* 33 *and* 37)

10. Solve $2x^2 - 5x - 7 = 0$ by two graphical methods:

(i) By plotting $y = 2x^2 - 5x - 7$ from $x = -4$ to $x = +4$.

(ii) By plotting $y = x^2$ and a linear graph. (*Chap.* 40)

Answers to Revision Examples, Set 3

1. $x = 2,\ y = 8;\ x = -2,\ y = -8$.

2. 72·8.

4. (*a*) $l\sin\theta$. (*b*) 44° 25′. (*c*) 40·45 ft.

(*d*) $\frac{\sqrt{(b^2 - a^2)}}{b}$, $\frac{a}{\sqrt{(b^2 - a^2)}}$.

5. 124·3 ft.

6. (*a*) $2y + x = 7,\ y = 3x - 7,\ 3y + 5x + 7 = 0$.

(*b*) $y + 4x = 0,\ 5y - x = 7$.

7. $W = 4E - 14$.

8. 4·5 cm.

9. (*a*) 15°, 135°, 177·6°.

(*b*) 100π radians per sec.

(*c*) A and $-A$.

10. (i) $x = 3·5,\ x = -1$.

(ii) Line, $y = \frac{5x}{2} + \frac{7}{2}$, values of x as above.

SET 4

1. $ABCDEF$ is a regular hexagon. FB and BE are joined. Write the values of angles AFB, ABF.

If a side of the hexagon is of length a, write the areas of the triangles AFB, FBE, and of the figure $BEDC$.

(*Chaps.* 24 *and* 30)

2. When a body is immersed in a liquid it appears to lose weight, the loss in weight being equal to the weight of liquid displaced. Find the apparent weight of a solid cylinder of brass, radius 1·5 cm., length 3·5 cm., of specific gravity 8·7, when completely immersed in (*a*) water, (*b*) methylated spirit of specific gravity 0·8.

(*Chap.* 31)

3. A rectangle of sides x and y has a perimeter of 48 in. Show that the areas of all rectangles fulfilling these conditions are given by

$$\text{Area} = 24x - x^2.$$

Plot area, vertically, against x, horizontally, from $x = 0$ to $x = 24$ in units of 2, and from the graph find the dimensions of the rectangle which has the greatest area. (*Chap.* 28)

4. A beam 10 ft. long rests on two supports at its ends. The shear force (s) at any point x ft. from the left-hand end is given by $s = 25 - 5x$, while the bending moment (M) at any point x ft. from the left-hand end is given by $M = 25x - \frac{5x^2}{2}$. Plot s against x, and M against x on the same axes, and from the graph determine

(i) the maximum value of M,

(ii) the value of s at the same point on the beam,

(iii) the values of s when $M = 0$. (*Chaps.* 27 *and* 28)

5. (*a*) A weight W is hung by two strings each inclined to the vertical at 35°. The tension in each string is 5 lb. wt. The value of W is the sum of the vertical components of the tensions. Find W.

(*b*) What angle would the strings make with the vertical if their tensions remained the same, and the value of W was 8 lb.?

(*c*) If $V = 250 \sin\theta$, write, without tables, the values of V when $\theta = 30°$, $\theta = 60°$, $\theta = 90°$, $\theta = 120°$. (*Chaps.* 33 *and* 36)

6. A pyramid whose sloping edges are 3 in. long stands on a square base of side 2 in.

Find (*a*) the angle between a sloping edge and the horizontal,

(*b*) the angle between the horizontal and a line drawn down the centre of a sloping face. (*Chaps.* 30 *and* 32)

7. Values for the current (I) through a tangent galvanometer, and the deflection ($\theta°$) are given. Test by a linear graph for a relation of the form $I = K \tan\theta$, and find the value of K.

$\theta°$	35	40	45	50	55
I (amperes)	0·11	0·13	0·16	0·19	0·23

(*Chap.* 39)

8. Half of a regular octagon of side 4 in. is folded to form the sloping sides of a square pyramid. Find the area of the sloping sides, and the volume of the pyramid. (*Chaps.* 29, 30 *and* 34)

9. (*a*) Express as radians in terms of π: $22\frac{1}{2}°$, 18°, 54°, 270°.

(*b*) The arm of a rotary sprinkler at a sewage works is 30 ft. long, and completes one revolution in 45 sec. Express its angular velocity in radians per second, and hence find the area it covers in 1 sec. (*Chap.* 37)

10. (*a*) Write the equation of the straight line which passes through (0, 2) and (1, −7).

(*b*) Plot $y = 4x^2 + x - 4$ from $x = -4$ to $x = +4$. On the same axes plot the line in (*a*) above, and write the abscissae of the points of intersection of the two graphs. Find the quadratic equation of which these values are the roots.

(*Chaps.* 27 *and* 40)

Answers to Revision Examples, Set 4

1. $30°$, $30°$, $\frac{a^2\sqrt{3}}{4}$, $\frac{a^2\sqrt{3}}{2}$, $\frac{3a^2\sqrt{3}}{4}$.

2. (*a*) 190·5 g. (*b*) 195·4 g.

3. Length and breadth 12 in.

4. (i) $M = 62\frac{1}{2}$. (ii) $s = 0$. (iii) $s = 25$, $s = -25$.

5. (*a*) 8·192 lb. wt. (*b*) $36° 52'$.
   (*c*) $V = 125$, $V = 125\sqrt{3}$, $V = 250$, $V = 125\sqrt{3}$.

6. (*a*) $61° 54'$. (*b*) $69° 17'$.

7. $I = 0{\cdot}16 \tan \theta$.

8. 38·63 sq.in., 23·28 cu.in.

9. (*a*) $\frac{\pi}{8}$, $\frac{\pi}{10}$, $\frac{3\pi}{10}$, $\frac{3\pi}{2}$. (*b*) $\frac{2\pi}{45}$ rad. per sec., 62·84 sq.ft.

10. (*a*) $y = 2 - 9x$.
    (*b*) $x = \frac{1}{2}$, $x = -3$, $2x^2 + 5x - 3 = 0$.

SET 5

1. Find the weight of a cylinder of brass, length 3·5 cm. and radius 2·5 cm., if the specific gravity of brass is 8·7.

A small cavity is drilled in the cylinder and the space filled with lead of specific gravity 11·4, so that the solid weighs 600 g. Find the volume of the cavity. (*Chaps.* 29 *and* 31)

2. The table gives the fusing currents (I amperes) for wires of different diameters (d in.) in the case of tin.

Plot d against I and from the graph find approximately (i) the fusing current for diameter 330×10^{-4} in., (ii) the diameter for a fusing current of 14 amps. (*Chap.* 26)

d	48×10^{-4}	108×10^{-4}	148×10^{-4}	220×10^{-4}
I	0·55	1·84	2·96	5·36

d	280×10^{-4}	360×10^{-4}	400×10^{-4}	480×10^{-4}
I	7·69	11·2	13	17·2

3. A screen containing a triangular hole is placed 6 in. from an arc light. The image of the hole is caught on a white screen placed 5 ft. 6 in. from the arc lamp. If the triangle is right-angled and contains two sides of length 1 in., find the area of the image on the white screen. (*Chaps.* 23 *and* 24)

4. (*a*) Solve graphically for x and y:

$$y - 3x + 3 = 0, \quad 9y + x - 15 = 0.$$

(*b*) Find the equations to the lines passing through the point of intersection in (*a*), and having gradients of 1 and -1. (*Chaps.* 27 *and* 38)

5. (*a*) If $\sin\theta = \frac{5}{13}$, $\cos\phi = \frac{24}{25}$, find the value of

$$4\cos\theta + 3\sin\phi.$$

(*b*) Prove

$$(\cot A + 1)(\tan A - 1) = \frac{(\sin A - \cos A)(\sin A + \cos A)}{\sin A \cos A}.$$

(*c*) Evaluate, without tables,

$$3\cos^2 30^\circ + 5\sin^2 45^\circ - 2\cos^2 60^\circ.$$

(*Chaps.* 33 *and* 35)

6. A stretch of straight road runs due north. From one milestone the bearing of a building is N. 35° W., and from the next milestone the bearing is N. 42° W. Find the distance of the building from the road and from the first milestone, in yards. (*Chaps.* 34 *or* 36)

7. The sum of the length and the circumference of a cylinder is 6 ft. Prove that the volume is given by $V = 2\pi r^2(3 - \pi r)$.

Find the volume when the diameter is 1 ft. (*Chap.* 29)

8. (*a*) Find the area of a rhombus whose sides are 13 in. long, and one of whose diagonals is 10 in.

(*b*) A rhombus of side 13 in. is pin-jointed at the corners. Find (i) the greatest area it can have; (ii) the greatest and least values of the sum of the lengths of the diagonals.

(*Chaps.* 24 *and* 30)

9. The path of a projectile thrown with velocity V, at an angle θ to the horizontal, is given by

$$y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta},$$

where $g = 32$.

Plot the path when $V = 32$, $\theta = 45°$, taking x from 0 to 32 ft. in units of four.

Find from the graph (i) the highest wall the projectile could clear, (ii) how far from the point of projection is a wall 7 ft. 6 in. high, if the projectile is to strike its flat top. (*Chaps.* 28 *and* 32)

10. The vertical section through a cone is an equilateral triangle of side l. Prove that the shape of its curved surface is a semicircle, and find its volume. (*Chaps.* 29 *and* 30)

Answers to Revision Examples, Set 5

1. 597·8 g.; 0·815 c.c.

2. (i) 10 amp. (ii) 420×10^{-4} in.

3. 60·5 sq.in.

4. (*a*) $x = 1{\cdot}5$, $y = 1{\cdot}5$. (*b*) $y = x$, $y + x = 3$.

5. (*a*) $\frac{1473}{325}$. (*c*) $4\frac{1}{4}$.

6. 5543 yd., 9663 yd.

7. 2·246 cu.ft.

8. (*a*) 120 sq.in. (*b*) (i) 169 sq.in.; (ii) 36·764, 26.

9. (i) 8 ft. (ii) 20 ft.

10. $\dfrac{\pi l^2\sqrt{3}}{24}$.

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FOUR-FIGURE TABLES

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	·0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	·0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
12	·0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	·1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	·1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	14	17	20	23	26
15	·1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	·2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	·2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	·2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	4	7	9	11	14	16	18	21
19	·2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	·3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	·3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	·3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	·3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	·3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	·3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	·4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	·4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	·4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	·4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	·4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	·4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	·5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	·5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	·5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	·5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	·5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	·5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	·5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	·5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	·6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	·6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	·6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	·6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	·6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	·6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	·6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	·6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	·6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	·6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	·6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	·7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	·7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	·7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	·7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	·7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	·7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	·7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	·7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	·7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	·7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	·7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	·7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	·7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	·8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	·8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	·8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	·8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	·8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	·8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	·8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	·8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	·8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	·8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	·8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	·8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	·8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	·8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	·8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	·8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	·9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	·9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	·9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	·9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	·9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	·9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	·9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	1	2	3	3	4	4	5
87	·9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	·9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	·9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	·9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	·9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	·9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	·9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	·9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	·9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	·9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	·9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	·9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	·9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTI-LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTI-LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

NATURAL SINES

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	14
5	·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	11	14
10	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	13
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	13
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

NATURAL SINES

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	·7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	4	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	1
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	0	0
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	·9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0
90	1·000														

NATURAL COSINES

SUBTRACT

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	1·0000	1·000	1·000	1·000	1·000	1·000	**9999**	**9999**	**9999**	**9999**					
1	·9998	9998	9998	9997	9997	9997	9996	9996	9995	9995					
2	·9994	9993	9993	9992	9991	9990	9990	9989	9988	9987					
3	·9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	·9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	·9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	1
6	·9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	·9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	·9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	·9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10	·9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	·9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	·9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	·9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	·9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	·9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	·9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	·9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	4	4
18	·9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	·9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20	·9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	3	4	5
21	·9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	·9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	·9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	·9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	·9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	·8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	·8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	·8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	·8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30	·8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	·8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	·8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	·8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	·8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	·8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	·8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	·7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	·7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	·7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40	·7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	9
41	·7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	·7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	·7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	·7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10

The black type indicates that the integer changes.

NATURAL COSINES

SUBTRACT

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	·7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	·6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	·6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	·6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	·6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50	·6428	6414	6401	6388	6374	6361	6347	6334	6320	6307	2	4	7	9	11
51	·6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	·6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	·6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	·5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	9	12
55	·5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2	5	7	10	12
56	·5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	·5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	·5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	·5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60	·5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3	5	8	10	13
61	·4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	·4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	·4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	·4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	·4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	3	5	8	11	13
66	·4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	13
67	·3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	13
68	·3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	·3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70	·3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	3	5	8	11	14
71	·3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	·3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	·2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3	6	8	11	14
74	·2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3	6	8	11	14
75	·2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3	6	8	11	14
76	·2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	·2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	9	11	14
78	·2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	·1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3	6	9	11	14
80	·1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3	6	9	11	14
81	·1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	·1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	·1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	·1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	·0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	3	6	9	12	14
86	·0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	·0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	15
88	·0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	·0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15
90	·0000														

NATURAL TANGENTS

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	0·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	0·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	0·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	0·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	0·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	0·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	0·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	0·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	0·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	0·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	0·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	0·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	0·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	0·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	0·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	0·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	0·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	0·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20	0·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	0·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	0·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	0·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	0·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	0·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	0·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	0·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	0·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	0·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	0·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	0·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	0·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	0·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	0·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	0·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	0·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	0·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	0·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	0·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	0·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	0·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	0·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	0·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	0·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

ALLAMA IQBAL LIBRARY

30863

NATURAL TANGENTS

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	26	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	**0057**	**0145**	**0233**	**0323**	**0413**	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	91
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	130
71	2·9042	9208	9375	9544	9714	9887	**0061**	**0237**	**0415**	**0595**	29	58	87	116	144
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	97	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	203
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	53	107	160	214	267
77	4·3315	3662	4015	4373	4737	5107	5483	5864	6252	6646	62	124	186	248	310
78	4·7046	7453	7867	8288	8716	9152	9594	**0045**	**0504**	**0970**	73	146	220	293	366
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140	87	175	263	350	438
80	5·671	5·730	5·789	5·850	5·912	5·976	6·041	6·107	6·174	6·243					
81	6·314	6·386	6·460	6·535	6·612	6·691	6·772	6·855	6·940	7·026					
82	7·115	7·207	7·300	7·396	7·495	7·596	7·700	7·806	7·916	8·028					
83	8·144	8·264	8·386	8·513	8·643	8·777	8·915	9·058	9·205	9·357					
84	9·51	9·68	9·84	10·02	10·20	10·39	10·58	10·78	10·99	11·20					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95			Differences untrustworthy here		
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					
90	∞														

The black type indicates that the integer changes.

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